About Stefan’s definition of a foliation with singularities : a reduction of the axioms


<http://www.numdam.org/item?id=BSMF_1990__118_4_391_0>
ABOUT STEFAN'S DEFINITION OF A FOLIATION WITH SINGULARITIES : A REDUCTION OF THE AXIOMS

BY

JAN KUBARSKI (*)

RÉSUMÉ. — L'article présent concerne la définition d'un feuilletage de Stefan. Le résultat principal de cet article est le fait qu'un axiomme de la définition d'un feuilletage de Stefan [4, chap. 1] est une conséquence des autres.

ABSTRACT. — The aim of this paper is to give an accurate proof of the fact formulated in [3, p. 45] that one of the axioms of Stefan's foliations [4, chap. 1] follows from the remaining ones.

The following definitions of a foliation with singularities comes from the work by P. Stefan [4].

Suppose $V$ is a connected Hausdorff $C^\infty$ and paracompact (equivalently and with a countable basis) manifold of dimension $n$. By a foliation of $V$ with singularities we mean a partition $\mathcal{F}$ of $V$ into sets such that:

1. for each element $L \in \mathcal{F}$, there exists a structure of differentiable manifold $\sigma$ on $L$ such that
   - (i) $(L, \sigma)$ is a connected immersed submanifold of $V$,
   - (ii) $(L, \sigma)$ is a leaf of $V$ with respect to all locally connected topological spaces, i.e. if $X$ is an arbitrary locally connected topological space and $f : X \rightarrow V$ is a continuous function such that $f[X] \subset L$, then $f : X \rightarrow (L, \sigma)$ is continuous;
2. for each $x \in V$, there exists a local chart $\varphi$ on $V$ around $x$ with the following properties:
   - (a) $\varphi$ is a surjection $D_\varphi \rightarrow U_\varphi \times W_\varphi$ where $U_\varphi, W_\varphi$ are open neighbourhoods of 0 in $\mathbb{R}^k$ and $\mathbb{R}^{n-k}$, respectively, and $k$ is the dimension of the leaf through $x$ (denoted by $L_x$);
   - (b) $\varphi(x) = (0, 0)$.

(*) Texte reçu le 19 mai 1988, révisé le 26 septembre 1990.
J. KUBARSKI, Institute of Mathematics, Technical University of Lodz, Al. Politechniki 11, 90-924 Lodz, Pologne.

BULLETIN DE LA SOCIÉTÉ MATHÉMATIQUE DE FRANCE 0037-9484/1990/391/$ 5.00 © Société mathématique de France
(c) if $L \in \mathcal{F}$, then $\varphi[L \cap D_\varphi] = U_\varphi \times \ell_{\varphi,L}$ where

$$\ell_{\varphi,L} = \{w \in W_\varphi : \varphi^{-1}(0,w) \in L\}.$$  

A chart $\varphi$ which fulfils the above condition is called distinguished around $x$.

**Theorem.** — Let $\mathcal{F}$ be a partition of $V$ into connected immersed submanifolds of $V$, fulfilling (2). Then $\mathcal{F}$ is a foliation with singularities.

**Remark.** — This theorem is formulated in [3, p. 45] without an accurate proof. The author say that it easily follows in the same way as in the case without singularities, indicating [1]. It turns out that this theorem needs a subtler proof. The reasoning as in [1] gives the proof provided some added assumption

$$\frac{\partial}{\partial \varphi^i} \bigg|_{y} \in T_y(L_x) \text{ for } i \leq k \text{ and all } y \in L_x \cap D_\varphi, \ k = \dim L_x,$$  

is satisfied, which is exactly the body of Stefan’s lemma [4, Lemma 3.1]. That this added condition follows from the remaining ones is the aim of our paper.

**Proof of the Theorem:** according to Stefan ([4, Lemma 3.1]), it is sufficient to show that each distinguished chart $\varphi = (\varphi^1, \ldots, \varphi^n)$ around $x$ has the property (1).

Assume to the contrary that, for a distinguished chart $\varphi$ around $x$, this property does not hold at a point $y_0 \in L_x \cap D_\varphi$. Then, of course, there exists a vector $v \in T_{y_0}(L_x)$ such that

$$\varphi_{*y_0}(v) \notin T_{(\tilde{y}_0, c_0)}(U_\varphi \times \{c_0\})$$

where $(\tilde{y}_0, c_0) = \varphi(y_0), \tilde{y}_0 \in U_\varphi, c_0 \in W_\varphi$.

Take any smooth curve

$$c : (-\varepsilon, \varepsilon) \longrightarrow L_x, \quad \varepsilon > 0,$$  

such that $c(0) = y_0, \dot{c}(0) = v$ and $\operatorname{Im} c \subset D_\varphi$. Consider the curve $\varphi \circ c : (-\varepsilon, \varepsilon) \rightarrow \mathbb{R}^n$. Let $pr_2 : U_\varphi \times W_\varphi \rightarrow W_\varphi$ denotes the projection onto the second factor. By (2):

$$(pr_2)_*(((\varphi \circ c)(0))) \neq 0.$$
Diminishing $\epsilon > 0$, if necessary, we may assume that

$$pr_2 \circ \varphi \circ c : (-\epsilon, \epsilon) \rightarrow W_\varphi$$

is an embedding. Denote the set $\text{Im}(pr_2 \circ \varphi \circ c)$ by $I$. Of course,

$$U_\varphi \times I \subset U_\varphi \times \ell_\varphi, L_x$$

(because $I \subset pr_2 \circ \varphi[D_\varphi \cap L_x] = \ell_\varphi, L_x$) and $U_\varphi \times I$ is a $k + 1$-dimensional hypersurface of $\mathbb{R}^n$, thus a locally compact space. Put — for brevity —

$$M := L_x \cap D_x$$

understanding it as an open submanifold of $L_x$, and consider the injective immersion

$$\tilde{\varphi} : M \rightarrow \mathbb{R}^n, \quad x \mapsto \varphi(x).$$

By the above $\tilde{\varphi}[M] \supset U_\varphi \times I$.

For each point $x \in M$, we choose a neighbourhood $U(x) \subset M$ of $x$ such that

$$\tilde{\varphi}|_{U(x)} : U(x) \rightarrow \mathbb{R}^n$$

is an embedding. By the assumption of the second axiom of countability of $V$, each connected immersed submanifold of $V$ fulfils this axiom (see Appendix). Then $M$, as an open submanifold of the manifold $L_x$, has a countable basis. Choose a countable open covering $\{U_i ; i \in \mathbb{N}\}$ of $M$ such that each $\bar{U}_i$ is compact and contained in some $U(x_i)$. We prove that

$$\tilde{\varphi}[\bar{U}_i] \cap (U_\varphi \times I)$$

— as a subset of the space $U_\varphi \times I$ — has no interior. We have a little more, namely that the set $\tilde{\varphi}[U(x_i)] \cap (U_\varphi \times I)$ has no interior. If it were not, then by taking an nonempty and open subset $U \subset U_\varphi \times I$ such that $X \subset \tilde{\varphi}[U(x_i)]$, we would obtain the mapping

$$\left(\tilde{\varphi}|_{U(x_i)}\right)^{-1}|_X : X \rightarrow U(x_i)$$

from a $(k + 1)$-dimensional manifold to a $k$-dimensional one, being an immersion, which is not possible. Thus $U_\varphi \times I$ is an union of a countable sequence of nowhere dense sets

$$\left\{\tilde{\varphi}[\bar{U}_i] \cap (U_\varphi \times I) ; i \in \mathbb{N}\right\},$$

which leads to a contradiction with Baire's theorem for locally compact spaces. The theorem is proved.  \[\square\]
Appendix: The following theorem is well known; here we give a simple proof of it.

**Theorem.** Each connected immersed submanifold $L$ of a $C^\infty$ Hausdorff paracompact manifold $V$ has a countable basis.

**Proof.** Let $f: L \to V$ be an immersion. The assumptions imply the existence of a Riemann tensor $G$ on $V$. Its pullback $f^* G$ is a Riemann tensor on $L$. A connected manifold which possesses a Riemann tensor is separable [2], therefore it has a countable basis. $lacksquare$

**BIBLIOGRAPHIE**


