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**A DIOPHANTINE PROBLEM ON ALGEBRAIC  
CURVES OVER FUNCTION FIELDS OF  
POSITIVE CHARACTERISTIC**

BY

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RÉSUMÉ. — Soit  $K$  un corps de fonctions d'une variable sur un corps fini de caractéristique  $p$ . On détermine les courbes algébriques sur  $K$  ayant une fonction  $K$ -rationnelle dont leurs valeurs dans une infinité de points  $K$ -rationnels sont des puissances  $p$ -èmes. On en déduit la finitude de l'ensemble des points rationnels des courbes sur  $K$  qui changent de genre sous une extension de corps de base.

ABSTRACT. — Let  $K$  be a function field in one variable over a finite field of characteristic  $p$ . We determine the algebraic curves over  $K$  having a  $K$ -rational function on it whose value at infinitely many  $K$ -rational points is a  $p$ -th power. From this we deduce the finiteness of the set of  $K$ -rational points of curves over  $K$  that change genus under ground-field extension.

**1. Introduction**

Let  $K$  be a function field in one variable over a finite field of characteristic  $p$ . The purpose of this paper is to characterize the algebraic curves  $X/K$  and the rational functions  $f \in K(X)$  such that  $f(P) \in K^p$  for infinitely many rational points  $P \in X(K)$ . This problem ties up with a question left open by SAMUEL [2] in his extension to positive characteristic of GRAUERT's proof of MORDELL's conjecture for function fields of characteristic zero. The question occurs when the relative genus of  $X/K$  is different from the absolute genus of  $X$  in the sense of [2] (or equivalently when  $K(X)$  is a non-conservative function field in the sense of [1]).

The genus of a curve  $X$  defined over  $K$ , relative to  $K$ , can be defined as follows. It is the integer  $g$  for which  $\ell(D) = \deg D + 1 - g$ , for divisors  $D$ , defined over  $K$ , with degree  $\deg D$  sufficiently large, where  $\ell(D)$  is the dimension, as a  $K$ -vector space, of the space of rational functions on  $X$ ,

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defined over  $K$ , with polar divisor bounded by  $D$ . The above definition of the genus depends on  $K$ . The genus of  $X$ , relative to  $K$ , does not change under separable extensions of  $K$  but may decrease under inseparable extensions. The absolute genus of  $X$  is thus defined as the genus of  $X$  relative to the algebraic closure of  $K$ . A standard example, for  $p \geq 3$ , is the curve  $y^2 = x^p - a$ . If  $a \in K \setminus K^p$ , then its genus, relative to  $K$ , is  $\frac{1}{2}(p-1)$  and its absolute genus is 0.

SAMUEL showed that, with notation as above,  $X(K)$  is finite if the absolute genus of  $X$  is at least two [2, Chapitre III, Theorem 1 and app. 2] and therefore the problem above is trivial for those curves. The question left open by SAMUEL [2, page 3] is whether curves with relative genus at least two and absolute genus 0 or 1 have finitely many rational points and we solve this question in the affirmative. Note that we have shown previously [4] that curves with relative genus 1 and absolute genus 0 have finitely many rational points (this will also follow from THEOREM 1 below). Hence all curves that admit genus change have finitely many rational points.

The paper is organized as follows. In sections 2 and 3 we solve our basic problem for rational curves and elliptic curves, respectively, and in section 4 we use these results to show that curves that admit genus change have finitely many rational points. Finally, we obtain the general solution to our problem.

## 2. Rational curves

Recall that  $K$  is a function field in one variable over a finite field of characteristic  $p$ . Let  $t \in K \setminus K^p$  and  $\delta = d/dt$ , a derivation of  $K$ . If  $x$  is a variable over  $K$ , we extend  $\delta$  to  $K(x)$  by  $\delta(x) = 0$ . We shall also use the notation  $r^\delta(x)$  for the action of  $\delta$  on  $r(x) \in K(x)$ .

**THEOREM 1.** — *Let  $r(x) \in K(x)$  be a rational function such that the set  $\{a \in K \mid r(a) \in K^p\}$  is infinite. Then, there exists  $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in GL_2(K)$  such that  $r((\alpha x + \beta)/(\gamma x + \delta)) \in K^p(x)$ .*

*Proof.* — Multiplying, if necessary,  $r(x)$  by the  $p$ -th power of its denominator, we can assume that  $r(x)$  is a polynomial. Let  $n$  be the degree of  $r(x)$  and assume first that  $p \nmid n$ .

Let  $r(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n$ . By changing, if necessary, the variable  $x$  to  $a_0^m x$ , where  $mn + 1 \equiv 0 \pmod{p}$ , we can assume that  $a_0 \in K^p$ . Further, dividing  $r(x)$  by  $a_0$ , we can also assume that  $a_0 = 1$ . Finally, changing  $x$  to  $x - a_1/n$ , we can assume that  $a_1 = 0$ .

If  $a \in K$  is such that  $r(a) \in K^p$ , then

$$(*) \quad 0 = \delta(r(a)) = r'(a)\delta a + r^\delta(a).$$

Note that  $r^\delta(a) = \delta a_2 x^{n-2} + \dots + \delta a_n$  is of degree at most  $(n-2)$ . If  $r^\delta(x)$  is identically zero, then  $r(x) \in K^p[x]$ , as desired. Assume then that  $r^\delta(x) \neq 0$ .

Let  $v$  be a place of  $K$  with  $v(a_i) \geq 0$ ,  $i = 0, \dots, n$  and  $v(dt) = 0$ . If  $a \in K$  is such that  $v(a) < 0$  then, clearly,  $v(r^\delta(a)) \geq (n-2)v(a)$  and  $v(r'(a)) = (n-1)v(a)$ , whence  $v(\delta a) \geq 0$ , from (\*). If  $v(a) \geq 0$  then, obviously,  $v(\delta a) \geq 0$ , as well. Thus  $v(\delta a) \geq 0$  for all but finitely many places of  $K$ .

Further, the rational function  $-r^\delta(x)/r'(x)$  has a zero at infinity. Thus, for any place  $v$  of  $K$ , if  $a$  has a sufficiently large pole at 0 then  $\delta a = -r^\delta(a)/r'(a)$  satisfies  $v(\delta a) \geq 0$ , say. On the other hand, if  $v(a)$  is bounded below, then  $v(\delta a)$  is also bounded below. The conclusion of the above discussion is that there exists a divisor  $D$  of  $K$  such that  $\delta a \in L(D)$  for any  $a \in K$  with  $r(a) \in K^p$ . Hence,  $\delta a$  can assume finitely many values  $b_1, \dots, b_N$  for those  $a$ . The polynomial equations  $r'(x)b_i + r^\delta(x) = 0$ ,  $i = 1, \dots, N$ , have finitely many solutions unless one of them is identically zero. In the latter case, looking at the coefficient in  $x^{n-1}$ , it follows that  $b_i = 0$  (recall that  $p \nmid n$ ) and so  $r^\delta(x) = 0$ , contrary to the hypothesis. This proves the result when  $p \nmid n$ .

Let now  $r(x)$  be a polynomial of degree  $n \equiv 0 \pmod{p}$  satisfying the hypothesis of the theorem. Let  $a \in K$  be such that  $r(a) \in K^p$ . To prove the theorem for  $r(x)$  it suffices to prove the theorem for the polynomial  $x^n(r(1/x+a) - r(a))$ , which has degree strictly less than  $n$ . The theorem now follows by induction on  $n$ .

REMARK 1. — Let  $r(x) \in K(x)$  be such that there exists  $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in GL_2(K)$  with  $r((\alpha x + \beta)/(\gamma x + \delta)) \in K^p(x)$ . Then  $r(a) \in K^p$  for infinitely many  $a \in K$ . Indeed  $r((\alpha x^p + \beta)/(\gamma x^p + \delta)) = (s(x))^p$  for some  $s(x) \in K(x)$ . This also shows that the curve  $y^p = r(x)$  is parametrizable over  $K$ , that is, has relative genus zero over  $K$ .

REMARK 2. — THEOREM 1 contains, as special cases, the results of [4]. The proof of THEOREM 1 is an extension of the techniques of [4].

### 3. Elliptic curves

We keep the notation of section 2. In particular, recall the derivation  $\delta$  of  $K$ . If  $E/K$  is an elliptic curve given by the Weierstrass equation  $y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$ , let  $E^{(p)}/K$  be the elliptic curve with Weierstrass equation  $y^2 + a_1^p xy + a_3^p y = x^3 + a_2^p x^2 + a_4^p x + a_6^p$  and  $F : E \rightarrow E^{(p)}$  be the Frobenius map defined by  $F(x, y) = (x^p, y^p)$ . Let also  $V : E^{(p)} \rightarrow E$  be the isogeny dual to  $F$ . We extend  $\delta$  to a derivation on  $K(E^{(p)}) = K(x, y)$  by  $\delta(x) = \delta(y) = 0$ . As in section 2 we also denote by  $r^\delta$  the action of  $\delta$  on  $r \in K(E^{(p)})$ .

**THEOREM 2.** — *Notation as above. If  $r \in K(E^{(p)})$  is such that the set  $\{P \in E^{(p)}(K) \mid r(P) \in K^p\}$  is infinite, then there exists  $P_0 \in E^{(p)}(K)$  such that the function  $P \mapsto r(P + P_0)$  belongs to  $K^p(E^{(p)})$ . If  $r \in K(E)$  is such that the set  $\{P \in E(K) \mid r(P) \in K^p\}$  is infinite, then there exists  $P_0 \in E(K)$  such that the function  $P \mapsto r(V(P) + P_0)$  belongs to  $K^p(E^{(p)})$ .*

*Proof.* — Let  $r \in K(E^{(p)})$  satisfy the hypothesis of the theorem. As  $E^{(p)}(K)/F(E(K))$  is finite (by the Mordell-Weil theorem) it follows that there exists  $P_0 \in E^{(p)}(K)$  such that, for infinitely many  $P \in F(E(K))$ ,  $r(P + P_0) \in K^p$ . Let  $s \in K(E^{(p)})$  be defined by  $s(P) = r(P + P_0)$ . If  $P \in F(E(K))$ , its  $x, y$  coordinates are  $p$ -th powers, hence  $\delta(s(P)) = s^\delta(P)$ . If, furthermore,  $s(P) \in K^p$  then  $s^\delta(P) = 0$ . But  $s^\delta$  has finitely many zeros unless it is identically zero. We therefore conclude that  $s^\delta = 0$ , that is,  $s \in K^p(E^{(p)})$ , as desired.

Let  $r \in K(E)$  satisfy the hypothesis of the theorem. Again by Mordell-Weil,  $E(K)/V(E^{(p)}(K))$  is finite : there exists  $P_1 \in E(K)$  such that there exists infinitely many  $P \in V(E^{(p)}(K))$  with  $r(P + P_1) \in K^p$ . Thus, the function  $P \mapsto r(V(P) + P_1)$  on  $E^{(p)}$ , satisfies the hypothesis of the theorem and, by what was proved above, there exists  $P_2$  such that the function  $P \mapsto r(V(P + P_2) + P_1)$  belongs to  $K^p(E^{(p)})$  and the theorem follows with  $P_0 = P_1 + V(P_2)$ .

**REMARK 3.** — If  $r \in K(E^{(p)})$  is such that  $P \mapsto r(P + P_0)$  belongs to  $K^p(E^{(p)})$  for some  $P_0 \in E^{(p)}(K)$  then  $r(P) \in K^p$  for all  $P \in E^{(p)}(K)$ ,  $P - P_0 \in F(E(K))$ . Indeed  $r(F(P) + P_0) = (s(P))^p$  for some  $s \in K(E)$ . Thus the cover of  $E^{(p)}$  defined by the equation  $z^p = r$  has genus 1 over  $K$ , since it is covered by  $E$  by the map  $P \mapsto (F(P) + P_0, s(P))$ . A similar phenomenon occurs for  $r \in K(E)$  such that  $P \mapsto r(V(P) + P_0)$  belongs to  $K^p(E^{(p)})$ . Indeed,  $r(pP + P_0) = s(P)^p$  for some  $s \in K(E)$ , since  $V \circ F$  is multiplication by  $p$  on  $E$ .

#### 4. Mordell's conjecture for non-conservative curves

An algebraic curve  $X/K$  is said to be conservative if its genus does not change under base-field extension from  $K$  to its algebraic closure  $\bar{K}$  (see [2, page 3] or [1], [3]). Otherwise  $X$  is called non-conservative. The genus of  $X$  over  $K$  is called the relative genus of  $X$  and the genus of  $X$  over  $\bar{K}$ , the absolute genus of  $X$ .

We retain the notation as above, in particular  $K$  is a function field in one variable over a finite field.

**THEOREM 3.** — *A non-conservative algebraic curve  $X/K$  has finitely many  $K$ -rational points.*

*Proof.* — Let  $X_n/K$ ,  $n = 0, 1, \dots$  be the algebraic curve whose function field is  $K \cdot (K(X))^{p^n}$  and denote by  $g_n$  the genus of  $X_n$ .

The sequence  $g_n$  is non-increasing and the constant value of  $g_n$  for all  $n$  sufficiently large is the absolute genus  $\bar{g}$  of  $X$  (see *e.g.* [3]). If  $\bar{g} \geq 2$ , then the theorem was already proved by SAMUEL. Assume that  $\bar{g} = 0$  or 1 and let  $n$  be such that  $g_{n-1} > g_n = \bar{g}$ . Let  $z \in K(X_{n-1}) \setminus K(X_n)$ , then  $r = z^p \in K(X_n)$  and  $K(X_{n-1}) = K(X_n)(z)$ . This means that  $X_{n-1}$  is the cover of  $X_n$  given by  $z^p = r$  and it is easy to see that all but finitely many rational points of  $X_{n-1}$  correspond to rational points  $P \in X_n$  for which  $r(P) \in K^p$ . From THEOREM 1 (and REMARK 1) and THEOREM 2 (and REMARK 3) it follows that there exists only finitely many such points and thus  $X_{n-1}(K)$  is finite. It then follows by a similar and easier argument that  $X_{n-2}(K), \dots, X_0(K) = X(K)$  are all finite, proving the theorem.

**REMARK 4.** — Returning to our basic problem, staded in the introduction, if  $X/K$  is an algebraic curve and  $f \in K(X)$  is such that  $\{P \in K(K) \mid f(P) \in K^p\}$  is infinite, then by THEOREM 3 and the Grauert-Samuel Theorem (*i.e.* the Mordell conjecture in characteristic  $p$ ) it follows that  $X$  is rational or elliptic, and in these cases the problem is solved by THEOREMS 1 and 2.

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