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A note on functional equations of the $p$-adic polylogarithms


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A NOTE ON FUNCTIONAL EQUATIONS OF
THE P-ADIC POLYLOGARITHMS

BY

ZDZISŁAW WOJTKOWIAK (*)

0. Introduction

Let \( n \) be an integer. The series \( \sum_{k=1}^{\infty} \frac{z^k}{k^n} \) converges on the open unit disc around 0 in the field of complex numbers \( \mathbb{C} \). Hence it determines an analytic function on this disc. This function can be extended by an

analytic continuation to a multivalued analytic function on $\mathbb{C} \setminus \{0,1\}$. We denote this function by $\text{Li}_n(z)$ and we call it the $n$-th order polylogarithm.

The functions $\text{Li}_n(z)$ are special cases of Chen iterated integrals (see [Ch]). We recall their definition. Let $\omega_1, \ldots, \omega_n$ be one-forms on a smooth manifold $M$ and let $\gamma : [0,1] \to M$ be a smooth path from $x$ to $z$. Let $\gamma^t : [0,1] \to M$ be a restriction of $\gamma$. We define by a recursive formula

$$
\int_\gamma \omega_1, \ldots, \omega_n := \int_\gamma \left( \int_{\gamma^t} \omega_1 \right) \omega_2, \ldots, \omega_n.
$$

If $x$ is fixed and $\omega_1, \ldots, \omega_n$ are closed one-forms on $M$ such that all possible products $\omega_{i_1} \wedge \cdots \wedge \omega_{i_k}$ vanish, then $F(z) = \int_\gamma \omega_1, \ldots, \omega_n$ is an analytic multivalued function on $M$. We shall write also $\int_x^z \omega_1, \ldots, \omega_n$ or $\int_x^z \omega_1, \ldots, \omega_n$ to denote the multivalued function $F(z)$.

It is clear that $\text{Li}_n(z) = \int_0^z \frac{dz}{1-z}, \frac{dz}{z}, \ldots, \frac{dz}{z}$.

Let $p$ be a finite prime of $\mathbb{Q}$ and let $\mathbb{C}_p$ denote a completion of an algebraic closure of $\mathbb{Q}$ at some place above $p$. Then the series

$$
\ell_{n,p}(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^n}
$$

determines an analytic function on the open unit disc around 0 in $\mathbb{C}_p$. However one cannot use analytic continuation to extend this function because the open unit disc is the maximal analytic domain for it.

The global $p$-adic analogs of $\text{Li}_n(z)$ are constructed in the framework of rigid analysis. Our basic reference is the paper of Coleman (see [C]). We briefly sketch the necessary results from [C], asking the reader to consult [C] for any details.

To define iterated integrals in the $p$-adic realm we consider the following system of differential equations

(*)

$$
f'_1 = \frac{1}{z - a_1}, \quad f'_2 = \frac{f_1}{z - a_2}, \quad \ldots, \quad f'_n = \frac{f_{n-1}}{z - a_n}.
$$

Let $a \in \mathbb{C}_p \setminus \{a_1, \ldots, a_n\}$. We pose the following initial conditions

(**)

$$
f_1(a) = 0, \quad f_2(a) = 0, \quad \ldots, \quad f_n(a) = 0.
$$

We set $D = \mathbb{C}_p \setminus \{a_1, \ldots, a_n\}$. The following result is the direct consequence of [C] (Theorem 4.3, Lemma 5.2 and the whole section V in [C]).
THEOREM A. — Let us choose a locally analytic homomorphism
\[ \log : \mathbb{C}_p^* \to \mathbb{C}_p. \]

There exists a logarithmic F-crystal \( M(D) \) on \( D = \mathbb{C}_p \setminus \{a_1, \ldots, a_n\} \) such that the system of differential equations \((*)\) has a unique solution \( f_1(z), \ldots, f_n(z) \) in \( M(D) \) which satisfies the initial conditions \((**)\).

It follows from the theory presented in [C] that the functions \( f_k(z) \) are locally analytic. The function \( f_n(z) \) we shall denote by
\[ \int_a^z \frac{dz}{z-a_1}, \frac{dz}{z-a_2}, \ldots, \frac{dz}{z-a_n} \]
and we shall call it an iterated integral in the p-adic realm.

The p-adic polylogarithms are defined in the section VI of [C]. We recall here their definition. Let \( D = \mathbb{C}_p \setminus \{0,1\} \). We consider the following system of differential equations
\[(*)_1 \quad \ell'_1 = \frac{1}{z-1}, \quad \ell'_2 = \frac{1}{z}, \quad \ldots, \quad \ell'_n = \frac{1}{z}. \]

We pose the following initial conditions
\[ (**)_2 \quad \lim_{z \to 0} \ell_k(z) = 0. \]

THEOREM A'. — Let us choose a locally analytic homomorphism \( \log : \mathbb{C}_p^* \to \mathbb{C}_p \). Then there exists a logarithmic F-crystal \( M(D) \) on \( D = \mathbb{C}_p \setminus \{0,1\} \) such that the system of differential equations \((*_1)\) has a unique solution \( \ell_1(z), \ldots, \ell_n(z) \) in \( M(D) \) which satisfies the initial condition \((**)_2\). The function \( \ell_k(z) \) extends to a locally analytic function on \( \mathbb{C}_p \setminus \{1\} \) such that \( \ell_k(0) = 0 \).

We shall denote \( \ell_k(z) \) by \( \text{Li}_k(z) \) and we shall call it the \( k \)-th p-adic polylogarithm. The function \( \text{Li}_k(z) \) is analytic at 0 and has the convergent Taylor expansion
\[ \sum_{n=1}^{\infty} \frac{z^n}{n^k} \]
at 0. In fact we shall use only the fact that functions \( \text{Li}_n(z) \) and \( \int_a^z \frac{dz}{z-a_1}, \ldots, \frac{dz}{z-a_n} \) exist in the p-adic realm, that they are locally analytic, that their Taylor power series at some points "coincide" with the Taylor power series of the corresponding complex functions and that the logarithmic F-crystal, where live p-adic iterated integrals satisfies a uniqueness principal.
The complex polylogarithms \( \text{Li}_n(z) \) have a lot of remarkable properties. For example, for small \( n \), they have functional equations which generalize the functional equation

\[
\log xy = \log x + \log y
\]
satisfied by the logarithm. The dilogarithm

\[
\text{Li}_2(z) = \int_0^z \frac{-\log(1 - z)}{z} \, dz
\]
satisfies the functional equation

\[
\text{Li}_2\left( \frac{x}{1-x} \cdot \frac{y}{1-y} \right) = \text{Li}_2\left( \frac{y}{1-x} \right) + \text{Li}_2\left( \frac{x}{1-y} \right) - \text{Li}_2(x) - \text{Li}_2(y) - \log(1-x)\log(1-y)
\]
(see [A]). In Lewin's book one can find more examples (see [L1]). The basic reference for \( p \)-adic polylogarithms is the paper of Coleman (see [C]). For more general review of various aspects of polylogarithms and iterated integrals one can consult [Ca].

In this paper we give some sufficient and necessary conditions to have functional equations of polylogarithms. We discuss complex polylogarithms and \( p \)-adic polylogarithms as well. One of the main results is the following theorem.

**Theorem.** Let \( K \) be the field of complex numbers or a \( p \)-adic completion of the algebraic closure of \( \mathbb{Q} \) at some place above \( p \). Let

\[
f_i : X = P^i(K) \setminus \{a_1, \ldots, a_n\} \to Y = P^1(K) \setminus \{0, 1, \infty\}
\]

\((i = 1, \ldots, N)\) be regular maps. Let \( n_1, \ldots, n_N \) be integers. There is a functional equation

\[
\sum_{i=1}^N n_i \text{Li}_n(f_i(z)) + \text{terms of lower degrees} = 0
\]

if and only if \( \sum_{i=1}^N n_i(f_i)_* = 0 \) in the group

\[
\text{Hom}\left( \Gamma^n(\pi_1(X, x)_{et})/\Gamma^{n+1}(\pi_1(X, x)_{et}); \Gamma^n(\pi_1(Y, y)_{et})/(\Gamma^{n+1}(\pi_1(Y, y)_{et}) + L_n) \right)
\]
where \( \pi_1(X, x)^\ell_{\text{et}} \) is the \( \ell \)-profinite quotient of the etale fundamental group of \( X \) and where \((f_i)_* \) are maps induced by \( f_i \) on the etale fundamental groups. \( L_n \) is a closed subgroup of \( \Gamma^n(\pi_1(Y, y)_{\text{et}}) \) defined in the following way. If \( K \) is the field of complex numbers \( \mathbb{C} \), then \( L_n \) is topologically generated by all commutators in \( e_0 \) (loop around 0) and \( e_1 \) (loop around 1) which contain \( e_1 \) at least twice. If \( K \) is a non-Archimedean field \( \mathbb{C}_p \) then any isomorphism \( \mathbb{C}_p \approx \mathbb{C} \) induces an isomorphism

\[
\pi_1(P^1(\mathbb{C}_p) \setminus \{0, 1, \infty\}, y)_{\text{et}}^\ell \cong \pi_1(P^1(\mathbb{C}) \setminus \{0, 1, \infty\}, y)_{\text{et}}^\ell
\]

and \( L_n \subset \Gamma^n(\pi_1(P^1(\mathbb{C}) \setminus \{0, 1, \infty\})_{\text{et}}) \) is the image of

\[
L_n \subset \Gamma^n\left(\pi^n(P^1(\mathbb{C}) \setminus \{0, 1, \infty\}, y)_{\text{et}}^\ell\right)
\]

under this isomorphism.

We give also a sufficient and necessary condition to have a functional equation in terms of a differential Galois group of a certain system of differential equations.

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This paper grew out from preprints [W1] and [W2]. We would like to point the attention to our preprint [W3] which, we hope, will be a chapter of a book “Properties of polylogarithms” of various authors, where we discuss functional equations of complex polylogarithms.

In the present paper we concentrate mostly on a \( p \)-adic situation, though quite often to prove something about \( p \)-adic polylogarithms, we must show an analogous result about complex polylogarithms first.

We point also that in some aspects the \( p \)-adic situation is simpler than the complex situation. The reader can look in chapter 4 where the results are due to the absence of \( 2\pi i \) in a non-Archimedean field \( \mathbb{C}_p \).
Plan

0. Introduction
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1. Canonical unipotent connection on a projective line minus several points

If \( p \) is any prime of \( \mathbb{Q} \) let \( \mathbb{C}_p \) denote a completion of an algebraic closure of \( \mathbb{Q} \) at some place above \( p \). This definition includes also the case when \( p = \infty \) and then \( \mathbb{C}_p = \mathbb{C} \) is the field of complex numbers.

Let \( X = \mathbb{P}^1(\mathbb{C}_p) \setminus \{a_1, \ldots, a_{n+1}\} \). Observe that \( X \) is an affine algebraic variety over \( \mathbb{C}_p \). Let \( \Omega^*(X) \) be the algebraic De Rham complex of smooth, algebraic differential forms on \( X \). Let \( A^1(X) \) be a \( \mathbb{C}_p \)-subspace of \( \Omega^1(X) \) generated by linear combinations with \( \mathbb{C}_p \)-coefficients of one forms \( \frac{dz}{z - a_i} \) for \( i = 1, \ldots, n + 1 \). Observe that

\[
A^1(X) = H^1_{DR}(X).
\]

Let \( H(X) \) be the dual of the \( \mathbb{C}_p \)-vector space \( A^1(X) \). Let \( \text{Lie}(H(X)) \) be a free Lie algebra over \( \mathbb{C}_p \) on \( H(X) \). Let

\[
L(X) := \lim_n \left( \text{Lie}(H(X))/\Gamma^n \text{Lie}(H(X)) \right)
\]

be the completion of \( \text{Lie}(H(X)) \) with respect to the filtration given by the lower central series. We equipped \( L(X) \) with a group law given by the Baker-Campbell-Hausdorff formula and a topology given by the inverse limit of finite dimensional \( \mathbb{C}_p \)-vector spaces with its natural \( p \)-adic topology if \( p < \infty \) and the complex topology if \( p = \infty \). We shall denote by \( \pi(X) \) this topological group. Observe that each quotient \( \pi(X)/\Gamma^n \pi(X) \) is an affine algebraic group, so \( \pi(X) \) is an affine pro-algebraic group and \( L(X) \) is its Lie algebra.

Definition. — The one form \( \omega_X \in A^1(X) \otimes H(X) \) corresponds to \( \text{id}_{A^1(X)} \) under the natural isomorphism

\[
A^1(X) \otimes (A^1(X))^* \approx \text{Hom}(A^1(X), A^1(X)).
\]
We consider $\omega_X$ as an element of $A^1(X) \otimes L(X)$.

Let $T(H(X))$ be a tensor algebra over $\mathbb{C}_p$ on $H(X)$. Let $I$ be an augmentation ideal of $T(H(X))$ and let

$$T[[H(X)]] := \lim_{n} T(H(X))/I^n$$

be the completed tensored algebra. Observe that $T(H(X))/I^n$ is a finite dimensional vector space over $\mathbb{C}_p$. Hence $T[[H(X)]]$ is equipped with the topology of an inverse limit of finite dimensional $\mathbb{C}_p$-vector spaces. Let $P(X)$ be a group of invertible elements in $T[[H(X)]]$ with leading term equal 1. From the discussion given above it follows that $P(X)$ is affine, pro-algebraic group over $\mathbb{C}_p$.

*Remark.* $T[[H(X)]]$ is nothing else but an algebra of non-commutative formal power series over $\mathbb{C}_p$ on $H(X)$.

In $T(H(X))$ and $T[[H(X)]]$ we consider the Lie algebras of Lie elements (possibly of infinite length in a case of $T[[H(X)]]$). These Lie algebras are naturally isomorphic with $\text{Lie}(H(X))$ and $L(X)$ respectively. After the identification of $L(X)$, which is the underlying set of $\pi(X)$ with the Lie elements (possibly of infinite length) in $T[[H(X)]]$ the exponential map

$$\exp : \pi(X) \longrightarrow P(X)$$

is defined by the standard formula

$$\exp(w) = 1 + \frac{w}{1!} + \frac{w^2}{2!} + \cdots$$

where we consider $w \in \pi(X)$ as a Lie element in $T[[H(X)]]$. The exponential map is a continuous monomorphism of topological groups, whose image is a closed subgroup of $P(X)$.

The inverse of $\exp$ is defined on the subgroup $\exp(\pi(X))$ of $P(X)$ and it is given by the formula

$$\log z = (z - 1) - \frac{1}{2}(z - 1)^2 + \frac{1}{3}(z - 1)^3 - \frac{1}{4}(z - 1)^4 + \cdots$$

and homomorphisms $\exp$ and $\log$ are mutually inverse isomorphisms

$$\exp : \pi(X) \cong \text{im}(\exp) : \log.$$

Let $p(X)$ be a Lie algebra of $P(X)$. We identify $v \in H(X)$ with a tangent vector to $P(X)$ given by $[0, 1] \ni t \mapsto 1 + t \cdot v \in P(V)$ if $\mathbb{C}_p = \mathbb{C}$. 

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is the field of complex numbers and by the differentiation in the direction of $v$ if $C_p$ is arbitrary.

After this identification we shall consider $\omega_X$ as an element of $A^1(X) \otimes p(X)$ and provisionally we shall denote it by

$$\tilde{\omega}(X) \in A^1(X) \otimes p(X).$$

**Lemma 1.2.** — The morphism $\text{id} \times \exp : X \times \pi(X) \to X \times P(X)$ maps $\omega_X$ into $\tilde{\omega}_X$.

**Proof.** — Let $v \in H(X)$. Then $\exp(tv)$ and $1 + tv$ define the same tangent vector. If $C_p$ is non-archimedean one observes that $\exp$ transforms the differentiation in the direction of $v$ on $\pi(X)$ in the differentiation in the direction of $v$ on $P(X)$.

It is clear that there is no need to distinguish between $\omega_X$ and $\tilde{\omega}_X$, hence we shall denote both forms by $\omega_X$.

Let $X = P^1(C_p) \setminus \{x_1, \ldots, x_{r+1}\}$ and let $Y = P^1(C_p) \setminus \{y_1, \ldots, y_{s+1}\}$. Let

$$f(z) = \alpha \prod_{i=1}^{n} (z - a_i)^{n_i} \prod_{j=1}^{m} (z - b_j)^{m_j}$$

be a rational function. Let us assume that $f$ restricts to a regular map $f : X \to Y$. Then $f$ induces

$$f^* : A^1(Y) \to A^1(X) \quad \text{and} \quad f_*: H(X) \to H(Y).$$

The map $f_*$ induces the following five maps which we shall denote by the same letter $f_*$

$$f_* : \text{Lie}(H(X)) \to \text{Lie}(H(Y));$$

$$f_* : L(X) \to L(Y), \quad f_* : \pi(X) \to \pi(Y);$$

$$f_* : p(X) \to p(Y), \quad f_* : P(X) \to P(Y).$$

Hence $\pi(\ )$ and $P(\ )$ are functors on the category of pointed projective lines and regular maps. We shall denote by $G(\ )$ any of them. In this way we avoid formulations of separated statements for $\pi(\ )$ and for $P(\ )$.

**Lemma 1.3.** — Let $X, Y$ and $f : X \to Y$ be as above. Let

$$f \times f_* : X \times G(X) \to Y \times G(Y)$$

be induced by \( f \). Then we have
\[
(id \otimes f_*) \omega_X = (f^* \otimes \text{id}) \omega_Y,
\]
where \( f^* : A^1(Y) \to A^1(X) \).

**Proof.** — The form \( \omega_X \) (resp. the form \( \omega_Y \)) corresponds to \( \text{id}_{A^1(X)} \) (resp. to \( \text{id}_{A^1(Y)} \)). The lemma follows immediately if we observe that \( \text{id}_{A^1(X)} \circ f^* = f^* \circ \text{id}_{A^1(Y)} \).

**2. Horizontal sections of the canonical connection**

Let \( X = P^1(C_p) \setminus \{a_1, \ldots, a_{n+1}\} \). Let us consider a principal \( P(X) \)-bundle
\[
X \times P(X) \to X
\]
equipped with the integrable connection given by \( \omega_X \).

Let us choose a base of \( A^1(X) \) given by one-forms
\[
\omega_1 = T_1(z) dz, \quad \ldots, \quad \omega_n = T_n(z) dz.
\]
Let \( X_1, \ldots, X_n \) be a dual base of \( H(X) \). Then \( P(X) \) is a multiplicative group of non-commutative, formal power series with constant terms equal 1 in non-commutative variables \( X_1, \ldots, X_n \).

Let \( p \) be a finite prime. Let us choose a locally analytic homomorphism \( \log : C_p^* \to C_p \). Then it follows from section 0 (THEOREM A) that there is a logarithmic \( F \)-crystal \( M(X) \) on \( X \) such that iterated integrals \( \int_x^{z} \omega_{i_1}, \ldots, \omega_{i_m} \) \((x \in X, i_1, \ldots, i_m \in \{1, 2, \ldots, n\})\) exist in \( M(X) \).

**Proposition 2.1.** — Let \( p \) be any prime of \( \mathbb{Q} \). Let
\[
X = P^1(C_p) \setminus \{a_1, \ldots, a_{n+1}\}
\]
and let \( x \in X \). Let \( \omega_1, \ldots, \omega_n \) and \( X_1, \ldots, X_n \) be as above.

(i) Let \( p \) be a finite prime. Then the map
\[
W \ni z \mapsto \left( z, 1 + \sum \left\{ (-1)^k \int_x^{z} \omega_{i_1}, \ldots, \omega_{i_k} \} X_{i_k} \cdots X_{i_1} \right\} \right) \in X \times P(X)
\]
(the summation is over all non-commutative monomials in \( X_1, \ldots, X_n \)) is a horizontal section of a principal \( P(X) \)-bundle \( X \times P(X) \to X \) equipped with an integrable connection given by \( \omega_X \). We shall denote this map shortly by
\[
X \ni z \mapsto \left( z, \lambda_X(z; x) \right) \in X \times P(X).
\]
(ii) Let $\mathbb{C}_p$ be the field of complex numbers $\mathbb{C}$. Let $\gamma$ be a path in $X$ from $x$ to $z$. Then the map

$$X \ni z \mapsto \left( z, 1 + \sum \left\{ (-1)^k \int_{x, \gamma}^z \omega_{i_1}, \ldots, \omega_{i_k} \right\} X_{i_k} \cdots X_{i_1} \right) \in X \times P(X)$$

is a horizontal section of a principal $P(X)$-bundle $X \times P(X) \to X$ equipped with an integrable connection given by $\omega_X$. We shall denote this map shortly by

$$X \ni z \mapsto (z, \lambda_X(z; x, \gamma)) \in X \times P(X)$$

or by

$$X \ni z \mapsto (z, \lambda_X(z; x)) \in X \times P(X).$$

(iii) The initial condition $\lambda_X(x; x) = 1$ determines $\lambda_X(z; x)$ (and $\lambda_X(z; x, \gamma)$ if $\mathbb{C}_p = \mathbb{C}$) uniquely.

Proof. — The system of differential equations for the coefficient $f_k(z)$ at $X_{i_k} \cdots X_{i_2} \cdot X_{i_1}$ of the horizontal section is the following

\[
(*) \quad \frac{df_1}{dz} = -\omega_{i_1}, \quad \frac{df_2}{dz} = -f_1 \omega_{i_2}, \quad \ldots, \quad \frac{df_k}{dz} = -f_{k-1} \omega_{i_k}
\]

with the initial condition

$$f_1(x) = 0, \quad f_2(x) = 0, \quad \ldots, \quad f_k(x) = 0.$$  

If $\mathbb{C}_p = \mathbb{C}$ the solution of the system $(*)$ is given by the iterated integrals $(-1)^\ell \int_{x, \gamma}^z \omega_{i_1}, \ldots, \omega_{i_\ell}$ for $\ell = 1, \ldots, k$ where $\gamma$ is a path from $x$ to $z$. If $p$ is finite then the functions $(-1)^\ell \int_{x, \gamma}^z \omega_{i_1}, \ldots, \omega_{i_\ell}$, $\ell = 1, \ldots, k$, exist in the logarithmic $F$-crystal $M(X)$ and satisfy the system $(*)$.

The uniqueness principal is valid for analytic functions on a connected open set in the complex situation and for functions in $M(X)$ in the $p$-adic situation (see [C], Theorem 5.7). This implies (iii).

We shall denote by

$$X \ni z \mapsto (z, \ell_X(z; x)) \in X \times \pi(X)$$

a horizontal section of a principal $\pi(X)$-bundle $X \times \pi(X) \to X$ equipped with the connection form $\omega_X$ which satisfies the initial condition $\ell_X(x; x) = 0$. If $\mathbb{C}_p = \mathbb{C}$ we shall also write $\ell_X(z; x, \gamma)$ instead of $\ell_X(z; x)$ to indicate the dependence on a path $\gamma$. It follows from Lemma 1.2 that

$$\exp(\ell_X(z; x)) = \lambda_X(z; x).$$

Hence we have

$$\ell_X(z; x) = \log(\lambda_X(z; x)).$$

This implies that $\ell_X(z; x)$ exists (in $M(X)$ if $p$ is finite) and it is uniquely determined by the initial conditions.
COROLLARY 2.2.
Let $X = P^1(C_p) \setminus \{a_1, \ldots, a_{n+1}\}$ and let $Y = P^1(C_p) \setminus \{b_1, \ldots, b_{m+1}\}$. Let $f : X \to Y$ be a regular map. The map $f \times f^* : X \times G(X) \to Y \times G(Y)$ maps horizontal sections of the bundle $X \times G(X) \to X$ equipped with the connection $\omega_X$ into horizontal sections of the bundle $Y \times G(Y) \to Y$ equipped with the connection $\omega_Y$ i.e. we have

\begin{align}
(2.2.1) \quad f_* (\ell_X(z; x)) &= \ell_Y(f(z); f(x)) \quad \text{if} \quad G() = \pi()
\end{align}

and

\begin{align}
(2.2.2) \quad f_* (\lambda_X(z; x)) &= \lambda_Y(f(z); f(x)) \quad \text{if} \quad G() = P()
\end{align}

Proof. — The corollary is an immediate consequence of LEMMA 1.3 and PROPOSITION 2.1.

3. Functional equations

Let $X$ be a projective line $P^1(C_p)$ minus a finite number of points. We recall from section 1 that $G(X)$ is an affine, pro-algebraic group. Let $\text{Alg}(G(X))$ be an algebra of polynomial $C_p$-valued functions on $G(X)$.

Let $X = P^1(C_p) \setminus \{a_1, \ldots, a_{n+1}\}$ and $Y = P^1(C_p) \setminus \{b_1, \ldots, b_{m+1}\}$. Let $f : X \to Y$ be a regular map. Let $x, z \in X$. Our principal tool to derive functional equations are equalities from COROLLARY 2.2

\begin{align}
(2.2.1) \quad f_* (\ell_X(z; x)) &= \ell_Y(f(z); f(x))
\end{align}

and

\begin{align}
(2.2.1) \quad f_* (\lambda_X(z; x)) &= \lambda_Y(f(z); f(x)).
\end{align}

THEOREM 3.1. — Let $f_1, \ldots, f_N$ be regular functions. Let $\mathcal{T}_1, \ldots, \mathcal{T}_N$ belong to $\text{Alg}(G(Y))$ and let $p(t_1, \ldots, t_n)$ be a polynomial in variables $t_1, \ldots, t_n$.

(i) Let $G() = \pi()$. There is a functional equation

\begin{align}
(1) \quad p\left\{ \mathcal{T}_1\left(\ell_Y\left(f_1(z), f_1(x)\right)\right), \ldots, \mathcal{T}_n\left(\ell_Y\left(f_N(z), f_N(x)\right)\right) \right\} = 0
\end{align}

if and only if

\begin{align}
(2) \quad p(\mathcal{T}_1 \circ f_1, \ldots, \mathcal{T}_N \circ f_N) = 0.
\end{align}

(ii) Let $G() = P()$. If $p(\mathcal{T}_1 \circ f_1, \ldots, \mathcal{T}_N \circ f_N) = 0$ then

\begin{align}
(2) \quad p\left\{ \mathcal{T}_1\left(\lambda_Y\left(f_1(z), f_1(x)\right)\right), \ldots, \mathcal{T}_n\left(\lambda_Y\left(f_N(z), f_N(x)\right)\right) \right\} = 0.
\end{align}
Proof. — Let us assume that we have the identity (2). The identity (2.2.1) implies that
\[ T_i(f_i^*(\ell_X(z;x))) = T_i(\ell_Y(f_i(z);f_i(x))). \]

Substituting \( T_i(f_i^*(\ell_X(z;x))) \) by \( T_i(\ell_Y(f_i(z);f_i(x))) \) in the formula (2) we get the functional equation (1). The same arguments show also part (ii).

Let us assume that we have a functional equation (1). Let \( C_p = \mathbb{C} \) be the field of complex numbers. Observe that the subset
\[ \{ \ell_X(x;x,\gamma) \in \pi(X) \mid \gamma \in \pi_1(X,x) \} \]
of \( \pi(x) \) is Zariski dense in \( \pi(X)/\Gamma^2\pi(X) \). Hence this subset is Zariski dense in \( \pi(X)/\Gamma^k\pi(X) \) for any \( k \). The vanishing of a regular function \( p(T_1 \circ f_1^*,\ldots,T_N \circ f_N^*) \) on a Zariski dense subset implies that this regular function is the zero function.

Now we shall assume that \( p \) is finite. Let us choose an isomorphism of fields \( \alpha : C_p \approx \mathbb{C} \). If
\[ q(t_1,\ldots,t_n) = \sum a_{i_1,\ldots,i_n} (t_1)^{i_1} (t_2)^{i_2} \cdots (t_n)^{i_n} \in C_p[[t_1,\ldots,t_n]] \]
then we set
\[ q^\alpha(t_1,\ldots,t_n) := \sum \alpha(a_{i_1,\ldots,i_n}) (t_1)^{i_1} (t_2)^{i_2} \cdots (t_n)^{i_n} \in \mathbb{C}[[t_1,\ldots,t_n]]. \]
If \( X = P^1(C_p) \setminus \{a_1,\ldots,a_{n+1}\} \) then we set
\[ X^\alpha := P^1(\mathbb{C}) \setminus \{\alpha(a_1),\ldots,\alpha(a_{n+1})\}. \]

Let us identify \( \left( \frac{dz}{z - a_i} \right)^* \) with \( \left( \frac{dz}{z - \alpha(a_i)} \right)^* \). After this identification, if \( T \in \text{Alg}(\pi(X)) \) then \( T^\alpha \in \text{Alg}(\pi(X^\alpha)) \).

Let \( q_i(z) \) be a Taylor series of \( T_i(\ell_Y(f_i(z);f_i(x))) \) at \( x \in C_p \). Then if follows from (1) that \( p(q_1(z),\ldots,q_N(z)) = 0 \) and consequently also \( p^\alpha(q_1^\alpha(z),\ldots,q_N^\alpha(z)) = 0 \). The power series \( q_i^\alpha(z) \) is a Taylor power series of \( T_i^\alpha(\ell_Y \alpha(f_i^\alpha(z);f_i^\alpha(\alpha(x)))) \) at \( \alpha(x) \in \mathbb{C} \). Hence locally, in a neighbourhood of \( \alpha(x) \) we have a functional equation
\[ p^\alpha\left\{ T_1^\alpha\left( \ell_Y \alpha(f_1^\alpha(z);f_1^\alpha(\alpha(x))) \right),\ldots,T_N^\alpha\left( \ell_Y \alpha(f_N^\alpha(z);f_N^\alpha(\alpha(x))) \right) \right\} = 0. \]

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By the principle of analytic continuation we have
\[ p^\alpha \left\{ T_1^\alpha \left( \ell_Y \alpha (f_1^\alpha (z); f_1^\alpha (\alpha(x)), f_1^\alpha (\gamma)) \right), \ldots \right\} = 0 \]
for any smooth path \( \gamma \) from \( \alpha(x) \) to \( z \). Hence we have
\[ p^\alpha \{ T_1^\alpha \circ f_1^\alpha, \ldots, T_N^\alpha \circ f_N^\alpha \} = 0 \]
by the result proved above for the field of complex numbers. This implies
\[ p\{ T_1 \circ f_1^\ast, \ldots, T_N \circ f_N^\ast \} = 0. \]

We recall that \( \text{Lie} \, H(Y) \) is a free Lie algebra on
\[ Y_1 = (\omega_1)^\ast, \ldots, Y_m = (\omega_m)^\ast \]
where \( \omega_1, \ldots, \omega_m \) is a base of \( A^1(Y) \). We fixed a base \( B_Y \) of \( \text{Lie} \, H(Y) \) given by basic Lie elements corresponding to the ordering \( Y_1, \ldots, Y_m \) (see [MKS], Theorem 5.8). Let \( v \in B_Y \) and let \( v^* \in \text{Hom} \left( \text{Lie} \, H(Y), \mathbb{C} \right) \) be a linear functional on \( \text{Lie} \, H(Y) \) dual to \( v \) with respect to the base \( B_Y \). We consider the linear functional \( v^* \in \text{Lie} \, H(Y) \) as an element of \( \text{Alg}(\pi(Y)) \). We set
\[ \mathcal{L}_{v,B_Y}(z; x) := v^* (\ell_Y(z; x)). \]
If the choice of the base \( B_Y \) is clear we shall omit the subscript \( B_Y \) and we shall write \( \mathcal{L}_v(z; x) \) instead of \( \mathcal{L}_{v,B_Y}(z; x) \).

The following results are immediate corollaries of THEOREM 3.1.

COROLLARY 3.2. — Let \( f_1, \ldots, f_N \) be regular functions, let \( n_1, \ldots, n_N \) be integers and let \( v_1, \ldots, v_N \) be homogeneous of degree \( n \) and let they belong to the base \( B_Y \) of \( \text{Lie}(H(Y)) \). There is a functional equation
\[ \sum_{i=1}^N n_i \mathcal{L}_{v_i}(f_i(z); f_i(x)) = 0 \]
if an only if
\[ \sum_{i=1}^N n_i (v_i^* \circ (f_i)_*) = 0 \]
in \( \text{Hom} \left( \Gamma^n\pi(X)/\Gamma^{n+1}\pi(X); \mathbb{C}_p \right) \), where
\[ (f_i)_*: \Gamma^n\pi(X)/\Gamma^{n+1}\pi(X) \longrightarrow \Gamma^n\pi(Y)/\Gamma^{n+1}\pi(Y) \]
is induced by \( f_i \).
COROLLARY 3.3. — Let $B_X$ be a base of Lie $H(X)$ given by basic Lie elements. The functions \{\mathcal{L}_v(z; x_0) \mid v \in B_X\} are algebraically independent on $X$.

Now we shall concentrate on polylogarithms. Let

$$Y = P^1(C_p) \setminus \{0; 1; \infty\}.$$ 

Let $B_Y$ be a base of Lie $H(Y)$ given by basic Lie elements corresponding to the ordering $e_0 = \left(\frac{dz}{z}\right)^*$ and $e_1 = \left(\frac{dz}{z-1}\right)^*$. Let us set $e_2 := [e_1, e_0]$ and $e_{n+1} := [e_n, e_0]$. Let $e_n^*$ denote the linear functional on Lie $H(Y)$ dual to $e_n$ with respect to the base $B_Y$. We shall consider $e_n^*$ as an element of Alg($\pi(Y)$).

We recall that $\mathcal{L}_{e_n}(z; x) = e_n^*(\ell_Y(z; x))$. To simplify notation we set

$$\mathcal{L}_n(z; x) := \mathcal{L}_{e_n}(z; x).$$

Let $T_n : P(Y) \hookrightarrow C_p[[e_0, e_1]]^* \rightarrow C_p$ associate to an element of $P(Y)$ its coefficient at $e_0^n \cdot e_1$. We set

$$L_n(z; x) := (-1)^{n-1} T_{n-1}(\lambda_Y(z; x)).$$

It is an easy observation that

$$L_n(z; x) = \int_x^z \frac{-dz}{z-1}, \frac{dz}{z}, \cdots, \frac{dz}{z},$$

where $dz/z$ appears $(n - 1)$ times.

We shall express the function $\mathcal{L}_n(z; x)$ by functions $L_i(z; x).$ Let

$$\lambda = \exp(a e_0) + \sum_{n=0}^{\infty} b_{n+1} e_0^n e_1 \in P(Y).$$

We recall that

$$\log t = (t - 1) - \frac{1}{2}(t - 1)^2 + \frac{1}{3}(t - 1)^3 - \frac{1}{4}(t - 1)^4 + \cdots.$$ 

Let $c_{n+1}$ be a coefficient at $e_0^n \cdot e_1$ in the power series $\log \lambda$. We have

$$c_{n+1} = b_{n+1} - \frac{1}{2} \left( \sum_{1 \leq i \leq n} \frac{a^i}{i!} b_{n+1-i} \right) + \frac{1}{3} \left( \sum_{i=j=1}^{i+j \leq n} \frac{a^{i+j}}{i!j!} b_{n+1-i-j} \right)$$

$$- \frac{1}{4} \left( \sum_{i=j=k=1}^{i+j+k \leq n} \frac{a^{i+j+k}}{i!j!k!} b_{n+1-i-j-k} \right) + \cdots.$$
We shall compute the coefficient at $a^k b_{n+1-k}$. Observe that this coefficient is equal to a coefficient at $z^k$ in a power series
\[
\varphi(z) = -\frac{1}{2}(e^z - 1) + \frac{1}{3}(e^z - 1)^2 - \frac{1}{4}(e^z - 1)^3 + \cdots.
\]
We have
\[
(e^z - 1) + \varphi(z)(e^z - 1) = \log e^z = z.
\]
Hence
\[
\varphi(z) = \frac{z}{e^z - 1} = \sum_{n=1}^{\infty} \frac{B_n}{n!} z^n
\]
where $B_n$ are Bernoulli numbers ($B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{12}$, $B_3 = 0, \ldots$). The immediate consequence of this discussion is the following lemma.

**Lemma 3.4.** — We have
\[
c_{n+1} = b_{n+1} + \sum_{k=1}^{n} \frac{B_k}{k!} a^k b_{n+1-k}.
\]
This Lemma implies the following result.

**Corollary 3.5.** — We have
\[
\mathcal{L}_{n+1}(z;x) = \text{Li}_{n+1}(z;x) + \sum_{k=1}^{n} \frac{B_k}{k!} \left( \int_{x}^{z} \frac{dz}{z} \right)^k \text{Li}_{n+1-k}(z;x).
\]
(Observe that $\mathcal{L}_2(z) := \text{Li}_2(z) + \frac{1}{2} \log z \log(1-z)$ is the Rogers function (see [R]).)

**Theorem 3.6.**
Let $X = P^1(C_p) \setminus \{a_1, \ldots, a_{n+1}\}$ and let $Y = P^1(C_p) \setminus \{0, 1, \infty\}$. Let $f_1, \ldots, f_N : X \to Y$ be regular functions and let $n_1, \ldots, n_N$ be integers. There is a functional equation
\[
\sum_{i=1}^{N} n_i \mathcal{L}_n(f_i(z); f_i(x)) = 0
\]
if and only if one of the following equivalent conditions is satisfied:

1. \[
\sum_{i=1}^{N} n_i e^*_n \circ (f_i)_* = 0 \text{ in the group } \text{Hom}(\Gamma^n \pi(X)/\Gamma^{n+1} \pi(X); C_p);
\]
(2) \[ \sum_{i=1}^{N} n_i(f_i)_* = 0 \text{ in the group} \]
\[ \text{Hom}(\Gamma^n\pi(X) \backslash \Gamma^{n+1}\pi(X); \Gamma^n\pi(Y)/(\Gamma^{n+1}\pi(Y) + L_n)), \]

where \( L_n \) is a \( C_p \)-vector subspace of \( \Gamma^n\pi(Y)/\Gamma^{n+1}\pi(Y) \) generated by all commutators in \( e_0 \) and \( e_1 \) which contain \( e_1 \) at least twice;

(3) \[ \sum_{i=1}^{N} n_i(f_i)_* = 0 \text{ in the group} \]
\[ \text{Hom}(\Gamma^n(\pi_1(X,x)_{et}^\ell)/\Gamma^{n+1}(\pi_1(X,x)_{et}^\ell); \Gamma^n(\pi_1(Y,y)_{et}^\ell)/(\Gamma^{n+1}(\pi_1(Y,y)_{et}^\ell) + L_n)) \]

where \( \pi_1(X,x)_{et}^\ell \) is the \( \ell \)-profinite quotient of the etale fundamental group of \( X \) and where \( (f_i)_* \) are maps induced by \( f_i \) on etale fundamental groups. \( L_n \) is a closed subgroup of \( \Gamma^n(\pi_1(Y,y)_{et}^\ell) \) defined in the following way.

If \( C_p \) is the field of complex numbers \( C \) then \( L_n \) is topologically generated by all commutators in \( e_0 \) (loop around \( 0 \)) and \( e_1 \) (loop around \( 1 \)) which contain \( e_1 \) at least twice. If \( C_p \) is a non-archimedean field then any isomorphism \( C_p \cong \mathbb{C} \) induces an isomorphism

\[ \pi_1(P^1(C_p) \setminus \{0,1,\infty\},y)_{et}^\ell \cong (P^1(C) \setminus \{0,1,\infty\},y)_{et}^\ell \]

and \( L_n \subset \Gamma^n(\pi_1(P^1(C_p) \setminus \{0,1,\infty\},y)_{et}^\ell) \) is the image of

\[ L_n \subset \Gamma^n(\pi_1(P^1(C) \setminus \{0,1,\infty\},y)_{et}^\ell) \]

under this isomorphism.

**Proof.** — It follows immediately from **COROLLARY 3.2** that (0) and (1) are equivalent. Observe that \( L_n = \ker e_n^* \). This implies that (1) and (2) are equivalent. Observe that the map induced by \( f_i \) on quotient groups \( \Gamma^n\pi(X)/\Gamma^{n+1}\pi(X) \) “coincides” with the map induced by \( f_i \) on \( \Gamma^n(\pi_1(X,x)_{et}^\ell)/\Gamma^{n+1}(\pi_1(X,x)_{et}^\ell) \). This implies that conditions (2) and (3) are equivalent.

**Definition 3.7.** — Let \( n \) be a natural number. We note \( \text{ltd}(n) \) a polynomial in variables \( L_{ik}(g_i(z)) \), where \( k < n \) and \( g_i(z) \) are rational functions.
Corollary 3.8. — There is a functional equation

\[ \sum_{i=1}^{N} n_i (\text{Li}_n(f_i(z)) - \text{Li}_n(f_i(x))) + \text{ldt}(n) = 0 \]

if and only if

\[ \sum_{i=1}^{N} n_i (f_i)_* = 0 \]

in the group \( \text{Hom}(\Gamma^n\pi(X)/\Gamma^{n+1}\pi(X); \Gamma^n\pi(Y)/(\Gamma^{n+1}\pi(Y) + L_n)) \).

Proof. — Let us assume that \( \sum_{i=1}^{N} n_i (f_i)_* = 0 \). Theorem 3.6 implies that \( \sum_{i=1}^{N} n_i \mathcal{L}_n(f_i(z); f_i(x)) = 0 \). It is a trivial observation that \( \text{Li}_n(z; x) = \text{Li}_n(z) - \text{Li}_n(x) + \text{ldt}(n) \). Hence Corollary 3.5 implies that

\[ \sum_{i=1}^{N} n_i \left[ \text{Li}_n(f_i(z)) - \text{Li}_n(f_i(x)) \right] + \text{ldt}(n) = 0. \]

Let us assume that (1) holds. Let \( \mathbb{C}_p = \mathbb{C} \) be the field of complex numbers. Calculating the monodromy of the function \( \text{Li}_n(z) \) on elements of \( \Gamma^n\pi_1(Y, y) \) we get a linear function \( \tilde{e}_n \) from \( \Gamma^n\pi_1(Y, y)/\Gamma^{n+1}\pi_1(Y, y) \) to \( (2\pi i)^n \cdot \mathbb{Z} \) which after the identification of \( e_0 \in \pi(Y) \) (resp. \( e_1 \in \pi(Y) \)) with a loop around 0 (resp. loop around 1) coincides with

\[ (2\pi i)^n \cdot e^*_n : \Gamma^n\pi(Y)/\Gamma^{n+1}\pi(Y) \rightarrow \mathbb{C}. \]

Calculating the monodromy of \( \text{Li}_n(f_i(z)) \) on \( \Gamma^n\pi_1(X, x) \) we get a linear function

\[ \tilde{e}_n \circ (f_i)_* : \Gamma^n\pi_1(X, x)/\Gamma^{n+1}\pi_1(X, x) \rightarrow (2\pi i)^n \cdot \mathbb{Z} \]

where \( (f_i)_* : \pi_1(X, x) \rightarrow \pi_1(Y, y) \) is the map induced by \( f_i \). The functional equation (1) implies that we have \( \sum_{i=1}^{N} n_i \tilde{e}_n \circ (f_i)_* = 0 \) in \( \text{Hom}(\Gamma^n\pi_1(X, x)/\Gamma^{n+1}\pi_1(X, x); (2\pi i)^n\mathbb{Z}) \). This condition is of course equivalent to (2).

Let \( \mathbb{C}_p \) be a non-archimedean field. We rewrite the equation (1) in the form

\[ \sum_{i=1}^{N} n_i \text{Li}_n(f_i(z); f_i(x)) + \text{ldt}(n) = 0 \]

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where \( ldt(n) \) is a polynomial in \( L_k(g_j(z); g_j(x)) \) (here \( k < n \) and \( g_j(z) \) are rational functions) and constants. We replace the functions \( \text{Li}_n(f_1(z); f_1(x)) \) and \( \text{Li}_k(g_j(z); g_j(x)) \) by their Taylor power series at \( x \). We choose an isomorphism \( \alpha : \mathbb{C}_p \cong \mathbb{C} \) and we interpret the Taylor power series over \( \mathbb{C}_p \) as Taylor power series of complex functions \( \text{Li}_k(\cdot; \cdot) \). Hence we get a functional equation of complex polylogarithms

\[
\sum_{i=1}^{N} n_i \text{Li}_n(f_i^{\alpha}(z); f_i^{\alpha}(x)) + ldt(n) = 0
\]

and this situation we have already considered.

Now we shall show that the ideal of polynomial relations between functions \( \text{Li}_n(f_i(z)) \), where \( f_i(z) \) are rational functions is generated by linear relations from Theorem 3.6.

Let \( f_1(z), \ldots, f_N(z) \) be rational functions. Let \( p(x_1, \ldots, x_N, t_1, \ldots, t_R) \) be a polynomial whose degree with respect to \( x_1, \ldots, x_N \) is strictly smaller than \( k \). We set

\[
LDT_k(n) := p\left( \text{Li}_n(f_1(z)), \ldots, \text{Li}_n(f_N(z)), T_1, \ldots, T_R \right)
\]

where \( T_1 = ldt(n), \ldots, T_R = ldt(n) \).

We recall that \( I \) is a homogeneous ideal in \( \mathbb{C}_p[x_1, \ldots, x_m] \) if the following two conditions holds:

(i) for any two homogeneous elements in \( I \) of the same degree, their sum is in \( I \);

(ii) for any homogeneous element \( a \) in \( I \) and any homogeneous element \( c \) in \( \mathbb{C}_p[x_1, \ldots, x_m] \), the element \( c \cdot a \) is in \( I \).

**Theorem 3.9.** — Let \( f_1(z), \ldots, f_N(z) \) be rational functions. Let

\[
I_n(f_1, \ldots, f_N)
= \left\{ p(x_1, \ldots, x_N) \in \mathbb{C}_p[x_1, \ldots, x_N] \mid p(x_1, \ldots, x_N) \text{ homogeneous of degree } k > 0, \right. \\
\left. p(\text{Li}_n(f_1(z)), \ldots, \text{Li}_n(f_N(z))) + LDT_k(n) = 0 \right\}.
\]

Then \( I_n(f_1, \ldots, f_N) \) is a homogeneous ideal generated by a finite number of linear forms in \( x_1, \ldots, x_N \).
Proof. — It is clear that $I_n(f_1, \ldots, f_N)$ is a homogeneous ideal. Let $X = P^1(C_p) \setminus \{a_1, \ldots, a_l\}$ be such that the maps $f_i : X \to P^1(C_p) \setminus \{0, 1, \infty\}$ are regular for $i = 1, \ldots, N$. It follows from Theorem 3.1 and the fact that $p(x_1, \ldots, x_N)$ is homogeneous that

$$p(Li_n(f_1(z)), \ldots, Li_n(f_N(z))) + LDT_k(n) = 0$$

if and only if $p(e_n^* \circ (f_1)_*, \ldots, e_n^* \circ (f_N)_*) = 0$ in $\text{Alg}(\pi(X))$. This is equivalent to the condition $p(e_n^* \circ (f_1)_*, \ldots, e_n^* \circ (f_N)_*) = 0$ in $S(V^*)$ where $V = \Gamma^n \pi(X) / \Gamma^{n+1} \pi(X)$ and $S(V^*)$ is the symmetric algebra over $C_p$ on the vector space $V^* = \text{Hom}(V, C_p)$.

Then $I_n(f_1, \ldots, f_N)$ is the maximal homogeneous ideal contained in $\ker(C_p[\{a_1, \ldots, a_l\}] \to S(V^*))$, where $\pi(x_i) = e_n^* \circ (f_i)_*$. The map $\pi$ is induced by a linear map $\bigoplus_{i=1}^N C_p x_i \to V^*$. Hence the ideal $I_n(f_1, \ldots, f_N)$ is generated by one-forms.

4. Sometimes it is easier without $2\pi i$

In this section $p$ is a finite prime, so we are working in a $p$-adic realm. Let

$$P_{n+1}(z) = \sum_{i=0}^{n} \alpha_i (\log z)^i Li_{n+1-i}(z)$$

$(Li_1(z) = -\log(1 - z))$. Let $V_{n+1} \subset C_p^{n+1}$ be given by

$$V_{n+1} = \{(a_0, \ldots, a_n) \in C_p^{n+1} \mid \sum_{i=0}^{n} \frac{a_k}{(n+1-k)!} = 0\}.$$

Lemma 4.1. — Let

$$\frac{d}{dz}(P_{n+1}(z)) = \alpha_n \left[ \frac{(\log z)^{n-1} \log(1 - z)}{z} + \frac{\log z^n}{1 - z} \right]$$

$$= \sum_{i=0}^{n-1} \beta_i \frac{(\log z)^i Li_{n-i}(z)}{z}.$$

If $(\alpha_0, \ldots, \alpha_n) \in V_{n+1}$ then $(\beta_0, \ldots, \beta_{n-1}) \in V_n$.

Proof. — We have

$$\beta_k = a_k + (k+1)a_{k+1} \quad \text{if} \quad k < n - 1,$$

$$\beta_{n-1} = a_{n-1} + (n+1)a_n.$$
Hence
\[ \sum_{i=0}^{n-1} \frac{\beta_i}{(n-i)!} = \sum_{i=0}^{n-1} \frac{a_i + (i+1)a_{i+1}}{(n-i)!} + a_n \]
\[ = \frac{a_0}{n!} + \sum_{i=1}^{n} \frac{(n+1)}{(n+1-i)!}a_i = 0. \]

The $p$-adic $k$-th polylogarithm satisfies the functional equation
\[
(*) \quad \text{Li}_k \left( \frac{1}{z} \right) = (-1)^{k+1} \text{Li}_k (z) + (-1)^{k+1} \left( \log z \right)^k / k!
\]
(see [C], Proposition 6.4).

**Lemma 4.2.** — Let $P_{n+1}(z)$ be such that $(\alpha_0, \ldots, \alpha_n) \in V_{n+1}$. Then
\[ P_{n+1} \left( \frac{1}{z} \right) + (-1)^{n+1} P_{n+1}(z) = 0. \]

**Proof.** — This follows immediately from (*).

Following [C], we set
\[ \lim' f(z) = \lim_{z \to a} f(z) \]
if all the limits on the right side exist and coincide, for an arbitrary finitely ramified extension $K$ of $\mathbb{Q}_p$ such that coordinates of $a$ are in $K$.

**Lemma 4.3.** — Let $P_{n+1}(z)$ be such that $(\alpha_0, \ldots, \alpha_n) \in V_{n+1}$. Then
\[ \lim_{z \to 0} P_{n+1}(z) = 0 \text{ and } \lim_{z \to \infty} P_{n+1}(z) = 0. \]

**Proof.** — The fact that $\log z$ is bounded on any finitely ramified extension of $\mathbb{Q}_p$ and $\text{Li}_k(0) = 0$ implies that the first limit vanishes. It follows from Lemma 4.2 that the second limit vanishes.

Now we give some examples of functions $P_{n+1}(z)$ such that the sequence of coefficients $(\alpha_0, \ldots, \alpha_n) \in V_{n+1}$.

**Example 1.** — Let
\[ \alpha_i = \frac{(-1)^i}{i!} \quad \text{if} \quad i < n \quad \text{and} \quad \alpha_n = \frac{(-1)^n}{n!} + \frac{(-1)^{n+1}}{(n+1)!}. \]

We must show that the expression $\sum_{i=0}^{n+1} (-1)^i / i! \cdot (n + 1 - i)!$ vanishes. Observe that this expression is a coefficient at $z^{n+1}$ in the power series $e^{-z} \cdot e^z = 1$, hence
\[ \sum_{i=0}^{n+1} \frac{(-1)^i}{i! \cdot (n + 1 - i)!} = 0. \]
The function

$$\sum_{i=0}^{n} \frac{(-1)^i}{i!} (\log z)^i \text{Li}_{n-i}(z) + \frac{(-1)^{n+1}}{(n+1)!} (\log z)^n \text{Li}_1(z)$$

appeared in [L2] while its single valued analogue in [W4].

**Example 2.** — Let \( \alpha_i = \frac{\beta_i}{i!} \) where \( \beta_i \) are Bernoulli numbers \( \beta_0 = 1, \beta_1 = -\frac{1}{2}, \beta_2 = \frac{1}{6}, \beta_3 = 0, \ldots \) Observe that \( \sum_{i=0}^{n} \beta_i / i! (n+1-i)! \) is a coefficient at \( z^{n+1} \) in the power series \( \frac{z}{e^z - 1} \cdot (e^z - 1) \), hence

$$\sum_{i=0}^{n} \frac{\beta_i}{i! (n+1-i)!} = 0.$$ 

The function \( \sum_{i=0}^{n} \frac{\beta_i}{i!} (\log z)^i \text{Li}_{n+1-i}(z) \) appeared in [D2] and in a non explicit way in [W2] as a solution of a system of differential equations defining horizontal sections.

**Example 3.** — Let

$$\alpha_0 = 1, \quad \alpha_i = 0 \quad \text{for} \quad 0 < i < n \quad \text{and} \quad \alpha_n = -\frac{1}{(n+1)!}.$$ 

The corresponding function is \( \text{Li}_{n+1}(z) + \frac{1}{(n+1)!} (\log z)^n \log(1-z) \).

Observe that \( \dim V_{n+1} = n \). Hence for \( n = 1 \) there is only one function (up to a multiplication by a constant) such that its sequence of coefficients belongs to \( V_2 \). This is the Rogers function \( \text{Li}_2(z) + \frac{1}{2} \log(z) \cdot \log(1-z) \).

We hope that a function \( P_{n+1}(z) \) such that its sequence of coefficients \( (\alpha_0, \ldots, \alpha_n) \in V_{n+1} \), has all functional equations without lower degree terms. We give a partial result in this direction. We shall imitate D. Zagier (see [Z1]).

Let \( \mathcal{A}_{\text{loc}}(C_p) \) be a ring of functions which are locally analytic on some \( C_p \) several points. Let \( \text{Sym}^k(C_p(z)^*) \) be the \( k \)-th symmetric power of the multiplicative group \( C_p(z)^* \). Let us set

$$L_{n+1}(C_p(z)^*) := \text{Sym}^{n-1}(C_p(z)^*) \otimes (C_p(z)^* \wedge C_p(z)^*) \otimes \mathbb{Q}/\mathbb{R}$$

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where $R$ is generated by expressions of the form

\[(**)
\[f_1 \odot \cdots \odot f_{n-2} \odot a \odot b \odot c
\[+ f_1 \odot \cdots \odot f_{n-2} \odot b \odot c \odot a
\[+ f_1 \odot \cdots \odot f_{n-2} \odot c \odot a \odot b
\]

and

\[(***)
\[c_1 \odot \cdots \odot c_{n-1} \odot c_n \odot c_{n+1}
\]

where $c_i \in \mathbb{C}_p$ for $i = 1, \ldots, n + 1$. Let $K_{n+1} : L_{n+1}(\mathbb{C}_p(z)^*) \to A_{\text{loc}}(\mathbb{C}_p)$ be given by

\[K_{n+1}(\(f_1 \odot f_2 \odot \cdots \odot f_{n-1} \odot f_n \wedge f_{n+1} \odot a\))
\[= \alpha A(f_1) \cdots A(f_{n-1}) \cdot \left( A(f_{n+1}) \cdot B(f_n) - A(f_n) \cdot B(f_{n+1}) \right)
\]

where $A(f) = \log f$ and $B(g) = g'/g$.

Let $B(\mathbb{C}_p(z)^*)$ be a free abelian group on the set $\mathbb{C}_p(z)^*$. We shall denote by $[f]$ the generator corresponding to $f \in \mathbb{C}_p(z)^*$. Let

\[b_{n+1} : B(\mathbb{C}_p(z)^*) \rightarrow L_{n+1}(\mathbb{C}_p(z)^*)
\]

be a homomorphism given by

\[b_{n+1}([f]) = f \odot \cdots \odot f \odot f \odot 1 - f.
\]

**Proposition 4.4.** — Let

\[P_{n+1}(z) = \sum_{i=0}^{n} \alpha_i (\log z)^i \text{Li}_{n+1-i}(z)
\]

be such that $(\alpha_0, \ldots, \alpha_n) \in V_{n+1}$. If $f = \sum_{k=1}^{m} n_k [f_k] \in \ker b_{n+1}$, then we have

\[\sum_{k=1}^{m} n_k \left( P_{n+1}(f_k(z)) - P_{n+1}(f_k(z_0)) \right) = 0
\]

where $z_0$ is a fixed element of $\mathbb{C}_p$.

**Proof.** — Let $n = 1$. The space $V_2$ is one-dimensional generated by $(1, -\frac{1}{2})$. Then for $P_2(z) = \text{Li}_2(z) + \frac{1}{2} \log z \log(1 - z)$ we have

\[\frac{d}{dz} \sum_{k=1}^{m} n_k \left( P_2(f_k(z)) \right) = -\frac{1}{2} \sum_{k=1}^{m} n_k \left\{ \frac{f_k'(z)}{f_k(z)} \log(1 - f_k(z))
\[= \frac{1}{2} \left( 1 - f_k(z) \right) \frac{f_k'(z)}{f_k(z)} \log(f_k(z)) \right\}
\[= -\frac{1}{2} K_2(b_2(f)) = 0.
\]
Hence $\sum_{k=1}^{m} n_k (P_2(f_k(z)))$ is constant. Assume that the theorem holds for $n$.

We have

$$\frac{d}{dz} \sum_{k=1}^{m} n_k P_{n+1}(f_k(z)) = \sum_{k=1}^{m} n_k \left( \frac{f'_k(z)}{f_k(z)} \right) \cdot Q_n(f_k(z))$$

$$+ \alpha_n \sum_{k=1}^{m} n_k \left\{ \frac{f'_k(z)}{f_k(z)} \left( (\log f_k(z))^{n-1} \log (1 - f_k(z)) \right) - \frac{(1 - f_k(z))'}{1 - f_k(z)} (\log f_k(z))^n \right\}$$

where $Q_n(z) = \sum_{i=0}^{n-1} \beta_i (\log z)^i \text{Li}_{n-i}(z)$ and

$$\beta_i = \alpha_i + (i + 1) \alpha_{i+1} \quad \text{if} \quad i < n - 1, \quad \beta_{n-1} = \alpha_{n-1} + (n + 1) \alpha_n.$$

The second summand is equal to $\alpha_n K_{n+1}(b_{n+1}(f)) = 0$. Let $v_{z-a}(f(z))$ be a valuation of a rational function $f(z)$ at $z - a$. The first summand

$$\sum_{k=1}^{m} n_k f'_k(z) / f_k(z) \cdot Q_n(f_k(z))$$

is equal to

$$\sum_{a \in \mathbb{C}_p \cup \{\infty\}} \frac{1}{z-a} \sum_{k=1}^{m} n_k v_{z-a}(f_k(z)) Q_n(f_k(z)).$$

For any $a \in \mathbb{C}_p$ the element

$$\sum_{k=1}^{m} n_k v_{z-a}(f_k(z)) [f_k(z)] \in B(\mathbb{C}_p(z)^*)$$

belongs to $\ker b_n$. By Lemma 4.1 and the inductive assumption the expression $\sum_{k=1}^{m} n_k v_{z-a}(f_k(z)) Q_n(f_k(z))$ is constant. It follows from Lemma 4.3 that this constant is zero. Hence $(d/dz) \sum_{k=1}^{m} n_k P_{n+1}(f_k(z))$ vanishes and the function $\sum_{k=1}^{m} n_k P_{n+1}(f_k(z))$ is constant.

COROLLARY 4.5 (Conjectured by L. Lewin (see [L2], pp. 7–8)). — Let $f = \sum_{i=1}^{N} n_i [f_i(z)] \in B(\mathbb{C}_p(z)^*)$ belong to $\ker b_{n+1}$. The lower degree terms of the functional equation of $\text{Li}_{n+1}(z)$ for $f$ involve only constants and logarithms.
Proof. — The sequence \((1, 0, \ldots, 0, -1/(n + 1)!\) belongs to \(V_{n+1}\). Let 
\[
P_{n+1}(z) = \text{Li}_{n+1}(z) + \frac{1}{(n + 1)!} \log z \log(1 - z).
\] 
It follows from Proposition 4.4 that
\[
\sum_{i=1}^{N} n_i \left( P_{n+1}(f_i(z)) - P_{n+1}(f_i(x)) \right) = 0.
\]
This implies the corollary.

5. Differential Galois groups and functional equations

Let
\[
(*) \quad X' = AX
\]
be a linear system of differential equations on \(P^1(C_p)\) where \(X(z) := (X_1(z), \ldots, X_n(z))\) and \(A(z) = (A_{ij}(z))_{i,j=1,\ldots,n}\). Assume that the elements of the matrix \(A\) are in \(K = C_p(z - a)\). Assume also that the functions \(A_{ij}(z)\) for \(i, j = 1, \ldots, n\) have no poles at \(a \in C_p\). Then there exists \(n\)-solutions \(Y_1, \ldots, Y_n\) of (*) in \(C_p[[z - a]]\) linearly independent over \(C_p\). The subfield \(F = C_p(z - a)(Y_1, \ldots, Y_n)\) of the field of fractions of \(C_p[[z - a]]\) is preserved by the derivation \(\partial = \frac{d}{d(z - a)}\). The differential Galois group of \(F/K\) is the group \(\text{Aut}_\partial(F/K)\) of automorphism of \(F\) which commute with \(\partial\) and fix \(K\) (see [An]).

Our fundamental example is the following system of differential equations
\[
(**) \quad \begin{cases} 
T'_0 = 0, & \Psi = \frac{T_0}{z}, \\
T'_1 = \frac{T_0}{1 - z}, & T'_2 = \frac{T_1}{z}, \quad \ldots, \quad T'_n = \frac{T_{n-1}}{z}
\end{cases}
\]
with initial conditions \(T_0 = 1, \Psi(a) = 0\) and \(T_k(a) = 0\) for \(k > 0\).

Its differential Galois group \(G_n\) is given by the following automorphisms of \(F = C_p(z - a)(\Psi, T_1, \ldots, T_n)\):

\[
\theta(\alpha, \beta_1, \ldots, \beta_n) : \\
\Psi \mapsto \Psi + \alpha \\
T_k \mapsto T_k + \sum_{i=1}^{k} \frac{\beta_i}{(k - i)!} \Psi^{k-i}
\]
for $k = 1, \ldots, n$, where $\alpha, \beta_1, \ldots, \beta_n$ are independent parameters. Observe that

$$\theta(\alpha, \beta_1, \ldots, \beta_n) \circ \theta(\alpha', \beta_1', \ldots, \beta_n')$$

$$= \theta(\alpha + \alpha', \beta_1 + \beta_1', \beta_2 + \beta_2' + \beta_1 \frac{\alpha'}{1!},$$

$$\beta_3 + \beta_3' + \beta_2 \frac{\alpha'}{2!} + \beta_1 \frac{\alpha''}{2!}, \ldots).$$

The group $G_n$ is a nilpotent, affine, algebraic group over $\mathbb{C}_p$. The nilpotence class of $G_n$ is $n$, $G_n^{ab} = \mathbb{C}_p \oplus \mathbb{C}_p$ and, for each $k \leq n$, $\Gamma^k G_n/\Gamma^{k+1} G_n \approx \mathbb{C}_p$ is generated by the class of some $\theta(0, \ldots, 0, \beta_k, \ldots)$ with $\beta_k \neq 0$, $\alpha = 0$ and $\beta_i = 0$ for $i < k$. Observe that

$$Gala(F/\mathbb{C}_p(z - a)(\Psi, T_1, \ldots, T_k)) \approx \Gamma^{k+1} G_n.$$

Let $f_1(z), \ldots, f_m(z)$ be rational functions on $P^1(\mathbb{C}_p)$. We consider the following system of differential equations

$$\begin{cases}
T' = 0, & \Psi'_i = T \cdot \frac{f'_i}{f_i} \\
T_{1,i} = T \cdot \frac{f'_i}{1 - f_i}, & T_{k,i} = T_{k-1,i} \cdot \frac{f'_i}{f_i}
\end{cases}$$

(k = 2, \ldots, n, i = 1, \ldots, m) with initial conditions

$$T = 1, \quad \Psi_i(a) = 0 \quad \text{and} \quad T_{k,i}(a) = 0.$$

Let $G$ be a differential Galois group of the system (***)). Let

$$F_s = \mathbb{C}_p(z - a)(\Psi_i, T_{k,i})_{i=1,\ldots,m, \ k=1,\ldots,s}.$$ 

Then $Gala(F_n/\mathbb{C}_p(z - a)) = G$ and $Gala(F_n/F_s) \approx \Gamma^{s+1} G$.

**THEOREM 5.1.** — There is a functional equation

$$\sum_{i=1}^{m} n_i T_{n,i}(z) + p(z) = 0$$

where $p(z) \in F_{n-1}$ and $\sum_{i=1}^{m} |n_i| > 0$ if and only if $\dim \Gamma^n G < m$. 

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Proof. — The differential Galois group $\mathcal{G}$ is given by the following automorphisms of the field $F_n$:

$$\Psi_i \mapsto \Psi_i + \alpha_i$$

$$\theta((\alpha_i)_{i=1,\ldots,n},(\beta_k)_{k=1,\ldots,m})$$

$$T_{k,i} \mapsto T_{k,i} + \sum_{\ell=1}^{k} \frac{\beta_{k,i}}{(k-\ell)!} \Psi_i^{k-\ell}.$$ 

The parameters $\alpha_i, \beta_{k,i}$ need not be longer independent hence we have $\dim \mathcal{G} \leq m(n + 1)$. We shall denote by $\theta(\beta_{k_{1,0}})$ the element $\theta$ of $\text{Aut}_\theta(F_n/C_p(z - a))$ such that all $\alpha_i = 0$ and all $\beta_{k_i} = 0$ but $\beta_{k_{1,0}} \neq 0$.

Let us assume that we have a functional equation (****). Applying $\theta(\beta_{n,1}) \circ \theta(\beta_{n,2}) \circ \cdots \circ \theta(\beta_{n,m})$ to (****), we get

$$\sum_{i=1}^{n} n_i (T_{n,i}(z) + \beta_{n,i}) + p(z) = 0.$$ 

Hence $\sum_{i=1}^{n} n_i \beta_{n,i} = 0$. Observe that the group $\Gamma^n \mathcal{G}$ is generated by elements $\theta(\beta_{n,i})$ for $i = 1,\ldots,m$. This implies that $\dim \Gamma^n \mathcal{G} \leq m$.

The group $\Gamma^n \mathcal{G}$ is an abelian group, quotient of $C_m^m$, generated by the elements $\theta(\beta_{n,i})$ for $i = 1,\ldots,m$. Hence every element $\theta \in \Gamma^n \mathcal{G}$ has the form

$$\theta(\beta_{n,1}) \circ \theta(\beta_{n,2}) \circ \cdots \circ \theta(\beta_{n,m}).$$

Let us assume that $\dim \Gamma^n \mathcal{G} < m$. Then there is a non-trivial relation

$$\sum_{i=1}^{m} n_i \beta_{n,i} = 0$$

for any $\theta = \theta(\beta_{n,1}) \circ \theta(\beta_{n,2}) \circ \cdots \circ \theta(\beta_{n,m}) \in \Gamma^n \mathcal{G}$.

Observe that the function $\sum_{i=1}^{m} n_i T_{n,i}(z)$ is fixed by $\Gamma^n \mathcal{G}$, because

$$\theta : \sum_{i=1}^{m} n_i T_{n,i}(z) \mapsto \sum_{i=1}^{m} n_i (T_{n,i}(z) + \beta_{n,i})$$

$$= \sum_{i=1}^{m} n_i T_{n,i}(z) + \sum_{i=1}^{m} n_i \beta_{n,i}.$$ 

The isomorphism $\text{Aut}_\theta(F_n/F_{n-1}) \approx \Gamma^n \mathcal{G}$ implies that

$$\sum_{i=1}^{m} n_i T_{n,i}(z) \in F_{n-1}.$$
Remark. — Solutions of (**) are

\[ T_0 = 1, \quad \Psi(z) = \int_a^z \frac{dz}{z}, \quad T_k(z) = \int_a^z \frac{dz}{1-z}, \frac{dz}{z}, \ldots, \frac{dz}{z} \]

and solutions of (***) are

\[ T = 1, \quad \Psi_i(z) = \int_{f_i(a)}^{f_i(z)} \frac{dz}{z}, \quad T_{k,i}(z) = \int_{f_i(a)}^{f_i(z)} \frac{dz}{1-z}, \frac{dz}{z}, \ldots, \frac{dz}{z}. \]

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