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# ON A PROBLEM OF TAMAS VARGA

BY

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RÉSUMÉ. — Dans la première partie, on considère des propriétés quantitatives des nombres $q$ où un développement en base $q$ de 1 possède un nombre non borné de chiffres 0 consécutifs. Dans la seconde partie, on étudie la distribution des sommes finies $\sum \varepsilon_i q^i$, où $\varepsilon_i = 0$ ou 1 pour des valeurs spéciales de $q$. La troisième partie est consacrée à l'étude de la distribution des chiffres dans les développements gloutons des nombres $x$ presque partout dans $[0, 1]$. Finalement, on pose des problèmes ouverts.

ABSTRACT. — In the first part we investigate the quantitative properties of the numbers $q$ for which there exists an expansion of 1 in base $q$ where the length of consecutive 0-digits is not bounded. In the second part we study the distribution of the finite sums $\sum \varepsilon_i q^i$, $\varepsilon_i = 0$ or 1 for special values $q$. The third part is devoted to the study of the digit distribution of the greedy expansion of a.e. $x \in [0, 1]$. Finally we give some open problems.

*Dedicated to academician Vera T.* SÓS
*on the occasion of her birthday*

During his marvellous mathematical teaching activity Tamás VARGA found a lot of deep new problems. We mention the following one : in a heads or tails game repeated $n$ times how long sequences of consecutive heads can be found? In other words, if we consider the dyadic expansion

$$x = \sum_{1}^{\infty} \frac{\varepsilon_k(x)}{2^k}$$

of a randomly chosen number $0 \leq x \leq 1$, what can be asserted about the

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longest sequence of consecutive 0-digits (resp. 1-digits) between the first $n$ digits? This problem has thoroughly been investigated by many authors, see e.g. [1] and [2].

**1.**—In the paper [4], one of the authors modified the problem as follows. Let $1 < q < 2$ be an arbitrary number and consider the expansions of the number 1 of the following type :

$$1 = \sum_{i=1}^{\infty} \frac{1}{q^{n_i}}, \qquad n_i \in \mathbb{N} \text{ are different (natural numbers).} \tag{1}$$

For some values $q$ this expansion is not unique so the uniqueness problem can be investigated as well (see [5], [6]). Recently the second and third author proved in [7] that in the case $q = \sqrt{2}$ there exists an expansion 1 with the property :

$$\sup_i (n_{i+1} - n_i) = \infty. \tag{2}$$

The authors of the present paper, V. KOMORNIK and M. HORVÁTH solved the uniqueness problem of the expansion (1) in [5] and [6], further P. ERDŐS and I. JOÓ studied in [8] the properties of the sequence $(n_{i+1} - n_i)$.

In this paper consider the following properties of the expansion (1)

$$\sup_i \frac{n_i}{i} = \infty, \tag{2'}$$

$$\lim_i (n_{i+1} - n_i) = \infty, \tag{3}$$

$$\lim_i \frac{n_i}{i} = \infty. \tag{3'}$$

Obviously (2′) ⇒ (2) and (3) ⇒ (3′), further the reverse statements do not hold in general.

THEOREM 1 (cf. [8]). — *The set*

$$A := \Big\{ q \in ]1, 2[ : \text{ there exists an expansion (1) satisfying (2)} \Big\}$$

*is residual and of full measure in* $]1, 2[$.

*Problem 1.* — Does the statement of the THEOREM 1 remain true after substituting (2′) in place of (2) in the definition of $A$?

THEOREM 2. — *The set*

$$B := \Big\{ q \in \,]1,2[\colon \text{ there exists an expansion } (1) \text{ satisfying } (3') \Big\}$$

*is of first category and of measure zero.*

*Proof.* — It is enough to make the proof for the sets

$$B \cap \,]1+\delta, 2[, \qquad \delta > 0.$$

Consider arbitrary numbers $1+\delta < q_1 < q_2 < 2$, and sufficiently large $t \in \mathbb{N}$ :

$$1 = \sum_{i=1}^{n} \frac{\varepsilon_i}{q_1^i} = \sum_{i=1}^{n} \frac{\varepsilon_i}{q_2^i} + \frac{1}{q_2^{n+t}}. \tag{4}$$

It follows from $1+\delta < q_1$ that there exists $k \le c(\delta)$ with $\varepsilon_k = 1$. Consequently

$$\begin{aligned} \frac{1}{q_2^{n+t}} &= \sum_{i=1}^{n} \varepsilon_i \Big( \frac{1}{q_1^i} - \frac{1}{q_2^i} \Big) \ge \frac{1}{q_1^k} - \frac{1}{q_2^k} \\ &= \frac{\big[q_1 + (q_2 - q_1)\big]^k - q_1^k}{q_1^k q_2^k} \ge \frac{k(q_2-q_1)}{q_1 q_2^k} \ge \frac{q_2 - q_1}{q_2^{k+1}} \end{aligned}$$

i.e.

$$q_2 - q_1 \le \frac{1}{q_2^{n+t-k-1}} \le \frac{1}{q_2^{n+t-c(\delta)}} \le \frac{1}{(1+\delta)^{n+t-c(\delta)}}. \tag{5}$$

Denote $B_n$ the set of those $1+\delta < q < 2$ for which there exists an expansion of 1 satisfying $n_i/i \ge t$ for $n_i > n$. Take a number $N > 2n$. We see that between $\frac{1}{2}N$ and $N$ there exist $\ge \frac{1}{3}t$ consecutive zeros for any $q \in B_n$. Indeed, assume the contrary. Then between $\frac{1}{2}N$ and $N$ there exist

$$\ge \frac{\frac{1}{2}N}{\frac{1}{3}t} = \frac{3}{2}\,\frac{N}{t} \ \text{ 1-digits}$$

and then $i \ge \frac{3}{2}N/t$ and $n_i \le N$ would imply that $n_i/i \le \frac{2}{3}t$. The contradiction shows that there exists $\ge \frac{1}{3}t$ consecutive zeros between $\frac{1}{2}N$ and $N$; hence $q$ can be covered by an interval of length

$$\le (1+\delta)^{-N/2-t/3+c(\delta)},$$

see (5) above. For any $q \in B_n$ and $N$ we get an interval; the number of such intervals is not greater than $N$ times the number of the sequences $\varepsilon_1, \ldots, \varepsilon_N$ with $n_i \geq it$ and $n_i > n$. In particular $\varepsilon_1 + \ldots + \varepsilon_N \leq N/t$ (if $N > nt$). The number of choices of $N/t$ digits from $\varepsilon_1, \ldots, \varepsilon_N$ is

$$\begin{aligned}
\binom{N}{[N/t]} &= \frac{N(N-1)\cdots(n-[N/t]+1)}{[N/t]\,!} \\
&\leq c\,\frac{N^{[N/t]}}{\big([N/t]e^{-1}\big)^{[N/t]}\sqrt{[N/t]}} \\
&\leq c\,\frac{N^{[N/t]}}{\sqrt{[N/t]}\,\big(N/(2te)\big)^{[N/t]}} \\
&\leq c\sqrt{t/N}\,(2te)^{N/t} \\
&\leq c\sqrt{t}\,2^{c(1+\ln t)N/t} \\
&\leq c\sqrt{t}\,2^{N\varepsilon(t)} \qquad \big(\varepsilon(t) \to 0 \text{ when } t \to \infty\big).
\end{aligned}$$

Hence the sum of the length of the above intervals covering $B_n$ is not greater than $c\sqrt{t}\,(1+\delta)^{c(\delta)-t/3} \cdot N\big(2^{\varepsilon(t)}/\sqrt{1+\delta}\big)^N$. Given $\delta > 0$ we can choose $t \geq t(\delta)$ satisfying $2^{\varepsilon(t)}/\sqrt{1+\delta} < 1$. If we fix $t$ and let $N$ tend to infinity, we see that the set $B_n$ can be covered by finite systems of intervals of arbitrary small length sum. Consequently $B_n$ is nowhere dense and of measure zero.

Since $B \subset \bigcup_1^\infty B_n$ the proof of THEOREM 2 is complete. □

*Remark.* — By (3) ⇒ (3′) the same statement holds with (3) instead of (3′).

THEOREM 3. — *Define the set*

$$C := \Big\{q \in \,]1,2[\; : \; \textit{there exists an expansion } (3) \textit{ satisfying } (1)\Big\},$$

*For any interval* $I \subset \,]1,2[$ *the intersection* $I \cap C$ *has* $2^{\aleph_0}$ *many points.*

*Proof.* — Take any value $1 < q_0 < 2$ and any expansion $1 = \sum \varepsilon_i/q_0^i$. The set of $q$ for which there exists an expansion of 1 starting with $\varepsilon_1, \ldots, \varepsilon_n$, forms an interval whose length tends to zero when $n \to \infty$. This can be verified just as in (5). Consequently it is enough to prove that given $q_0$ and $n$ arbitrary we have $2^{\aleph_0}$ many $q \in C$ whose "good" expansion starts with $\varepsilon_1, \ldots, \varepsilon_n$. Fix a sequence $n < n_1 < n_2 \ldots$ satisfying (3) and construct a set $\mathcal{P}$ of infinite subsets of the set $\{n_1, n_2, \ldots\}$ such that $P_1, P_2 \in \mathcal{P}$ implies $P_1 \subset P_2$ or $P_2 \subset P_1$ and there are $2^{\aleph_0}$ elements of $\mathcal{P}$. This can be done in the usual way mapping the set $\{n_k\}$ onto the set of rational numbers $\mathbb{Q}$ in a one-to-one way and then consider the sets $\mathbb{Q} \cap ]-\infty, x[$, $x \in \mathbb{R}$. Now for every $P \in \mathcal{P}$ it corresponds to a $q = q_P$ by the rule

$$1 = \sum_{i=1}^{n} \frac{\varepsilon_i}{q^i} + \sum_{n_i \in P} \frac{1}{q^{n_i}}\cdot$$

Then $q_P \in C$ and for different $P$ the value $q_P$ is also different (in case $P_1 \subset P_2$ we have $q_{P_1} < q_{P_2}$).

THEOREM 3 is proved. □

**2.** — Now consider the following problem. For given $1 < q < 2$ define the sets

$$A_n := A_n(q) := \Big\{ \sum_{i=0}^{n-1} \varepsilon_i q^i : \varepsilon_i = 0 \text{ or } 1 \Big\}, \quad n = 1, 2, \ldots$$

If we arrange the sums $A_n$ in a sequence $y_1^{(n)} \leq \ldots \leq y_{2^n}^{(n)}$, we can write

$$A_n = \Big\{ y_k^{(n)} : 1 \leq k \leq 2^n \Big\}.$$

LEMMA 1. — *We have $y_{k+1}^{(n)} - y_k^{(n)} \leq 1$ for all $k$ and $n$.*

*Proof.* — Almost the same is proved in [6]. It runs as follows. We apply induction on $n$. If $n = 1$ then $A_n = \{0, 1\}$ so the statement is true. Suppose it for $A_n$ and prove for $A_{n+1}$. Obviously we have

$$A_{n+1} = A_n \cup (q^n + A_n). \tag{6}$$

Now if in $A_n$ there is an element larger than $q^n$, the smallest element of $q^n + A_n$, then the inductional hypothesis gives the statement by (6). If not, we have to check that the distance between the largest element of $A_n$ and $q^n$ is not larger than 1, i.e.

$$(1 + q + q^2 + \ldots + q^{n-1}) + 1 \geq q^n \quad \text{i.e.} \quad \frac{q^n - 1}{q - 1} \geq q^n - 1.$$

But this is true since $1 < q < 2$.

LEMMA 2. — *The polynomial*

$$P_r(x) := x^{r+1} - \sum_{k=0}^{r} x^k, \quad r \geq 1$$

*has exactly one zero* $\xi_r$ *in* $]1,2[$ *and* $\xi_r \to 2$ *monotone increasingly as* $r \to \infty$.

*Proof.* — Define the polynomial $Q_r$ by

$$P_r(x) = x^{r+1} - \frac{x^{r+1}-1}{x-1} = \frac{x^{r+2}-2x^{r+1}+1}{x-1} := \frac{Q_r(x)}{x-1}.$$

We see that the polynomial $Q_r$ decreases for $1 \leq x \leq x_0$, increases for $x_0 \leq x \leq 2$, where $x_0 = 2(r+1)/(r+2)$, further $Q_r(1) = 0, Q_r(2) = 1$. It shows that $Q_r(x)$ has exactly one zero $\xi_r$ in $]1,2[$ and $x_0 < \xi_r < 2$. It implies at once that $\xi_r \to 2$ as $r \to \infty$. On the other hand

$$Q_r(\xi_{r-1}) = \xi_{r-1}^{r+2} - 2\xi_{r-1}^{r+1} + 1 = 1 - \xi_{r-1} < 0$$

which shows that $\xi_{r-1} < \xi_r$.

LEMMA 3. — *Let* $n \geq 1$, $q = \xi_r$ *for some* $r \geq 1$ *and* $A_n = A_n(q)$. *Then we have* $A_n \cap ]q^n, 1+q^n[ = \emptyset$.

*Proof.* — By $q = \xi_r$ we have $P_r(q) = 0$, i.e.

$$1 = \frac{1}{q} + \frac{1}{q^2} + \cdots + \frac{1}{q^{r+1}}. \tag{7}$$

Iterating this we get the other representation

$$1 = \sum_{\substack{k=1 \\ k \nmid r+1}}^{\infty} \frac{1}{q^k}. \tag{8}$$

Next we show that

$$1 \geq \sum_{\substack{k=1 \\ k \neq r}}^{\infty} \frac{1}{q^k}, \tag{9}$$

and equality holds for $r = 1$. Indeed, we can transform the numbers $q^{-r}, q^{-2r-1}, q^{-3r-2}$, etc. of (8) by the aid of (7) to obtain

$$1 = \frac{1}{q} + \frac{1}{q^2} + \cdots + \frac{1}{q^{r-1}} + \frac{1}{q^{r+1}} + \left(\frac{2}{q^{r+2}} + \ldots + \frac{2}{q^{2r}}\right) + \frac{1}{q^{2r+1}} + \frac{1}{q^{2r+2}} + \left(\frac{2}{q^{2r+3}} + \cdots + \frac{2}{q^{3r+1}}\right) + \frac{1}{q^{3r+2}} + \frac{1}{q^{3r+3}} + \left(\frac{2}{q^{3r+4}} + \cdots + \frac{2}{q^{4r+2}}\right) + \cdots$$

which implies (9). This shows that if $y \in A_n$ and $y > q^n$ then the first $r$ digits $\varepsilon_{n-1}, \varepsilon_{n-2}, \ldots, \varepsilon_{n-r}$ of

$$y = \sum_{i=1}^{n-1} \varepsilon_i q^i$$

must be 1 (otherwise $y < q^n$). If $\varepsilon_{n-r-1} = 1$ then $q^n = \sum_{n-r-1}^{n-1} \varepsilon_i q^i$ and hence $y > q^n$ clearly implies $y \geq q^n + 1$. If $\varepsilon_{n-r-1} = 0$ then define

$$y_1 := y - \sum_{i=n-r}^{n-1} q^i \in A_{n-r-1}.$$

Since $q^{n-1} + \ldots + q^{n-r} = q^n - q^{n-r-1}$, this implies

$$q^{n-r-1} < y_1 < q^{n-r-1} + 1.$$

Iterating this process we finally find a value $1 \leq n \leq r+1$ and $y \in A_n$, $q^n < y < 1 + q^n$. But this is impossible since $n \leq r+1$ implies that $q^n \geq 1 + q + \ldots + q^{n-1}$. The contradiction proves the LEMMA 3. □

Now introduce the Fibonacci-type sequence $F_n^{(k)}$ by the recursion :

$$F_n^{(k)} = \begin{cases} 0 & \text{for } n < 0, \\ \sum_{i=1}^{k} F_{n-i}^{(k)} + 1 & \text{for } n \geq 0. \end{cases} \tag{10}$$

We see that

$$F_n^{(k)} = 2^n \quad \text{for} \quad 0 \leq n \leq k, \qquad F_{k+1}^{(k)} = 2^{k+1} - 1,$$
$$F_{k+2}^{(k)} = 2^{k+2} - 3, \qquad F_{k+3}^{(k)} = 2^{k+3} - 11.$$

THEOREM 4. — *Let $n, r = 1, 2 \dots$ and $q = \xi_r$. Then*

(a) $$|A_n(q)| = F_n^{(r+1)},$$

(b) $$\min_{\substack{1 \le k \le 2^n \\ y_{k+1}^{(n)} \ne y_k^{(n)}}} \left(y_{k+1}^{(n)} - y_k^{(n)}\right) \ge \frac{1}{q},$$

*and equality holds for $n \ge r + 1$.*

*Proof.* — Consider first the case $n \le r$. Then

$$q^n - \left(1 + q + \dots + q^{n-1}\right) \ge \frac{1}{q} > 0$$

hence

$$A_n \cap (q^n + A_n) = \emptyset$$

and then from the obvious relation

$$A_{n+1} = A_n \cup (q^n + A_n)$$

we see at once that

$$|A_{n+1}| = 2\,|A_n| = \cdots = 2^n\,|A_1| = 2^{n+1} = F_{n+1}^{r+1}$$

further (b) is also true and equality holds only for $n = r + 1$.

Now let $n \ge r + 1$. Then $A_n$ and $q^n + A_n$ has nonempty intersection since $q^n = q^{n-1} + \dots + q^{n-r-1}$. We show that in this case the sets $A_n$ and $q^n + A_n$ has overlapping maximal possible. Namely every $y \in A_n$, $y \ge q^n$ belongs to $q^n + A_n$. More precisely :

(*) $$\begin{cases} \text{Every } y \in A_n, y \ge q^n \text{ has an expansion} \\ y = q^{n-1} + \cdots + q^{n-r-1} + \displaystyle\sum_{k=0}^{n-r-2} \varepsilon_k q^k. \end{cases}$$

To prove (*) we apply induction on $n$. For $n = r + 1$ the only element $y \in A_n$ with $y \ge q^n$ is $q^n = q^{n-1} + \dots + q + 1$. Suppose (*) for $n$ and prove it for $n + 1$. Let $y \in A_{n+1}$ and $y \ge q^{n+1}$. If in the expansion $y = \sum_{k=0}^{n} \varepsilon_k q^k$ we have $\varepsilon_0 = 0$, then applying the induction hypothesis to $y/q \in A_n$, $y/q \ge q^n$ we are ready. If $y = q^{n+1}$ then $y = q^n + q^{n-1} + \dots + q^{n-r}$ which is also a good representation. Finally if $y > q^{n+1}$ and $\varepsilon_0 = 1$ then by Lemma 3, $y - 1 \ge q^{n+1}$, hence we can apply the induction hypothesis for $y-1/q \in A_n, y-1/q \geq q^n$ so (*) holds indeed. Consequently $A_{n+1} = 2A_n - A_{n-r-1}$, if $n \geq r+1$. So we can prove (a) by induction as follows :

$$\begin{aligned} |A_{n+1}| &= 2F_n^{(r+1)} - F_{n-r-1}^{(r+1)} \\ &= F_n^{(r+1)} + \Big(1 + \sum_{i=1}^{r+1} F_{n-i}^{(r+1)}\Big) - F_{n-r-1}^{(r+1)} \\ &= 1 + \sum_{i=1}^{r+1} F_{n+1-i}^{(r+1)} = F_{n+1}^{(r+1)}. \end{aligned}$$

The proof of (b) for $n \geq r+1$ is obvious : in $A_n$ and in $A_n + q^n$ the minimal distance is $1/q$ and they overlap maximally hence in $A_{n+1}$ the minimal distance is also $1/q$. THEOREM 4 is proved. □

**3.** — In the following part of this paper we consider two other problems related to the papers of ERDŐS, RÉNYI [3] and ERDŐS, RÉVÉSZ [1]. To formulate the first one, fix a number $1 < q < 2$ and expand any number $0 \leq x \leq 1$ by the so-called *greedy expansion*

$$x = \sum_1^\infty \frac{\varepsilon_n(x)}{q^n}, \qquad \varepsilon_n(x) = \begin{cases} 0 \\ 1. \end{cases}$$

We assert that

THEOREM 5. — *There exists a constant $c > 0$ with the following properties. Consider the set of those $x \in [0,1]$ for which the greedy expansion of $x$ contains a sequence of $\geq c\log n$ consecutive $0$-digits between the first $n$ digits $\varepsilon_1(x), \ldots, \varepsilon_n(x)$ for all indices $n \geq n_0(x)$. This set has full measure in $[0,1]$.*

The second problem arises in a heads or tails game with an asymmetric piece of money. We represent it as a random variable whose value is zero with probability $p, 0 < p < 1$ and 1 with probability $q = 1-p$. Consider a sequence $x_1, x_2, \ldots$ of independent random variables with such distributions. Introduce the quantities

$$\alpha_n := \log n - \log\log\log n + K$$

with some constant $K < 0$ to be specified later. We prove the

THEOREM 6. — *The following event has probability 1 : between the first $n$ digits $x_1, \ldots, x_n$ there exist $\alpha_n$ consecutive $0$ digits for sufficiently large $n > n_0$ ; here $n_0 = n_0(\omega)$ may depend on the concrete value of the sequence $\big(x_n(\omega)\big)_1^\infty$.*

We mention the following open

*Problem 2.* — THEOREM 5 does not remain true for large $c > 0$ (this is the case if $q = 2$, see ERDŐS, RÉNYI [3]).

For the proof of THEOREMS 5 and 6 we need some lemmas. Denote

$$P(\varepsilon_1, \dots \varepsilon_n) = \left|\left\{x \in [0,1] : \varepsilon_1 = \varepsilon_1(x), \dots, \varepsilon_n = \varepsilon_n(x)\right\}\right|$$

the probability of the event that the greedy expansion of $x$ begins with the digits $\varepsilon_1, \dots, \varepsilon_n$.

LEMMA 4.

(a) $$P(\varepsilon_1, \dots, \varepsilon_n, 1) \le \frac{1}{q-1} P(\varepsilon_1, \dots, \varepsilon_n, 0),$$

(b) $$P(\varepsilon_1, \dots, \varepsilon_n) \le \frac{q}{q-1} P(\varepsilon_1, \dots, \varepsilon_n, 0).$$

*Proof.* — (b) follows form (a) since

$$\begin{aligned} P(\varepsilon_1, \dots, \varepsilon_n) &= P(\varepsilon_1, \dots, \varepsilon_n, 0) + P(\varepsilon_1, \dots, \varepsilon_n, 1) \\ &\le \frac{q}{q-1} P(\varepsilon_1, \dots, \varepsilon_n, 0). \end{aligned}$$

To see (a) denote $I_n$ the length of the segment

$$\left\{x \in [0,1] : \varepsilon_1 = \varepsilon_1(x), \dots,\ \varepsilon_n = \varepsilon_n(x)\right\}.$$

The left endpoint is $x = \sum_{k=1}^{n} \frac{\varepsilon_k}{q^k}$. Hence :

i) If $I_n < \frac{1}{q^{n+1}}$, then

$$P(\varepsilon_1, \dots, \varepsilon_n, 0) = P(\varepsilon_1, \dots, \varepsilon_n), \quad P(\varepsilon_1, \dots, \varepsilon_n, 1) = 0.$$

ii) If $I_n \in \left[\frac{1}{q^{n+1}}, \frac{1}{q^{n+1}} \frac{q}{q-1}\right[$, then

$$P(\varepsilon_1, \dots, \varepsilon_n, 0) = \frac{1}{q^{n+1}}, \quad P(\varepsilon_1, \dots, \varepsilon_n, 1) = P(\varepsilon_1, \dots, \varepsilon_n) - \frac{1}{q^{n+1}}$$

and hence (a) follows. □

From LEMMA 4 we obtain immediately the

LEMMA 5.

$$P(\varepsilon_1, \dots, \varepsilon_n, \overset{1}{0}, \dots, \overset{\alpha_n}{0}) \geq \Big(\frac{q-1}{q}\Big)^{\alpha_n} P(\varepsilon_1, \dots, \varepsilon_n).$$

Now denote $S_k(x) := \sum_{j=1}^{k} \frac{\varepsilon_j(x)}{q^j}$. We obtain from LEMMA 5 by induction (in $[n/\alpha_n]$) the

LEMMA 6.

$$\Big|\Big\{x : S_{\ell\alpha_n}(x) \neq S_{(\ell+1)\alpha_n}(x),\ \ell = 0, 1, \dots, \Big[\frac{n}{\alpha_n}\Big]\Big\}\Big| \leq \Big(1 - \Big(\frac{q-1}{q}\Big)^{\alpha_n}\Big)^{[n/\alpha_n]+1}$$

*Proof of the Theorem 5.* — Let

$$\alpha_n := \log n - \log\log n - \log\log\log n - K$$

where log denotes the logarithm of base $q/(q-1)$ and $K = K(q) > 0$ is a constant "large enough". Then

$$\Big(1 - \Big(\frac{q-1}{q}\Big)^{\alpha_n}\Big)^{(q/(q-1))^{\alpha_n}} \to \frac{1}{e} \qquad (\text{as } n \to \infty),$$

hence

$$\Big(1 - \Big(\frac{q-1}{q}\Big)^{\alpha_n}\Big)^{(q/(q-1))^{\alpha_n}} \leq \frac{1}{e}.$$

(We know that $\sqrt[k+1]{1-(1-k^{-1})^k} \leq \dfrac{k}{k+1}$ i.e.

$$(1-k^{-1})^k \leq \Big(1 - \frac{1}{k+1}\Big)^{k+1}.)$$

We have

$$\Big(1 - \Big(\frac{q-1}{q}\Big)^{\alpha_n}\Big)^{[n/\alpha_n]+1} \leq e^{-((q-1)/q)^{\alpha_n}([n/\alpha_n]+1)}.$$

The exponent can be estimated as follows if $n > n_0$ :

$$\begin{aligned} -\Big(\frac{q-1}{q}\Big)^{\alpha_n}\Big(\Big[\frac{n}{\alpha_n}\Big]+1\Big) &\leq -\frac{1}{2}\Big(\frac{q-1}{q}\Big)^{\alpha_n}\frac{n}{\alpha_n} \\ &= -\frac{1}{2}\Big(\frac{q}{q-1}\Big)^{K}\frac{\log n \log\log n}{n}\,\frac{n}{\alpha_n} \\ &\leq -\frac{1}{3}\Big(\frac{q}{q-1}\Big)^{K}\log\log n \\ &\leq -R\log\log n\,; \end{aligned}$$

but if $K = K(q)$ is large enough, then the condition $n > n_0$ can be omitted. We obtained that

$$|P_n| = \left|\left\{x : S_{\ell\alpha_n}(x) \neq S_{(\ell+1)\alpha_n}(x),\ \ell = 0, 1, \dots, [n/\alpha_n]\right\}\right| \leq \frac{1}{(\log n)^2}$$

is $R$ is large enough, i.e. $K$ is large enough. This means that

$$\sum \left|P_{\left[\left(\frac{q}{q-1}\right)^m\right]}\right| < \infty$$

and according to the Borel-Cantelli lemma almost every $x \in [0,1]$ belongs to finitely many set $P_{[(q/(q-1))^m]}$, i.e. for a.e. $x \in [0,1]$ the first $[(q/(q-1))^m]$ digit contains 0-sequence of length

$$(*) \qquad cm \quad \text{if} \quad m > m_0(x).$$

Now if

$$\left[\left(\frac{q}{q-1}\right)^m\right] \leq n < \left[\left(\frac{q}{q-1}\right)^{m+1}\right],$$

then $m \asymp \log n$, i.e. it follows from (*) that for a.e. $x \in [0,1]$ among the first $n$ digits there exists 0-sequence of length $\geq c \log n$. Theorem 5 is proved. □

*Proof the Theorem 6.* — We need some lemmas.

LEMMA 7. — *The probability of the event that a sequence of length* $2n$ *contains a sequence of zeros of length* $n$ *is* $p^n(1 + nq)$.

*Proof.* — Consider the sequence of $n$ consecutive zeros with minimal first index. The probability of the event that this minimal index is the first is $p^n$, the probability of the event that it is $k$ $(2 \leq k \leq n+1)$ is $qp^n$ because the $(k-1)$th digit must be equal to 1. Hence LEMMA 7 follows. □

LEMMA 8. — *Let* $0 < \alpha_n < n$ *be arbitrary and consider the sets*

$$B_k := \left(S_{k+\alpha_n} = S_k\right), \qquad \left(k = 0, 1, \dots, n - \alpha_n\right),$$

$$C_\ell := \bigcup_{k=\ell\alpha_n}^{(\ell+1)\alpha_n} B_k, \qquad \left(\ell = 0, 1, \dots, 2\left([n/\alpha_n] - 1\right)\right),$$

$$D_n := D := \bigcup_{\ell=0}^{[n/\alpha_n]-1} C_{2\ell}.$$

*Then the probability of $\overline{D}$ is*

$$\left[1 - p^{\alpha_n}(1 + \alpha_n q)\right]^{[n/(2\alpha_n)]}.$$

*Proof.* — The events $C_{2\ell}$ are independent and one of them has probability $p^{\alpha_n}(1 + \alpha_n q)$ by LEMMA 7. □

Now we give upper estimate for the probability of $\overline{D}$. Because

$$p^{\alpha_n}(1 + \alpha_n q) \to 0 \quad \text{as} \quad \alpha_n \to \infty,$$

hence

$$\left[\left(1 - p^{\alpha_n}(1 + \alpha_n q)\right)^{(p^{\alpha_n}(1+\alpha_n q))^{-1}}\right]^{p^{\alpha_n}(1+\alpha_n q)[n/\alpha_n]} \leq \left(\frac{1}{e}\right)^{p^{\alpha_n}(1+\alpha_n q)[n/\alpha_n]} =: W.$$

Let $\alpha_n := \log n - \log\log\log n + K$ with some constant $K$, where log is of base $1/p$. Then we have

$$p^{\alpha_n} = p^K \, \frac{\log\log n}{n}$$

hence the exponent of $1/e$ in $W$ is

$$\geq p^K \, \frac{\log\log n}{n} \, q \log n \, \frac{n}{4\log n} = \tfrac{1}{4} \log\log n \, p^K q,$$

consequently

$$W \leq \left[\left(\frac{1}{e}\right)^{\log\log n}\right]^{p^K q/4}.$$

Choose $-K$ to be large enough, then the probability of $\overline{D}_{1/p^n}$ can be estimated as follows :

$$\left|\overline{D}_{1/p^n}\right| \leq \left[\left(\frac{1}{e}\right)^{\log\log(1/p^n)}\right]^{p^K q/4} \leq \frac{c_1}{n^{c_2 p^K}},$$

hence

$$\sum \left|\overline{D}_{1/p^n}\right| \leq c_1 \sum \frac{1}{n^{c_2 p^K}} < \infty$$

so according to Borel-Cantelli lemma, almost every $x$ belongs to finitely many $\overline{D}_{1/p^n}$ only i.e. for every $n > n_0(x)$ the first $n$ digits contain consecutive 0-s of length $\alpha_n$. THEOREM 6 is proved. □

At last we state the following questions.

*Problem* 3. — Investigate the behaviour of $n_{i+1}/n_i$ for the greedy, lazy or arbitrary expansions (see [8]).

*Problem* 4. — Is the set investigated in THEOREM 5 residual in $[0,1]$?

*Problem* 5. — If $q$ is not the root of the equations

$$1 = \frac{1}{q} + \frac{1}{q^2} + \cdots + \frac{1}{q^{r+1}}, \qquad r = 1, 2, \ldots$$

then

$$\inf(y_{n+1} - y_n) = 0,$$

where $y_n$ is the strictly increasing list of the values

$$\sum_{i=1}^{k} \varepsilon_i q^i \quad k = 1, 2, \ldots; \qquad \varepsilon_i = \begin{cases} 0 \\ 1. \end{cases}$$

*Problem* 6. — The statement analogous to THEOREM 5 with lazy expansions and consecutive 1 digits.

*Problem* 7. — THEOREM 3 for greedy expansion.

*Problem* 8. — By [6], THEOREM 2 the set of $q$ for which the greedy expansion of 1 contains consecutive 0-sequences of length $\geq \log_2 m$ between the first $m$ digits for infinitely many $m$, is residual and of full measure in $]1,2[$. Does it remain true if we require $\geq c \log m$ consecutive 0 between the first $m$ digits for every $m \geq m(q)$ (the constant $c > 0$ can be chosen appropriately small)?

*Problem* 9. — In [14] we showed, among others, that the value $q$ defined by

$$1 = \sum_{i=1}^{9} \frac{1}{q^i} + \sum_{j=1}^{n} \frac{1}{q^{9+10j}} + \sum_{k=1}^{\infty} \frac{1}{q^{9+10n+5k}} \qquad (n \geq 1)$$

has the property that 1 has exactly $n+1$ expansions. Describe the set of all $q$'s having this property.

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