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Ends of varieties


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ENDS OF VARIETIES

BY

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Introduction

Let $V$ be a subvariety of a domain $D$ in $\mathbb{C}^n$. The end of $V$, denoted $bV$, is the set $\overline{V} \setminus V$ contained in the boundary $bD$ of $D$. The terminology is due to GLOBEVNIK and STOUT, who studied the notion in a series of papers [12], [13], [14], [15]. Here we shall consider one-dimensional subvarieties, in strictly pseudoconvex domains, whose ends essentially have locally finite one-dimensional Hausdorff measure (which we shall refer to as “linear measure” and denote by $\mathcal{H}^1$).

Our first result concerns the general question of uniqueness. Given two irreducible subvarieties $V_1$ and $V_2$ of $D$, we want to conclude that $V_1 = V_2$ provided that $bV_1 \cap bV_2$ is, in some sense, sufficiently large. GLOBEVNIK and STOUT [13] showed that if $D$ is the unit ball in $\mathbb{C}^2$ and each of the subvarieties is the image of the unit disc under a proper holomorphic map and each end is a rectifiable Jordan curve, then the two subvarieties

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coincide, provided that the two ends meet in a set of positive linear measure.

We shall not require that our varieties be parameterized by the unit disk or that their ends be anywhere arc-like. We do however need a topological assumption of local connectedness. Recall that a continuum is, by definition, a compact connected set of more than one point. To formulate our assumptions, we define the subset $aV$ of $bV$ as the set of points $p$ in $bV$ for which there exists a continuum $X$ of finite linear measure contained in $bV$ such that $X$ is a (compact) neighborhood of $p$ in $bV$. If $X$ is a continuum of finite linear measure in $R^n$, then, by a theorem of Besicovitch [7], $X$ is arcwise connected and is a disjoint union of a countable set of rectifiable Jordan arcs and a set of linear measure zero; moreover, $X$ is locally connected, which can easily be proved directly or by using a result of Eilenberg and Harrold [6] to the effect that $X$ is the continuous image of the unit interval. Consequently, we can alternatively describe $aV$ as the set of points $p$ of $bV$ such that (i) $bV$ has locally finite linear measure at $p$ and (ii) $bV$ is locally connected at $p$. In particular, $aV$ is an open, sigma-compact subset of $bV$ and is a countable union of continua of finite linear measure. We have the following uniqueness result:

**Theorem 1.** — Let $V_1$ and $V_2$ be one-dimensional irreducible subvarieties of a strictly pseudoconvex domain $D$ in $\mathbb{C}^n$. If $aV_1 \cap aV_2$ has positive linear measure, then $V_1 = V_2$.

We should note that some sort of convexity condition is needed for this result; in our case, the subvarieties are both contained in a strictly pseudoconvex domain. Globevnik and Stout [13, Example 14] have indicated how the conclusion may otherwise fail, even if the ends are infinitely differentiable Jordan curves. Globevnik and Stout’s proof of uniqueness is restricted to $\mathbb{C}^2$ because it is based on a result of Berndtsson [4] which does not hold in higher dimensions. The different method used here is to find “good” projections: to do this, we need to consider the tangents to $bV$ at points of $aV$; these exist $H^1$ a.e. on $aV$. In the case that $bV$ is a $C^2$ Jordan curve in the boundary of the unit ball, Forstnerič [10] has shown that the tangents to $bV$ never lie in the complex tangential subspace of the tangent space to the unit sphere. It turns out that the existence of good projections is closely related to the existence of tangents which are not complex tangential to the boundary of the domain. Forstnerič’s result, however, may fail if $bV$ is only $C^1$, as was observed by Rosay [18]. That is, in the $C^1$ case, tangents to the curve may be complex tangential to $bD$. Nevertheless, the next result provides, in a quite general case, lots of tangents which are not complex tangential.
THEOREM 2. — Let $V$ be a one-dimensional irreducible subvariety of a strictly pseudoconvex domain $D$. Then $\mathcal{H}^1$ almost all points $p$ of $aV$ have the property that the tangent to $bV$ at $p$ exists and is not in the complex tangent space to $bD$ at $p$.

From this we obtain the following variant of the $C^2$ case obtained by Forstnerič [10]:

COROLLARY 1. — Let $\Gamma$ be a rectifiable Jordan curve in $bD$ with $D$ strictly pseudoconvex and $\overline{D}$ polynomially convex. Suppose that the tangents to $\Gamma$ are complex tangential to $bD$ at a set of points of $\Gamma$ of positive linear measure. Then $\Gamma$ is polynomially convex.

For the proof we note that, by [1] and [2], $\widehat{\Gamma} \setminus \Gamma$ is either empty or is an irreducible one-dimensional subvariety of $D$ whose end is exactly $\Gamma$. Theorem 2 rules out the latter. The corollary is false if the Jordan curve is replaced by a continuum $X$ of finite linear measure, even if $X$ is a union of two real analytic curves; the reason being that the part of the polynomial hull of $X$ inside $D$ may be a non-empty subvariety whose end is a proper subset of $X$.

According to a classical result of F. and M. Riesz [17], if $J$ is a Jordan domain in the plane whose boundary $bJ$ is a rectifiable Jordan curve, then harmonic measure on $bJ$ (for some interior point of $J$) and arc length measure on $bJ$ are mutually absolutely continuous measures. Part (a) of our next result can be viewed as a generalization to $\mathbb{C}^n$. A different formulation of the Riesz theorem, as, for example, given by Gamelin [11, p. 45], involves annihilating (“orthogonal”) measures. This relates to part (b) in our setting. The proof uses the “abstract” F. and M. Riesz theorem [11].

THEOREM 3. — Let $V$ be an irreducible one-dimensional subvariety of a strictly pseudoconvex domain $D$ in $\mathbb{C}^n$ such that $\mathcal{H}^1(bV \setminus aV) = 0$.

(a) Let $\mu_p$ be harmonic measure, with respect to $V$, on $bV$ for some point $p$ of $V$. Then $\mu_p$ and $\mathcal{H}^1|_{bV}$ (the restriction of linear measure to $bV$) are mutually absolutely continuous measures on $bV$.

(b) Let $\nu$ be a measure on $bV$ which is orthogonal to $A(V)$. Then $\nu$ is absolutely continuous with respect to $\mathcal{H}^1$.

(c) If $\mu$ is a representing measure on $bV$ for $p \in V$ for $A(V)$, then $\mu$ is absolutely continuous with respect to $\mathcal{H}^1$.

Here $A(V)$ denotes the algebra of functions continuous on $\overline{V}$ and holomorphic on $V$. We briefly introduce harmonic measure for $V$ below, in the usual way.
Our last results concern the accessibility of points on the boundary. Let $V$ be an irreducible subvariety of a domain $D$. We say that a point $p$ of $bV$ is accessible from $V$ if there is a real curve in $V$ which approaches $p$ asymptotically. Not every point of $bV$ need be accessible: we shall give an example when $D$ is the unit ball and $V$ is a properly imbedded disk. However, we shall show that at every point of $bV$ at which $bV$ locally has finite linear measure, is accessible. The question of non-tangential accessibility is more subtle. We say that $p$ is non-tangentially accessible from $V$ if there exists a real curve in $V$ which approaches $p$ asymptotically and which approaches $bD$ non-tangentially. Results on non-tangential accessibility for the case of holomorphic images of the unit disk were obtained by GLOBEVNIK and STOUT [13, Thm 9].

**Theorem 4.** — Let $V$ be an irreducible one-dimensional subvariety of a strictly pseudoconvex domain $D$ in $\mathbb{C}^n$. Every point of $aV$, except possibly for a set of linear measure zero, is non-tangentially accessible from $V$.

Here is an outline of what follows: in section 1 we consider projections to a real line of a continuum in $\mathbb{R}^n$ of finite linear measure. We show that if the continuum is regular and has a tangent line at a point, then the projection of the continuum to a line not orthogonal to the tangent line is close to being one-one, in an appropriate measure theoretic sense. We apply this to a complexified projection in section 2 to prove Theorem 2. Theorem 1 is proved in section 3. We discuss harmonic measure and prove Theorem 3 in section 4. We finish with a result on accessibility and a proof of Theorem 4 in section 5.

1.1. — We begin with some preliminary results on rectifiable Jordan arcs and continua of finite linear measure in $\mathbb{R}^n$. As usual, $B(p, r)$ denotes the open ball of radius $r$ about $p$ and $\|p\|$, $(p, q)$ denote the Euclidian norm and inner product. Let $\Gamma$ be an open Jordan arc in $\mathbb{R}^n$, parameterized by arc length $\gamma : (a, b) \to \mathbb{R}^n$. Then $\|\gamma(t_1) - \gamma(t_2)\| \leq |t_1 - t_2|$, $\gamma$ is absolutely continuous, $\gamma'$ exists a.e., $\|\gamma'(t)\| \leq 1$ wherever $\gamma'$ exists and $\|\gamma'(t)\| = 1$ a.e.

**Lemma 1.** — Suppose that $\gamma'(t_0)$ exists. Then $\|\gamma'(t_0)\| = 1$ if and only if $t_0$ is a Lebesgue point of $\gamma'(t)$.

**Proof.** — Suppose that $t_0$ is a Lebesgue point of $\gamma'$. By definition this means that

$$\frac{1}{|h|} \int_{t_0}^{t_0+h} \|\gamma(t) - \gamma'(t_0)\| \, dt \to 0 \quad \text{as} \quad h \to 0.$$
Then

\[ 1 - \|\gamma'(t_0)\| = \left| \frac{1}{h} \int_{t_0}^{t_0+h} \|\gamma'(t)\| \, dt \right| - \left| \frac{1}{h} \int_{t_0}^{t_0+h} \|\gamma'(t_0)\| \, dt \right| \]

\[ \leq \left| \frac{1}{h} \int_{t_0}^{t_0+h} \|\gamma'(t) - \gamma'(t_0)\| \, dt \right| \to 0. \]

Hence \(1 \leq \|\gamma'(t_0)\|\). Hence \(1 = \|\gamma'(t_0)\|\). Conversely, if \(\|\gamma'(t_0)\| = 1\), then

\[
\left\{ \frac{1}{h} \int_{t_0}^{t_0+h} \|\gamma'(t) - \gamma'(t_0)\| \, dt \right\}^2
\leq \left| \frac{1}{h} \int_{t_0}^{t_0+h} \|\gamma'(t) - \gamma'(t_0)\|^2 \, dt \right|
\]

\[ = 2 \times \left| 1 - \left( \frac{1}{h} \int_{t_0}^{t_0+h} \gamma'(t) \, dt, \gamma'(t_0) \right) \right|
\]

\[ = 2 \times \left| 1 - \left( \frac{\gamma(t_0 + h) - \gamma(t_0)}{h}, \gamma'(t_0) \right) \right| \to 2(1 - \|\gamma'(t_0)\|^2) = 0. \]

We next show that a continuum \(X\) in \(\mathbb{R}^n\) of finite linear measure has a tangent \(\mathcal{H}^1\) a.e. in a strong sense. Namely, in the sense that the tangent cone reduces to a line. For \(x \in \mathbb{R}^n\) and \(\alpha\) a unit vector in \(\mathbb{R}^n\) and \(0 < \varepsilon < \frac{1}{2} \pi\) we define the cone (two-sided) at \(x\) in direction \(\alpha\) and opening \(\varepsilon\) to be

\[ S(x, \alpha, \varepsilon) = \left\{ y \in \mathbb{R}^n : \left| (y - x, \alpha) \right| \geq \cos \varepsilon \cdot \|y - x\| \right\}. \]

We shall say that \(\alpha\) is a weak tangent to \(X\) at \(x\) if

\[ \lim_{r \to 0} \frac{1}{r} \mathcal{H}^1(X \cap (B(x, r) \setminus S(x, \alpha, \varepsilon))) = 0 \]

for all \(\varepsilon > 0\). According to [7], a continuum \(X\) of finite linear measure has a weak tangent at \(\mathcal{H}^1\) a.e. points of \(X\). In fact, in [7], the word “weak” is omitted. We shall reserve the word tangent for a stronger notion. We shall say that \(\alpha\) is a tangent of \(X\) at \(x\) if there exists a \(\delta_0\) such that \(X \cap B(x, r) \subseteq S(x, \alpha, \varepsilon)\) if \(r < \delta_0 = \delta_0(\varepsilon)\).

**PROPOSITION 1.** — **Let** \(X\) **be a continuum of finite linear measure in** \(\mathbb{R}^n\). **Then** \(X\) **has a tangent** \(\mathcal{H}^1\) **a.e. on** \(X\).

**Proof.** — By a theorem of Besicovitch (see [7]), \(X\) is the disjoint union of a countable set of open rectifiable Jordan arcs \(\{J_k\}\) and a set \(Z\).
with $\mathcal{H}^1(Z) = 0$. It suffices to show that $X$ has a tangent at $\mathcal{H}^1$ almost every point of $J_k$ for each $k$. Fix $k$. Let $\gamma : (a, b) \to \mathbb{R}^n$ parameterize $J_k$ by arc length. It suffices to show that $X$ has a tangent at each point $p_0 = \gamma(t_0)$ of $J_k$ such that (i) $X$ has a weak tangent at $p_0$ and (ii) $t_0$ is a Lebesgue point of $\gamma'$.

Put $\alpha = \gamma'(t_0)$. Then $\|\alpha\| = 1$ and clearly $\alpha$ is the weak tangent to $X$ at $p_0$, by uniqueness. Let $0 < \varepsilon < \frac{1}{2} \pi$. We must show that $X \cap B(p_0, r) \subseteq S(p_0, \alpha, \varepsilon)$ if $r$ is sufficiently small. Suppose not! Then there exists a sequence $\{x_n\} \subseteq X$, $x_n \to p_0$ and $x_n \notin S(p_0, \alpha, \varepsilon)$ for $n = 1, 2, \ldots$. Let $r_n = 2\|x_n\| > 0$. Then, since $\alpha$ is a weak tangent,

$$
\frac{1}{r_n} \mathcal{H}^1(X \cap B(p_0, r_n) \setminus S(p_0, \alpha, \frac{1}{2} \varepsilon)) \to 0.
$$

Let $Y_n$ be the connected component of $X \cap B(p_0, r_n) \setminus S(p_0, \alpha, \frac{1}{2} \varepsilon)$ which contains $x_n$. Since $X$ is connected, $Y_n$ has a non-empty intersection with $b[B(p_0, r_n) \setminus S(p_0, \alpha, \frac{1}{2} \varepsilon)]$. Because $\|x_n\| = \frac{1}{2} r_n$ and because $x_n \notin S(p_0, \alpha, \varepsilon)$ we conclude that distance of $x_n$ to this boundary is at least $\|x_n\| \cdot \sin\left(\frac{\pi}{2} \varepsilon\right) \equiv \eta_n$. Hence $\text{diam} (Y_n) \geq \eta_n$. Therefore $\mathcal{H}^1(Y_n) \geq \eta_n$. Then (1) implies $\eta_n/r_n \to 0$. But $\eta_n/r_n = \frac{1}{2} \sin\left(\frac{\pi}{2} \varepsilon\right) > 0$. Contradiction! This proves the proposition.

1.2. — Now suppose that $\gamma$ is a closed rectifiable Jordan arc in $\mathbb{R}^n$ and that $\gamma : [a, b] \to \mathbb{R}^n$ parameterizes $\gamma$ by arc length. Let $p_0 \in \gamma$, $p_0 = \gamma(t_0)$ and let $e$ be a unit vector in $\mathbb{R}^n$. We shall consider the projection $\pi : \mathbb{R}^n \to \mathbb{R}$ given by $\pi(p) = (p - p_0, e)$ and the associated function $u : [a, b] \to \mathbb{R}$, $u(t) = \pi(\gamma(t)) = (\gamma(t) - p_0, e)$. Then $u(t_0) = 0$, $u'(t) = (\gamma'(t), e)$ whenever $\gamma'(t)$ exists and $|u(t_1) - u(t_2)| \leq |t_1 - t_2|$. Therefore $u$ is absolutely continuous and $\mathcal{H}^1(u(A)) \leq \mathcal{H}^1(A)$ for all Borel sets $A \subseteq [a, b]$.

Proposition 2. — Let $t_0 \in (a, b)$ be such that

(a) $t_0$ is a Lebesgue point of $\gamma'$;

(b) $(\gamma'(t_0), e) \neq 0$;

(c) $\pi^{-1}(0) \cap \gamma = \{p_0\}$.

Let $n(x) = \#\{t \in [a, b] : u(t) = x\}$ and $E = \{x \in \mathbb{R} : n(x) = 1\}$. Let $T$ be a Borel subset of $[a, b]$ such that $t_0$ is a point of density of $T$. Then 0 is a point of density of the two subsets of $\mathbb{R}$:

(i) $E$, and

(ii) $u(T)$.
Remarks:

(a) By LEMMA 1, from (a), \( \gamma'(t_0) \) exists and is a unit vector. From (b), \( u'(t_0) \neq 0 \). Without loss of generality, we assume that \( u'(t_0) := \eta > 0 \).

(\beta) Hypothesis (a) implies that \( t_0 \) is a Lebesgue point for \( |u'(t)| \).

Indeed, for \( h > 0 \),

\[
\frac{1}{2h} \int_{t_0 - h}^{t_0 + h} \left| |u'(t)| - |u'(t_0)| \right| dt \leq \frac{1}{2h} \int_{t_0 - h}^{t_0 + h} \left| u'(t) - u'(t_0) \right| dt \\
\leq \frac{1}{2h} \int_{t_0 - h}^{t_0 + h} \left| \gamma'(t) - \gamma'(t_0) \right| dt \\
\leq \frac{1}{2h} \int_{t_0 - h}^{t_0 + h} \left| \gamma'(t) - \gamma'(t_0) \right| dt \to 0.
\]

In particular, \( \frac{1}{2h} I(h) := \frac{1}{2h} \int_{t_0 - h}^{t_0 + h} \left| u'(t) \right| dt \to |u'(t_0)| = \eta. \)

(\gamma) If \( u^{-1}((x_1, x_2)) \subseteq (t_1, t_2) \subseteq [a, b] \) then

\[
\int_{x_1}^{x_2} n(x) \, dx \leq \int_{t_1}^{t_2} \left| u'(t) \right| \, dt.
\]

This is the usual Banach indicatrix; see [8, Thm 2.10.13].

Proof. — We may assume that \( t_0 = 0 \) and that \( p_0 = 0 \in \mathbb{R}^n \). Let \( \varepsilon > 0 \). We claim

\[
(2) \quad (\exists \delta_0 > 0) \quad (\forall x \neq 0) \quad (\forall t \in [a, b]) \\
\quad \quad \left\{ \left| x \right| < \delta_0 \text{ and } u(t) = x \Rightarrow \left| t \right| < \left| x \right| (1 + \varepsilon)/\eta \right\}.
\]

Suppose not! Then for each positive integer \( n \) there exists \( x_n \neq 0 \) and \( t_n \in [a, b] \) such that \( \left| x_n \right| < 1/n, u(t_n) = x_n \) and \( |t_n| \geq (1 + \varepsilon)|x_n|/\eta. \)

If a subsequence \( \{t_{n_j}\} \) converges to \( t^* \in [a, b] \), \( u(t^*) = \lim u(t_{n_j}) = \lim x_{n_j} = 0 \). By (c), \( \gamma(t^*) = p_0 \) and so \( t^* = 0 \); i.e., we have \( t_n \to 0 \). Hence \( u(t_n)/t_n \to u'(0) = \eta. \) But \( |u(t_n)/t_n| = |x_n/t_n| \leq \eta/(1 + \varepsilon). \) This is a contradiction; (2) follows.

Now if \( 0 < \delta < \delta_0 \) then (2) implies

\[
u^{-1}(\{(-\delta, \delta)\}) \subseteq (-\delta(1 + \varepsilon)/\eta, \delta(1 + \varepsilon)/\eta).
\]

By Remark (\( \gamma \)) we have

\[
(3) \quad \int_{-\delta}^{\delta} n(x) \, dx \leq \int_{-\delta/\eta(1+\varepsilon)}^{\delta/\eta(1+\varepsilon)} \left| u'(t) \right| \, dt \equiv I(\delta(1 + \varepsilon)/\eta).
\]
Let $E'$ be the complement of $E$ in $\mathbb{R}$. If $\delta$ is sufficiently small, \( n(x) \geq 1 \) on $(-\delta, \delta)$ and so $n(x) \geq 1 + \chi_{E'}(x)$ on $(-\delta, \delta)$, where $\chi$ is the characteristic function. From (3) we get

$$2\delta + H^1((-\delta, \delta) \setminus E) \leq I(\delta(1 + \varepsilon)/\eta).$$

Hence

$$\frac{1}{2\delta} H^1((-\delta, \delta) \setminus E) \leq -1 + \frac{1 + \varepsilon}{\eta} I(\delta(1 + \varepsilon)/\eta)/2\delta(1 + \varepsilon)/\eta.$$  

By Remark $\beta$, $I(h)/(2h) \to \eta$ as $h \to 0$. We get

$$\limsup_{\delta \to 0} \frac{1}{2\delta} H^1((-\delta, \delta) \setminus E) \leq -1 + (1 + \varepsilon) = \varepsilon.$$

As $\varepsilon$ is arbitrary, (i) follows.

For (ii), we consider $S = [a, b] \setminus T$. If $\delta > 0$ is small,

$$(-\delta, \delta) \subseteq u([a, b]) \subseteq u(T) \cup u(S).$$

Hence

$$H^1(u(T) \cap (-\delta, \delta)) \geq 2\delta - H^1(u(S) \cap (-\delta, \delta)).$$

We have $u(S) \cap (-\delta, \delta) \subseteq u(S \cap u^{-1}((-\delta, \delta)))$. Let $\varepsilon > 0$. By (2) we get a $\delta_0$ such that $0 < \delta < \delta_0$ implies $u^{-1}((-\delta, \delta)) \subseteq (-\delta(1 + \varepsilon)/\eta, \delta(1 + \varepsilon)/\eta)$. Therefore $u(S) \cap (-\delta, \delta) \subseteq u(S \cap (-\delta(1 + \varepsilon)/\eta, \delta(1 + \varepsilon)/\eta))$. Hence

$$H^1(u(S) \cap (-\delta, \delta)) \leq H^1(S \cap (-\delta(1 + \varepsilon)/\eta, \delta(1 + \varepsilon)/\eta)).$$

Since $H^1(S \cap (-r, r))/(2r) \to 0$ as $r \to 0$ we conclude that

$$\frac{1}{2\delta} H^1(u(S) \cap (-\delta, \delta)) \to 0$$
as $\delta \to 0$. This, with (4), gives (ii).

We next extend the previous proposition to a continuum of finite linear measure. Suppose that $X$ is a continuum of finite linear measure in $\mathbb{R}^n$ and suppose that $\Gamma$ is a (necessarily rectifiable) Jordan arc contained in $X$, parameterized by $\gamma : [a, b] \to \mathbb{R}^n$ in arc length. Let $p_0 \in \Gamma$, $p_0 = \gamma(t_0)$ with $t_0 \in (a, b)$ and define $\pi : \mathbb{R}^n \to \mathbb{R}$ and $u : [a, b] \to \mathbb{R}$ as above, for a fixed unit vector $e$. 

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PROPOSITION 3. — Suppose that
(a) \(t_0\) is a Lebesgue point of \(\gamma'\);
(b) \(X\) has a tangent at \(p_0\);
(c) \(X\) is regular at \(p_0\);
(d) \((\gamma'(t_0), \epsilon) \neq 0\); and
(e) \(\pi^{-1}(0) \cap X = \{p_0\}\).

Let \(N(x) = \#\{p \in X : \pi(p) = x\}\) and let \(A = \{x \in \mathbb{R} : N(x) = 1\}\). Then \(0\) is a point of density of \(A\).

**Remark.** — \(X\) is regular at \(p_0\), means, by definition, that
\[
\lim_{r \to 0} \frac{1}{2r} \mathcal{H}^1(X \cap B(p_0, r)) = 1.
\]

Since \(X\) is a continuum of finite linear measure, it is regular at \(\mathcal{H}^1\) almost all of its points [7]. Also it follows easily from (a) and Lemma 1 that \(\Gamma\) is regular at \(p_0\).

**Proof.** — We define \(E\), as in PROPOSITION 2, as the set where \(\pi|\Gamma\) has multiplicity one. By PROPOSITION 2, \(E\) has 0 as a point of density and therefore its complement \(E'\) has density 0 at 0. Let \(C = \pi(X \setminus \Gamma)\). If \(\delta\) is sufficiently small, \(\pi(X) \supseteq (-\delta, \delta)\), and then \(A' \cap (-\delta, \delta) \subseteq (E' \cup C) \cap (-\delta, \delta)\), for if \(x \in A' \cap \pi(X) \setminus C\) then \(x \in \pi(\Gamma) \setminus E \subseteq E'\). We need to show that \(A'\) has density 0 at 0. It suffices to show that \(C\) has density 0 at 0.

Let \(\eta = (\gamma'(t_0), \epsilon) ; \eta \neq 0\) by (d). We may assume \(\eta > 0\). Let \(\epsilon > 0\). Then we claim that there exists \(\delta_0 > 0\) such that if \(0 < \delta < \delta_0\), then
\[
C \cap (-\delta, \delta) \subseteq \pi\left((X \setminus \Gamma) \cap B(p_0, (1 + \epsilon)\delta/\eta)\right).
\]
Suppose not! Then there exist \(\delta_n \downarrow 0\) and \(x_n \in \pi(X \setminus \Gamma) \cap (-\delta_n, \delta_n)\) but
\[
x_n \notin \pi\left((X \setminus \Gamma) \cap B(p_0, (1 + \epsilon)\delta_n/\eta)\right).
\]
Hence \(x_n = \pi(p_n), p_n \in X \setminus \Gamma, ||p_n - p_0|| \geq (1 + \epsilon)\delta_n/\eta\). Then \(p_n \to p_0\). For if a subsequence \(p_{n_j} \to p^* \in X\), then \(\pi(p^*) = \lim \pi(p_{n_j}) = \lim x_{n_j} = 0\). By (e), \(p^* = p_0\). Now, by (a) and (b), \(\gamma'(t_0)\) is the tangent to \(X\) at \(p_0\). Since \(p_n \to p_0\), some subsequence of \((p_n - p_0)/||p_n - p_0||\) \(\to \pm \gamma'(t_0)\). We may assume this is the case for the original sequence. Hence \(\alpha_n \equiv \left((p_n - p_0)/||p_n - p_0||, \epsilon\right) \to \pm \eta\). But
\[
|\alpha_n| = \frac{|x_n|}{||p_n - p_0||} \leq \frac{\delta_n}{(1 + \epsilon)\delta_n/\eta} = \frac{\eta}{(1 + \epsilon)}.
\]
This is a contradiction and we conclude that (5) holds. From (5) we get
\[ H^1(C \cap (-\delta, \delta)) \leq H^1 \left[ X \cap B(p_0, (1 + \varepsilon)\delta/\eta) \right] - H^1 \left[ \Gamma \cap B(p_0, (1 + \varepsilon)\delta/\eta) \right] \]
and therefore
\[ \frac{1}{2\delta} H^1(C \cap (-\delta, \delta)) \leq \frac{1 + \varepsilon}{\eta} \left\{ \frac{H^1 \left( X \cap B(p_0, (1 + \varepsilon)\delta/\eta) \right)}{2(1 + \varepsilon)\delta/\eta} - \left[ \frac{H^1 \left( \Gamma \cap B(p_0, (1 + \varepsilon)\delta/\eta) \right)}{2(1 + \varepsilon)\delta/\eta} \right] \right\}. \]

Since \( X \) and \( \Gamma \) are both regular at \( p_0 \), each of the two quotients in square brackets converges to 1 as \( \delta \to 0 \). Hence \( C \) has density 0 at 0.

2.1. — For the proof of Theorem 2 we shall need two lemmas. Let \( D \) be a strictly pseudoconvex domain in \( \mathbb{C}^n \). Recall that \( A(D) \) denotes the algebra of complex-valued functions which are continuous on \( \bar{D} \) and holomorphic on \( D \).

**Lemma 2.** — Let \( \Gamma \) be a rectifiable Jordan arc in \( \partial D \). Let \( E \) be a compact subset of \( \Gamma \) such that at each point of \( E \) the tangent to \( \Gamma \) exists and is complex tangential to \( \partial D \). Then \( E \) is a peak interpolation set for \( A(D) \).

**Lemma 3.** — Let \( \Omega \) be a bounded, open, simply connected subset of the complex plane with \( \mathcal{H}^1(\partial\Omega) < \infty \). Suppose that \( \omega \subseteq \partial\Omega \) is a Borel set with \( \mathcal{H}^1(\omega) > 0 \), that \( F \in H^\infty(\Omega) \) and that for all \( z \in \omega \) there is a path \( \sigma_z \) in \( \Omega \) which approaches \( z \) asymptotically such that the limit of \( F(\zeta) \) as \( \zeta \) approaches \( z \) along \( \sigma_z \) exists and equals 0. Then \( F \equiv 0 \).

We shall prove these lemmas after proving Theorem 2.

2.2. — We prove Theorem 2 by contradiction and suppose that the tangent to \( aV \), which exists \( H^1 \) a.e. on \( aV \), is complex tangential to \( bD \) on a set of positive \( H^1 \) measure. We claim then that there exists \( p_0 \in aV \), a continuum \( X \subseteq aV \) which is a neighborhood of \( p_0 \) in \( aV \) and with \( \mathcal{H}^1(X) < \infty \), a Jordan arc \( \Gamma \), \( p_0 \in \Gamma \subseteq X \) such that \( \gamma : (a, b) \to \mathbb{C}^n \) parametrizes \( \Gamma \) by arclength, \( \gamma(t_0) = p_0 \), and a compact set \( E \subseteq \Gamma \) such that the tangent to \( aV \) exists at each point of \( E \) and is complex tangential to \( bD \), \( \mathcal{H}^1(E) > 0 \), and that \( (a)-(c) \) of Proposition 3 hold with \( \epsilon = \gamma'(t_0) \) and such that \( t_0 \) is a point of density of \( T = \gamma^{-1}(E) \).

To see this, we use the fact that \( aV \) has a tangent and is regular \( H^1 \) a.e. and the fact that \( aV \) is a countable union of Jordan arcs and a null set to
get a $p_0$ and a $\Gamma$ satisfying (a), (b) and (c) with $aV$ in place of $X$, (d) and the statement on $T$. Then choose $X$ as a sufficiently small neighborhood of $p_0$ in $aV$, using (b) to achieve (e). Here we view $\mathbb{C}^n$ as being $\mathbb{R}^{2n}$, the real inner product $\langle \cdot, \cdot \rangle$ being the real part of the Hermitian inner product $\langle \cdot, \cdot \rangle$. In particular, we have the complex projection $g : \mathbb{C}^n \to \mathbb{C}$ given by $g(p) = \langle p - p_0, e \rangle$ and satisfying $\pi(p) = \text{Re}(g(p))$. Finally we can replace $\Gamma$ by a subarc to have $\Gamma \subseteq X$. As before we have $u : (a, b) \to \mathbb{R}$ given by $u(t) = \pi(t) = \text{Re}(g(t))$.

Since $e = \gamma'(t_0)$ is the tangent to $X$ at $p_0$ by (a) and (b), if $\delta_0 > 0$ is sufficiently small,

$$aV \cap B(p_0, \delta_0) = X \cap B(p_0, \delta_0) \subseteq S(p_0, e, \frac{1}{6} \pi).$$

Since

$$g(S(p_0, e, \frac{1}{6} \pi)) \subseteq \{ \lambda \in \mathbb{C} : |\text{Im} \lambda| \leq \frac{1}{\sqrt{3}} |\text{Re} \lambda| \}$$

we get

$$g(X \cap B(p_0, \delta_0)) \subseteq \{ \lambda \in \mathbb{C} : |\text{Im} \lambda| \leq \frac{1}{\sqrt{3}} |\text{Re} \lambda| \}.$$ (6)

For $0 < \delta < \delta_0$, consider $W = V \cap B(p_0, \delta)$, a subvariety of $B(p_0, \delta) \cap D$. Then

$$bW \subseteq (X \cap \overline{B}(p_0, \delta)) \cup (bB(p_0, \delta) \cap V),$$
and

$$g(bW) \subseteq g(X \cap B(p_0, \delta_0)) \cup g(bB(p_0, \delta) \cap V).$$

Since $\{ p \in V : g(p) = 0 \}$ is discrete and countable (otherwise $g \equiv 0$ on $V$ and so $g \equiv 0$ on $\Gamma \subseteq bV$ and so $u \equiv 0$, contradicting

$$u'(t_0) = (\gamma'(t_0), e) \neq 0$$

by (d)) we may choose $\delta$ so that $0 \notin g(bB(p_0, \delta) \cap V)$. From (6) and (e) we conclude that $g^{-1}(0) \cap bW = \{ p_0 \}$ and that $0 \notin g(bB(p_0, \delta) \cap \overline{V})$. Hence for $\rho > 0$ sufficiently small

$$Q_\rho \cap g(bB(p_0, \delta) \cap \overline{V}) = \emptyset$$

where

$$Q_\rho = \{ \lambda \in \mathbb{C} : |\text{Re} \lambda| \leq \rho \text{ and } |\text{Im} \lambda| \leq \rho \}.$$

Again from (6) we have

$$g(X \cap \overline{B}(p_0, \delta)) \cap \{ \lambda \in \mathbb{C} : |\text{Re} \lambda| < \rho \} \subseteq Q_\rho.$$
By Proposition 2, \( u(T) \) has 0 as a point of density and by Proposition 3, the set \( A \) defined there has 0 as a point of density. Set

\[
A_1 = A \cap u(T) \cap (-\rho, \rho).
\]

Then 0 is a point of density of \( A_1 \). For \( x \in \mathbb{R} \) we denote the vertical line \( \{ \lambda \in \mathbb{C} : \text{Re} \lambda = x \} \) by \( \ell_x \). Then by the definition of \( A \), for each \( x \in A_1 \), \( \ell_x \cap Q_\rho \cap g(X) \) contains exactly one point \( \lambda_x = x + iv_x \) with \( -\rho < v_x < \rho \). Moreover \( g^{-1}(\lambda_x) \cap X \) consists of a single point \( p_x \in \Gamma \). In particular the segment \( [-\rho, \rho] \times \{ \rho \} \) is disjoint from \( g(bW) \) and so is contained in a component \( \Omega_1 \) of \( \mathbb{C} \setminus g(bW) \). Likewise \( [-\rho, \rho] \times \{-\rho\} \) is contained in a component \( \Omega_2 \). Then for \( j = 1 \) and 2, \( g : g^{-1}(\Omega_j) \cap W \to \Omega_j \) is a proper holomorphic map and hence a branched cover of multiplicity \( m_j \), with \( m_j \geq 0 \).

Since the set of singular values of \( g|_V \) is countable, by removing a set of linear measure zero from \( A_1 \) we can assume that for every \( x \in A_1 \) (i) \( \ell_x \) contains no singular values of \( g \restriction V \) and (ii) every point of \( A_1 \) is a point of density of \( A_1 \).

Now fix \( x \in A_1 \). Let \( m = m(x) = \#(g^{-1}(\lambda_x) \cap W) \). Each neighborhood of each point in \( g^{-1}(\lambda_x) \cap W \) is mapped by \( g \) to a neighborhood of \( \lambda_x \in b\Omega_1 \cap b\Omega_2 \). We conclude that \( m \leq m_1 \) and \( m \leq m_2 \).

We claim that it is not true that \( m_1 = m = m_2 \). Suppose it were true! Then \( g^{-1}(\lambda_x) \cap W = \{p_1, p_2, \ldots, p_m\} \). Choose \( m + 1 \) disjoint neighborhoods \( N; N_1, N_2, \ldots, N_m \) of \( p_1, p_2, \ldots, p_n \), respectively in \( \mathbb{C}^n \). Then \( g(W \cap N_j) \) is a neighborhood of \( \lambda_x \) for \( j = 1, 2, \ldots, m \). Since \( g|_g^{-1}(\Omega_j) \cap W \) has multiplicity \( m = m_j \) over \( \Omega_j \), we conclude that \( g(W \cap N) \) is disjoint from \( \Omega_1 \cup \Omega_2 \). Since \( \Omega_1 \cup \Omega_2 \) contains, for each \( x' \in A_1 \), \( (\ell_{x'} \setminus \{\lambda_{x'}\}) \cap Q_\rho \), it follows that the open set \( g(W \cap N) \) is disjoint from \( \ell_{x'} \cap Q_\rho \) for \( x' \in A_1 \). We can take \( N \) of the form \( B(p, \delta) \subseteq B(p_0, \delta) \).

Arguing, as above, that \( \{p \in V : g(p) = \lambda_x \} \) is discrete, we get that \( \varepsilon \) can be chosen so that

\[
\lambda_x \notin g(bB(p, \varepsilon) \cap V).
\]

(We use the fact that \( g^{-1}(\lambda_x) \cap X = \{p_x\} \), since \( x \in A_1 \).) By (7) we can choose a small open rectangle \( R \) about \( \lambda_x \) with \( R \) disjoint from \( g(bB(p, \varepsilon) \cap V) \). We can take \( R \subseteq Q_\sigma \) of the form \( (x_1, x_2) \times (y_1, y_2) \) where

\[
-\sigma < x_1 < x < x_2 < \sigma, \quad x_1, x_2 \in A_1,
\]

\[
-\sigma < y_1 < y < y_2 < \sigma.
\]
Since $bR \subseteq \Omega_1 \cup \Omega_2 \cup \{\lambda_{x_j} : j = 1, 2\}$ if $(x_2 - x_1)$ is sufficiently small, it follows that every component of $g(W \cap N)$ which meets $R$ is contained in $R$. Since $p_x \in bV$, there exists $q \in V \cap N = W \cap N$ such that $g(q) \in R$. Let $W_1$ be the connected component of $W \cap N$ which contains $q$. Then $g(W_1)$ meets $R$ and so is contained in $R$. Now we claim that $bW_1 \subseteq X$. Indeed, $bW_1 \subseteq b(W \cap N) \subseteq X \cup S$ where $S = bB(p_x, \varepsilon) \cap V$. But by choice of $R$, $g(S) \cap \overline{R}$ is empty. The claim thus follows. It implies that $W_1$ is a subvariety of $D$ which is contained in $V$, which is irreducible. A contradiction!

Thus for all $x \in A_1$, either $m(x) < m_1$ or $m(x) < m_2$. Thus we may assume that the first condition holds on a Borel subset $B$ of $A_1$ of positive measure. In particular, we can now say that $m_1 > 0$. Set

$$\omega = \{\lambda_x : x \in B\} = (B \times [-\sigma, \sigma]) \cap g(X),$$

a Borel subset of $b\Omega_1$ with $\mathcal{H}^1(\omega) \geq \mathcal{H}^1(B) > 0$.

For $x \in B$, set $\gamma_x = \{x + it : v_x < t < \sigma\} \subseteq \ell_x \cap Q_{\sigma}$. Then $\gamma_x \subseteq \Omega_1$. Since $\gamma_x$ contains no critical values of $g|_V$, $g^{-1}(\gamma_x) \cap W$ is a disjoint union of $m_1$ Jordan arcs $\gamma_{1,x}^{\tau}, \ldots, \gamma_{m_1}^{\tau}$. Fix $j$, $1 \leq j \leq m_1$. The cluster set of $\{p \in \gamma_x^{\tau}\}$ in $\mathbb{C}^n$ as $g(p) \to \lambda_x$ is connected and is contained in $g^{-1}(\lambda_x) \cap \overline{W}$ which consists of $m + 1$ points, $m$ of them in $g^{-1}(\lambda_x) \cap W$ and one point in $bW$, namely, $p_x$. Therefore the cluster set reduces to a single point and so $\gamma_j^{\tau}$ approaches one of the $m + 1$ points asymptotically. Since $g$ maps neighborhoods in $W$ of each of the $m$ points of $g^{-1}(\lambda_x) \cap W$ homeomorphically to a neighborhood of $\lambda_x$, it follows, since $m < m_1$ that one of the $\gamma_j^{\tau}$ approaches $p_x$. Recall that $p_x \in E \subseteq \Gamma$, since $A_1 \subseteq \omega(T)$.

Now we apply Lemma 2 to $E \subseteq \Gamma$ and obtain a peak function $f_1 \in A(D)$. Set $f = 1 - f_1$. Then $f \neq 0$ on $\overline{D} \setminus E$; in particular, $f \neq 0$ on $V$ and $f = 0$ on $E$. For $\lambda \in \Omega_1$, $g^{-1}(\lambda) \cap W = \{w^1, w^2, \ldots, w^{m_1}\}$, counting multiplicity. Define a bounded function $F$ on $\Omega_1$ by

$$F(\lambda) = \prod_{g^{-1}(\lambda) \cap W = \{w^1, w^2, \ldots, w^{m_1}\}} f(w^j).$$

It is standard that $F$ is a well-defined bounded holomorphic function on $\Omega_1$. Let $\Omega$ equal to the component of $\Omega_1 \cap Q_\sigma$ which contains $[-\sigma, \sigma] \times \{\sigma\}$ in its closure. Then $b\Omega \subseteq bQ_{\sigma} \cup g(X)$ and so $\mathcal{H}^1(b\Omega) < \infty$. Also $\omega \subseteq b\Omega$ and $\gamma_x \subseteq \Omega$ for $x \in A_1$. Fix $\lambda_x \in \omega$. We have seen that some $\gamma_j^{\tau} \to p_x \in E$. Hence $f(w) \to 0$ as $w \in \gamma_j^{\tau} \to p_x$, since $f(p_x) = 0$. It
follows that $F(\lambda) \to 0$ as $\lambda \in \gamma_x \to \lambda_x$. By Lemma 3, $F \equiv 0$. But $f \neq 0$ on $V$ implies $F \neq 0$ on $\Omega$. This is a contradiction. Theorem 2 follows.

2.3. — We shall deduce Lemma 2 from a theorem of Davie and Øksendal [5]; the case when $\Gamma$ is smooth and $E$ a subarc was already noted in [5]. Following the notation of [5], we write $T(\zeta)$ for the complex tangent space to $bD$ at $\zeta \in bD$ and $L(\zeta)$ for its orthogonal complement in the (real) tangent space to $bD$ at $\zeta$. Also if $S$ is a real linear subspace of $\mathbb{C}^n$, and $Y$ is a subset, $d_S(Y)$ denotes the diameter of the orthogonal projection of $Y$ to $S$. Clearly $d_S(Y) \leq \text{diam}(Y)$.

Let $\zeta \in E$. Then, for all $\eta > 0$, there exists a subarc $J$ of $\Gamma$ containing $\zeta$ such that $H^1(J) < \eta$ and $d_L(\zeta)(J) \leq \eta \cdot \text{diam}(J)$. This follows easily from the fact that the tangent to $\Gamma$ at $\zeta$ exists and is complex tangential to $bD$ at $\zeta$. Let $W$ be an open subset of $\Gamma$ containing $E$ and such that $H^1(W) < 2H^1(E)$.

Let $\varepsilon > 0$ and set $\eta = \min(\varepsilon, \varepsilon/(4H^1(E)))$. For each $\zeta \in E$, choose an interval $J$ as above such that $d_L(S)(J) \leq \eta \cdot \text{diam}(J)$ and also such that $\text{diam}(J) < \eta$ and $J \subseteq W$. Let $J_1, J_2, \ldots, J_N$ be a finite subcover of $E$ with $\zeta_1, \zeta_2, \ldots, \zeta_N$ the corresponding points. Write $d_L(k)$ for $d_L(\zeta_k)(J_k)$ and similarly $d_T(k)$. By discarding some $J_i$'s, without changing their union, we may assume that no point belongs to more than two $J_i$'s; hence

$$\sum H^1(J_k) \leq 2H^1(W) < 4H^1(E).$$

Therefore $\sum \text{diam}(J_k) < 4H^1(E)$.

Finally we get

(i) $\sum d_T(k)^2 \leq \eta \sum d_T(k) \leq \eta \sum \text{diam}(J_k) < 4\eta H^1(E) \leq \varepsilon$, and

(ii) $\sum d_L(k) \leq \eta \sum \text{diam}(J_k) < 4\eta H^1(E) \leq \varepsilon$.

The lemma now follows from Theorem 1 of [5].

2.4 Proof of lemma 3. — Let $\psi : U \to \Omega$ be a Riemann map. Recall [3] that $\psi$ extends to be a continuous map of $bU$ onto $b\Omega$. Set $\omega_1 = \psi^{-1}(\omega) \subseteq bU$ and $F_1 = F \circ \psi$ in $H^{\infty}(U)$. Let $F_1^*$ denote the a.e. defined radial limit of $F_1$ on $bU$. Set

$$N = \left\{ e^{i\theta} : F_1^*(e^{i\theta}) \text{ exists and equals } 0 \right\}.$$

We claim that $\psi(N \cap \omega_1) = \omega$. Let $z \in \omega$ and set $\tilde{\sigma}_z = \psi^{-1} \circ \sigma_z$. Then $\tilde{\sigma}_z$ is a path in $U$. Its cluster set on $bU$ is connected, hence is a
subarc. But ψ maps this subarc to z. We conclude that the subarc reduces to a single point λ with ψ(λ) = z ∈ ω. Hence λ ∈ ω_1. Also F_1(ζ) → 0 as ζ → λ along σ_z. By a classical result of Lindelöf, F_1^*(λ) = 0; i.e. λ ∈ N. Thus λ ∈ N ∩ Ω_1 and this gives the claim.

Next, we claim that H^1(N) > 0. This implies that F_1 ≡ 0 by Fatou’s lemma and hence that F ≡ 0, as desired.

To verify the claim, we use the fact that ψ' ∈ H^1 (Hardy space) and so Re ψ and Im ψ are absolutely continuous on bD; cf. [3]. We have H^1(ψ(N)) ≥ H^1(ψ(N ∩ ω_1)) = H^1(ω) > 0. By the projection lemma of [3] since ψ(N) ⊆ bΩ, a continuum with H^1(bΩ) < ∞, either H^1(Re ψ(N)) > 0 or H^1(Im ψ(N)) > 0. Suppose the former, without loss of generality. Then, as Re ψ is absolutely continuous, it follows that H^1(N) > 0; cf. [3, Lemma 1]. This gives the lemma.

3.1. — We can now use Proposition 3 with Theorem 2 to prove Theorem 1. It will be convenient to assume henceforth that D is strictly convex. This is justified by the imbedding theorem of Fornaess [9] and Henkin [16]. Let N(p) be the outward unit normal vector to bD at p ∈ bD. Fix p_0 ∈ bD, we set e = iN(p_0) and define g : C^n → C by g(p) = (p - p_0, e) and π : C^n → R by π = Re o g. For p ∈ D, by strict convexity, Im g(p) ≥ 0 and Im g(p) > 0 for p ∈ D \ {p_0}.

Lemma 4. — Let X ⊆ bD be a continuum and suppose that the tangent to X exists at p_0 ∈ X and that this tangent is not complex tangential to bD. Then the tangent to g(X) ⊆ C at λ = 0 is the real axis \{λ ∈ C : Im λ = 0\}.

Proof. — Let λ_n(≠ 0) ∈ g(X) be such that λ_n → 0. Say λ_n = g(p_n), p_n ∈ X. Then the fact π^{-1}(0) ∩ X = {p_0} yields that p_n → p_0. Passing to a subsequence, (p_n - p_0)/∥p_n - p_0∥ → τ, the tangent to X at p_0. Since X ⊆ bD, (τ, N(p_0)) = 0. Hence Im〈τ, e〉 = 0. We have

$$\frac{λ_n}{∥p_n - p_0∥} = \left< \frac{p_n - p_0}{∥p_n - p_0∥}, e \right>.$$ 

Hence Im(λ_n/∥p_n - p_0∥) → Im〈τ, e〉 = 0. As τ is not a complex tangent, Re〈τ, e〉 ≡ b ≠ 0. Hence

$$Re(λ_n/∥p_n - p_0∥) \rightarrow Re〈τ, e〉 = b.$$ 

It follows that Im λ_n/Re λ_n → 0/b = 0. This gives the lemma.

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The next lemma describes the nice projections which will be used to prove the remaining theorems. As above, we write

\[ Q_\sigma = \{ \lambda \in \mathbb{C} : |\Re \lambda| < \sigma, \ |\Im \lambda| < \sigma \} \]

**Lemma 5.** — Let \( V \) be an irreducible subvariety of \( D \), \( p_0 \in aV \). Let \( X \) be a continuum in \( bD \) with \( \mathcal{H}^1(X) < \infty \). Suppose that \( X \) contains a neighborhood of \( p_0 \) in \( bV \) and that the tangent to \( X \) at \( p_0 \) exists and is not complex tangential to \( bD \). Let \( A \subseteq \mathbb{R} \) be defined, as in Proposition 3, as \( \{ x \in \mathbb{R} : \#(\pi^{-1}(x) \cap X) = 1 \} \). Let \( E \) be a compact subset of \( aV \cap X \). Suppose that 0 is a point of density of \( A \cap \pi(E) \). Then for all \( \sigma > 0 \) sufficiently small,

1. \( g(bV \setminus X) \subseteq \{ \lambda : \Im \lambda > 2\sigma \} \).
2. \( g(X) \cap Q_\sigma \subseteq \{ \lambda : \Im \lambda < \frac{1}{2} \sigma \} \).
3. \( L := g(X) \cup bQ_\sigma \) is connected and if \( \Omega \) is the component of \( \mathbb{C} \setminus L \) containing \( (-\sigma, \sigma) \times (\frac{1}{2} \sigma, \sigma) \), and \( W := g^{-1}(\Omega) \cap V \), then \( g : W \to \Omega \) is a homeomorphism, \( \Omega \) is simply connected and \( \mathcal{H}^1(b\Omega) < \infty \).
4. Let \( B = A \cap \pi(E) \setminus (-\sigma, \sigma) \). For \( x \in B \), \( g(X) \cap \ell_x \cap Q_\sigma \) is a single point \( \lambda_x = x + iv_x \in b\Omega \) and \( g^{-1}(\lambda_x) \cap V \) is a single point \( p_x \in E \subseteq aV \). Let \( \gamma_x \) be the segment \( \{ x + iy : v_x < y < \sigma \} \). Then \( g^{-1}(\gamma_x) \cap W \) is a Jordan curve in \( W \) which approaches \( p_x \) asymptotically. The set

\[ \omega = \{ \lambda_x : x \in B \} = (B \times (-\sigma, \sigma)) \cap g(X) \]

is a Borel subset of \( b\Omega \) and \( \mathcal{H}^1(\omega) > 0 \).

**Proof.** — Since \( g^{-1}(0) \cap bD = \{ p_0 \} \), and since \( X \) is a neighborhood of \( p_0 \) in \( bV \), (a) holds for small \( \sigma \). Lemma 4 gives (b) for small \( \sigma \). For (c) we note first that \( L \) is connected if \( g(X) \) and \( bQ_\sigma \) are not disjoint, which is true for small \( \sigma \). Then by (a) and (b) \( L \) is disjoint from \( (-\sigma, \sigma) \times (\frac{1}{2} \sigma, \sigma) \) for small \( \sigma \). Clearly then \( \Omega \) is a simply connected domain and \( b\Omega \subseteq L \) and so \( \mathcal{H}^1(b\Omega) \leq \mathcal{H}^1(L) < \infty \). Then \( g : W \to \Omega \) is a proper map and so is a branched cover of multiplicity \( m \geq 0 \). We must show that \( m = 1 \).

Suppose that \( m = 0 \). Fix \( x_1, x_2 \in B \) with \( -\sigma < x_1 < 0 < x_2 < \sigma \). Let \( W' = g^{-1}(Q_\sigma^0) \cap V \). Since \( p_0 \in bV \), there exists \( q \in W' \) such that \( g(q) \in Q_\sigma^0 \) and \( x_1 < \Re g(q) < x_2 \). Let \( W'' \) be the component of \( W' \) which contains \( q \). Since \( m = 0 \), \( g(W'') \) does not meet \( \Omega \) and therefore \( g(W'') \) does not meet \( \ell_x \cap Q_\sigma \) for \( x \in B \). Set \( R = (x_1, x_2) \times (-\sigma, \sigma) \). Then \( g(W'') \) does not meet \( bR \) but does meet \( R \). Hence, by connectness, \( g(W'') \subseteq R \). Since \( bW'' \subseteq X \cup g^{-1}(bQ_\sigma) \), it follows that \( bW'' \subseteq X \). Hence \( W'' \) is a subvariety of \( D \) and \( W'' \subseteq V \). This implies \( W'' = V \), a contradiction if \( \sigma \) is small.
Now suppose \( m > 1 \). By removing a countable set from \( B \) we can assume that \( \gamma_x \) contains only regular values of \( g|_V \) for each \( x \in B \) and so \( g^{-1}(\gamma_x) \cap W \) is a disjoint union of \( m \) Jordan arcs \( \gamma_1^x, \gamma_2^x, \ldots, \gamma_m^x \). For each \( x \in B \), \( g(V) \) is disjoint from the set \( \ell_x \cap \{ \lambda : \text{Im} \lambda < v_x \} \); this is because \( g(X) \) and \( g(bV \setminus X) \) and therefore \( g(bV) \) are disjoint from the set. As \( g|_V \) is an open map, \( \lambda_x \notin g(V) \). Hence \( g^{-1}(\lambda_x) \cap V \) is the unique point \( p_x \) of \( g^{-1}(\lambda_x) \cap aV \). We conclude that each \( \gamma_j^x \) approaches \( p_x \) asymptotically. Suppose that \( \lambda_0 \in \Omega \) is such that \( g^{-1}(\lambda_0) \cap W \) contains \( m > 1 \) distinct points \( w_1^0, w_2^0, \ldots, w_m^0 \). Choose a polynomial \( f \in \mathbb{C}^n \) which separates these \( m \) points. Define a function \( F \) on \( \Omega \) by

\[
F(\lambda) = \prod_{W \cap g^{-1}(\lambda) = \{p_1, p_2, \ldots, p_m\}} (f(p_i) - f(p_j))^2.
\]

Then \( F \) is a well-defined bounded holomorphic function on \( \Omega \) and \( F(\lambda_0) \neq 0 \). For \( x \in B \), \( f(\zeta) \to f(p_x) \) as \( \zeta \in \gamma_j^x \to p_x \), for \( 1 \leq j \leq m \).

It follows that \( F(\lambda) \to 0 \) as \( \lambda \in \gamma_x \to \lambda_x \in \omega \). \( \mathcal{H}^1(\omega) \geq \mathcal{H}^1(B) > 0 \).

By Lemma 3, \( F \equiv 0 \). Contradiction! We conclude that \( m = 1 \) and therefore \( \gamma_x^1 \) is a homeomorphism. Our arguments also give \( (d) \).

3.2. — We now prove Theorem 1. By hypothesis, \( \mathcal{H}^1(aV_1 \cap aV_2) > 0 \).

For \( j = 1 \) and 2, \( aV_j \) is a disjoint union of a countable set of rectifiable Jordan arcs and an \( \mathcal{H}^1 \)-null set. It follows that there exist rectifiable Jordan arcs \( \Gamma_j \subseteq aV_j \) such that \( \mathcal{H}^1(\Gamma_1 \cap \Gamma_2) > 0 \). Let \( \gamma_j : (a_j, b_j) \to \mathbb{C}^n \) parameterize \( \Gamma_j \) by arc length and set \( T_j = \gamma_j^{-1}(\Gamma_1 \cap \Gamma_2) \).

Lemma 6. — Every point \( p \in \Gamma_1 \cap \Gamma_2 \), except for a set of \( \mathcal{H}^1 \) measure zero, has the following properties:

(a) \( \gamma_j^{-1}(p) \) is a Lebesgue point of \( \gamma_j \) and is a point of density of \( T_j \), for \( j = 1 \) and 2.

(b) \( aV_1 \cup aV_2 \) has a tangent at \( p \).

(c) \( aV_1 \cup aV_2 \) is regular at \( p \).

(d) the tangent to \( aV_1 \cup aV_2 \) at \( p \) is not complex tangential to \( bD \).

Proof. — It suffices to show that each of these conditions hold \( \mathcal{H}^1 \) a.e. on \( \Gamma_1 \cap \Gamma_2 \). Part (a) follows from the fact that almost every point of \( (a_j, b_j) \) is a Lebesgue point of \( \gamma_j \) and almost every point of \( T_j \) is a point of density of \( T_j \).

At each point of \( aV_1 \cap aV_2 \), the set \( aV_1 \cup aV_2 \) is locally connected and has a neighborhood of finite linear measure. Hence Proposition 1 implies that (b) holds a.e. on \( \Gamma_1 \cap \Gamma_2 \). Likewise for (c) because continua of finite linear measure are regular \( \mathcal{H}^1 \) a.e. Finally, Theorem 2 gives (d).
Now fix $p_0 \in \Gamma_1 \cap \Gamma_2$ such that (a)-(d) of Lemma 6 hold at $p_0$. Let $X$ be a compact connected neighborhood of $p_0$ in $aV_1 \cup aV_2$ with $\mathcal{H}^1(X) < \infty$. Set $t_j = \gamma_j^{-1}(p_0)$, $j = 1, 2$. Then $aV_1, aV_2, \Gamma_1, \Gamma_2$ and $X$ all have the same tangent at $p_0$ and this tangent is not complex tangential to $bD$ at $p_0$; the tangent is $\gamma_1'(t_1) = \gamma_2'(t_2)$, with a possible change of orientation of $\gamma_2$. We set $e = iN(p_0)$ as usual. We can now apply Proposition 3 to $X$ and $\Gamma_1$ and to the set $A$ defined there to conclude that 0 is a point of density of $A$. Similarly, we apply Proposition 2 to $\Gamma_1$ and $T_1$ to obtain that 0 is a point of density of $\pi(\Gamma_1 \cap \Gamma_2)$; we note that in Proposition 2, for $u := \pi \circ \gamma_1$, $u(T_1) = \pi(\Gamma_1 \cap \Gamma_2)$. Hence 0 is a point of density of $A \cap \pi(E)$ where we have set $E := \Gamma_1 \cap \Gamma_2$.

We can now apply Lemma 5 twice. First to $V_1$ and $X$ and then to $V_2$ and $X$. We conclude that if $\sigma > 0$ is sufficiently small, then $g : W_j \to \Omega$ is a homeomorphism for $j = 1$ and 2, with $W_j = g^{-1}(\Omega) \cap V_j$. We claim that $V_1 = V_2$. This implies that $V_1 = V_2$ and completes the proof of Theorem 1. Suppose not! Then there exists $\lambda_0 \in \Omega$ such that $w_1^0 \neq w_2^0$ where $w_j^0 = g^{-1}(\lambda_0) \cap W_j$. Choose a polynomial $f$ such that $f(w_1^0) \neq f(w_2^0)$. Define a function $F$ on $\Omega$ by

$$F(\lambda) = f \circ (g_{|W_1})^{-1} - f \circ (g_{|W_2})^{-1}.$$ 

$F$ is a bounded holomorphic function on $\Omega$ and $F(\lambda_0) \neq 0$. Let $x \in B$ and set $\sigma_x^j = (g_{|W_j})^{-1}(\gamma_x)$. Then $\sigma_x^j \to p_x$ and so $f(\zeta) \to f(p_x)$ as $\zeta \in \sigma_x^j \to p_x$. Hence $F(\lambda) \to 0$ as $\lambda \in \gamma_x \to \lambda_x$. Hence $F \equiv 0$ by Lemma 3. This is the desired contradiction.

4.1. — We shall briefly recall the definition of harmonic measure in our setting. Let $V$ be an irreducible subvariety of complex dimension one of a strictly pseudoconvex domain $D$ in $\mathbb{C}^n$. Let $\tau : \tilde{V} \to V$ be the usual normalization. We shall say that a continuous real valued function $\phi$ on $V$ is subharmonic if $\tau^*(\phi) = \phi \circ \tau$ is subharmonic on the Riemann surface $\tilde{V}$. If $\phi$ is subharmonic on $V$ and continuous on $\overline{V} = V \cup bV$, then the usual maximum principle holds. For a real-valued continuous function $u$ on $bV$ we apply the usual Perron process to get a continuous function $\tilde{u}$ on $V$ such that $\tau^*(\tilde{u})$ is harmonic on $\tilde{V}$. Barrier functions exist at each point of $bV$; in fact the real parts of peaking functions in $A(D)$ can be employed. Consequently $\tilde{u}$ attains the boundary values $u$ and so extends to be continuous on $\overline{V}$. For $p \in V$, the functional $u \mapsto \tilde{u}(p)$ is positive and linear and therefore there is a unique positive measure $\mu_p$ on $bV$ such that $\tilde{u}(p) = \int_{\partial V} u d\mu_p; \mu_p$ is harmonic measure for $p$ on $bV$ (relative to $V$). It follows from Harnack's equality that for $p_1$ and $p_2$ in $V$ there exists $C > 0$ such that $\mu_{p_2} \leq C \mu_{p_1}$.
**Lemma 7.** — Let $F$ be a compact subset of $bV$ such that $\mu_p(F) = 0$ for some $p \in V$. Then there exists a real continuous function $h$ on $V$ such that $\tau^*(h)$ is harmonic on $\tilde{V}$ and such that, for each $\zeta \in F$,

$$\lim_{z \to \zeta, \ z \in V} h(z) = \infty.$$ 

Moreover, $h \geq 1$ on $V$.

**Proof.** — First, since $\mu_p(F) = 0$, there exists a pointwise increasing sequence of continuous real-valued functions $\{u_n\}$ on $bV$ such that $u_n \geq n$ on $F$, $1 \leq u_n$ on $bV$, and $\int u_n \, d\mu_p < 2$ for all $n$. Let $h$ be the limit of the increasing sequence $\tilde{u}_n$. Since $\tilde{u}_n(p) < 2$, $h(p) \leq 2$ and therefore, by Harnack, $h$ is continuous and finite on $V$ and $\tau^*(h)$ is harmonic on $\tilde{V}$. The limit statement follows from the fact that, as $z \in V \to \zeta \in F$,

$$\liminf h(z) \geq \lim h_n(z) \geq n \text{ for all } n.$$

Also $1 \leq u_n$ on $bV$ implies $1 \leq h$ on $V$.

**4.2 Proof of theorem 3a.** — We first show $\mathcal{H}^1|_{bV} \ll \mu_p$. Suppose not! Then there exists a compact subset $F$ of $bV$ such that $\mu_p(F) = 0$ and $\mathcal{H}^1(F) > 0$. Since $\mathcal{H}^1(bV \setminus aV) = 0$, there exists a rectifiable Jordan arc $\Gamma \subseteq aV$ such that $\mathcal{H}^1(\Gamma) > 0$, where $E = \Gamma \cap F$. Let $\gamma : (a, b) \to \mathbb{C}^n$ parametrize $\Gamma$ by arclength and set $T = \gamma^{-1}(E)$. Choose $t_0 \in T$ such that all of the following conditions hold: $t_0$ is a point of density of $T$, $t_0$ is a Lebesgue point of $\gamma'$, $p_0 = \gamma(t_0)$ is a regular point of $aV$, $\gamma'(t_0)$ is the tangent to $aV$ at $p_0$ and is not complex tangential to $bD$. In fact, by *Lemma 1, Proposition 1 and Theorem 2*, almost all points of $T$ will do. Set $\varepsilon = iN(p_0)$, $g(p) = \langle p - p_0, iN(p_0) \rangle$ and $\tau = \Re \circ g$, as usual. By Propositions 2 and 3, 0 is a point of density of $\pi(E) \cap A$, where $A$ is defined in Proposition 3. Arguing as in § 3.2, we can choose a continuum $X$ which is a neighborhood of $p_0$ in $aV$ such that $\mathcal{H}^1(X) < \infty$. The hypotheses of Lemma 5 are valid and we get a $\sigma > 0$ such that $(a)$–$(d)$ of Lemma 5 hold; in particular we have $Q_\sigma$, the homeomorphism $g : W \to \Omega$ and $\omega \subseteq b\Omega$.

Since $\mu_p(F) = 0$, Lemma 7 gives a function $h$ with $\tau^*(h)$ harmonic on $\tilde{V}$. Set $u = h \circ (g|_W)^{-1}$ on $\Omega$. Then $u$ is continuous and harmonic on $\Omega$, since possible isolated singular points for $u$ are removable. Let $v$ be a harmonic conjugate for $u$ on the simply connected domain $\Omega$. Consider $\lambda_x \in \omega$. By Lemma 7, $h(z) \to \infty$ as $z \in g^{-1}(\gamma_x) \to p_x \in F$, in the notation of Lemma 5. Hence $u(\lambda) \to \infty$ as $\lambda \in \gamma_x \to \lambda_x \in \omega$. Set $f(\lambda) = e^{-(c(\lambda) + iv(\lambda))}$ for $\lambda \in \Omega$. Then $f$ is a bounded holomorphic function on $\Omega$ and $f \neq 0$ in $\Omega$. By Lemma 3, $f \equiv 0$ in $\Omega$. Contradiction! We conclude that $\mathcal{H}^1|_{bV} \ll \mu_p$.
Next we show that $\mu_p \ll \mathcal{H}^1_{|bV}$. Let $E \subseteq bV$ be a Borel set with $\mathcal{H}^1(E) = 0$. Let $K$ be any compact subset of $E$. Then $\mathcal{H}^1(K) = 0$ and by the corollary to Theorem 2 of Davie and Øksendal [5], $K$ is a peak interpolation set for $A(D)$. Hence there exists a function $f \in A(D)$ such that $f = 1$ on $K$ and $|f| < 1$ on $\overline{D} \setminus K$. We have

$$0 = \lim f^n(p) = \lim \int f^n \, d\mu_p = \int_K 1 \, d\mu_p = \mu_p(K)$$

By the regularity of $\mu_p$, we get $\mu_p(E) = 0$; i.e., $\mu_p \ll \mathcal{H}^1_{|bV}$.

Remark. — The last paragraph is equally valid for any representing measure for evaluation at $p$ in place of $\mu_p$. This gives part (c) of Theorem 3. Combining this with part (a) we see that every representing measure for $p$ is absolutely continuous with respect to $\mu_p$. In the terminology of the abstract F. and M. Riesz theorem [11], $\mu_p$ is a “dominant” representing measure for evaluation at $p$.

4.3 Proof of Theorem 3b. — Fix $p \in V$. Then $\mu_p$ is a representing measure for evaluation at $p$ for the algebra $A(V)$. Let $\nu = \nu_a + \nu_s$ be the Lebesgue decomposition of $\nu$ with respect to $\mu_p$; $\nu_a \ll \mu_p$ and $\nu_s \perp \mu_p$. By the abstract F. and M. Riesz theorem [11, p. 44], $\nu_s$ is orthogonal to $A(V)$. By part (a), $\nu_a \ll \mathcal{H}^1_{|bV}$ and $\nu_s \perp \mathcal{H}^1_{|bV}$. Thus it suffices to show that $\nu_s = 0$. There exists a Borel set $E \subseteq bV$ such that $\nu_s$ is concentrated on $E$ and $\mathcal{H}^1(E) = 0$. Let $K$ be any compact subset of $E$. Then $\mathcal{H}^1(K) = 0$ and, as noted above, Davie and Øksendal proved that $K$ is a peak interpolation set for $A(D)$. This implies that $|\nu_s|(K) = 0$. Indeed, let $g \in A(D)$ be such that $g \equiv 1$ on $K$ and $|g| < 1$ on $\overline{D} \setminus K$ and let $u$ be any continuous function on $K$; extend $u$ to a function on $\overline{D}$ in $A(D)$ and note that $\int_K ud\nu_s = \lim_{n \to \infty} \int u^n \, dv_s = 0$, since $u^n \in A(D)$ and $\nu_s$ is orthogonal to $A(D)$. By the regularity of $\nu_s$ we conclude that $\nu_s = 0$.

5.1. — We next consider accessibility of points of $bV$.

Proposition 4. — Let $V$ be an irreducible one-dimensional subvariety of a strictly pseudoconvex domain $D$ in $\mathbb{C}^n$. Let $p_0 \in bV$ be such that there exists a neighborhood $N$ of $p_0$ in $bV$ with $\mathcal{H}^1(N) < \infty$. Then $p_0$ is accessible from $V$.

Proof. — As above, we may assume that $D$ is strictly convex. Also as above we have the projection $g : \mathbb{C}^n \to \mathbb{C}$, $g(p) = \langle p - p_0, iN(p_0) \rangle$ with $\text{Im } g(p) > 0$ for $p \in \overline{D} \setminus \{p_0\}$. Fix $g_0 \in V$ and choose $c > 0$ such that $g(bV \setminus N) \subseteq \{ \lambda \in \mathbb{C} : \text{Im } \lambda > c \}$ and $c < \text{Im } g(q_0)$.

Let $k_t$ denote the horizontal line $\{ \lambda \in \mathbb{C} : \text{Im } \lambda = t \}$ for $t \in \mathbb{R}$. Set $n(t) = \# \{ p \in bV : g(p) \in k_t \}$. Then, since $\mathcal{H}^1(N) < \infty$, $\int_0^c n(t) \, dt < \infty$.
and, in particular, \( n(t) \) is finite a.e. for \( 0 < t < c \). Choose \( \delta_n \downarrow 0, \delta_n < c \) such that \( n(\delta_n) \) is finite and such that \( k_{\delta_n} \) contains no singular values of \( g|_{V} \) for all \( n \). Set \( V_n = V \cap g^{-1}\{\lambda \in \mathbb{C} : \text{Im} \lambda < \delta_n\} \). Then \( V_n \) is non-empty for each \( n \), since \( p_0 \in bV \).

We claim that \( V_n \), a subvariety of \( \{ p \in \mathbb{C}^n : \text{Im} g(p) < \delta_n \} \cap D \), has a finite number of components. Indeed the finite set \( F_n = g(bV) \cap k_{\delta_n} \subseteq k_{\delta_n} \) divides \( k_{\delta_n} \) into a finite set of line segments \( \{ \gamma \} \) such that \( g^{-1}(\gamma) \cap V \) is again a finite union of disjoint arcs \( \{ \sigma \} \), each mapped homeomorphically to \( \gamma \) by \( g \). This is because \( g : V \cap g^{-1}(\Omega) \to \Omega \) is a branched cover for each component \( \Omega \) of \( \mathbb{C} \setminus g(bV) \) and there is no branching over \( k_{\delta_n} \). Each \( \sigma \) is contained in the closure of one and only one component of \( V_n \). Also each component of \( V_n \) is such that any of its points can be joined to \( q_0 \) by a path \( \alpha \) in \( V \). By connectedness, the path \( \alpha \), which can be chosen to avoid the finite set \( V \cap g^{-1}(F_n) \), must meet one of the \( \sigma \)'s. Consequently the closure of the component meets and therefore must contain one of the \( \sigma \)'s. Thus the number of components of \( V_n \) is at most the number of \( \sigma \)'s.

Next we shall choose inductively a sequence \( \{ W_n \} \) such that \( W_n \) is a component of \( V_n \), \( W_{n+1} \subseteq W_n \) and \( p_0 \in \overline{W}_n \), as follows. Since \( p_0 \in \overline{V}_1 \) and \( V_1 \) contains a finite number of components, choose \( W_1 \) as any component of \( V_1 \) with \( \overline{W}_1 \) containing \( p_0 \). Given \( W_1, W_2, \ldots, W_n \) as above, note that \( V_{n+1} \cap W_n \) is non-empty since \( p_0 \in \overline{W}_n \). Clearly the set of components of \( V_{n+1} \cap W_n \) is a subset of the finite set of components of \( V_{n+1} \) and consequently, one of these components contains \( p_0 \) in its closure. Choose this component to be \( W_{n+1} \).

Now choose any sequence \( \{ q_n \} \) such that \( q_n \in W_n \). Then clearly \( q_n \to p_0 \). We can join \( q_1 \) to \( q_2 \) by a path in \( W_1 \) and then join \( q_2 \) to \( q_3 \) by a path in \( W_2 \), etc. The sum of these paths gives a path in \( V \) which approaches \( p_0 \) asymptotically.

5.2. Example. — Set \( \rho(\theta) = \theta/(1 + 2\theta) \) for \( 0 \leq \theta \). Let \( S \) be the spiral \( \{ \rho(\theta)e^{i\theta} : 0 \leq \theta < \infty \} \) and let \( C \) be the circle \( \{ \lambda \in \mathbb{C} : |\lambda| = \frac{1}{2} \} \). Then \( \overline{S} = S \cup C \). Let \( \Omega \) be the bounded component of \( \mathbb{C} \setminus \overline{S} \). Then \( \overline{C} \) is (the underlying set of) a prime end of the simply connected domain \( \Omega \). There is a Riemann map \( f : \overline{U} \to \Omega \), where \( U \) is the open unit disk, such that \( f \) extends to be a continuous map \( \overline{U} \setminus \{ 1 \} \to \overline{\Omega} \setminus C \) and \( |f| \) extends continuously to \( \overline{U} \) such that \( |f|(1) = \frac{1}{2} \) and \( |f| \leq \frac{1}{2} \) on \( \overline{U} \). Hence there exists a continuous function \( g \) on \( \overline{U} \), holomorphic on \( U \), such that \( |g| = \sqrt{1 - |f|^2} \) on \( bU \). In particular, \( (\frac{1}{2})^2 + |\beta|^2 = 1 \), where \( \beta = g(1) \). Define \( \Phi : \overline{U} \to \mathbb{B}_2 \) = the open unit ball in \( \mathbb{C}^2 \) by \( \Phi(\lambda) = (f(\lambda), g(\lambda)) \). Then \( \Phi \) is a proper map and so its image is a complex submanifold \( V \).
of $\mathbb{B}_2$ with $bV = \Phi(bU \setminus \{1\}) \cup C \times \{\beta\}$. The points of $C \times \{\beta\} \subseteq bV$ are not accessible from $V$. Indeed, if $\gamma$ were a curve in $V$ approaching some point of $C \times \{\beta\}$, then $z_1 \circ \gamma$ would be a curve in $\Omega$ approaching a point of $C$. But no such curve in $\Omega$ exists.

The construction also shows that Theorem 1 cannot have its hypotheses greatly weakened. Namely, if $V_1$ and $V_2$ have connected ends of finite linear measure, then Theorem 1 can be applied to see that $V_1 = V_2$ provided that $\mathcal{H}^1(bV_1 \cap bV_2) > 0$.

This would not be true if we only knew that the ends had $\sigma$-finite $\mathcal{H}^1$ measure. Namely, take $V_1 = V$ as above and take $V_2 = \{(\lambda, \beta) \in \mathbb{C}^2 : |\lambda|^2 + |\beta|^2 < 1\}$, a subvariety of the unit ball $\mathbb{B}_2$. Then $bV_2 = C \times \{\beta\}$ and $bV_1 \cap bV_2 = C \times \{\beta\}$ has positive measure, $\mathcal{H}^1(bV_2) < \infty$, $bV_1$ is $\mathcal{H}^1$ $\sigma$-finite and is the union of two real analytic curves, but $V_1 \neq V_2$. One can also show that $\{ \text{the polynomial hull of } bV_1 \} \setminus bV_1 = V_1 \cup V_2$.

5.3. — For the proof of Theorem 4 we shall assume that $D$ is strictly convex. Choose a point $p_0$ such that there exists an open rectifiable Jordan arc $\Gamma$ in $aV$ continuing $p_0$, such that $aV$ is regular at $p_0$, such that the tangent to $aV$ exists and is not complex tangential to $bD$. By our previous arguments, $\mathcal{H}^1$ almost all points $p_0 \in aV$ will suffice. Then, taking $E$ as a compact neighborhood of $p_0$ in $aV$, the argument of Theorem 3 shows that the conclusion of Lemma 5 holds for $\sigma$ sufficiently small, where $g(p) \equiv \langle p - p_0, e \rangle$ and $e = iN(p_0)$.

Thus we have the homeomorphism $g : W = V \cap g^{-1}(\Omega) \to \Omega$. We identify $e^\perp$ in $\mathbb{C}^n$ with $\mathbb{C}^{n-1}$ and define a $\mathbb{C}^{n-1}$-valued holomorphic $\psi(\lambda)$ for $\lambda \in \Omega$ as follows. For $\lambda \in \Omega$, write $g^{-1}(\lambda) = w = p_0 + \lambda e + w' \in W$, with $\langle w', e \rangle = 0$, the orthogonal decomposition. Now define $\psi(\lambda)$ for $\lambda \in \Omega$ by $\psi(\lambda) = w'/\lambda \in e^\perp = \mathbb{C}^{n-1}$; more explicitly,

$$\psi(\lambda) = \frac{1}{\lambda} [g^{-1}(\lambda) - p_0 - \lambda e].$$

Lemma 8. — $\psi$ is a bounded $\mathbb{C}^{n-1}$-valued holomorphic function on $\Omega$.

Assuming the lemma, say $\|\psi(\lambda)\| \leq M$ for $\lambda \in \Omega$, we shall complete the proof of Theorem 4. We know that $it \in \Omega$ for $0 < t < \sigma$. Define the curve $\gamma$ in $V$ by $\gamma(t) = g^{-1}(it) \in V$ for $0 < t < \sigma$. Then $\gamma \to p_0$ as $t \downarrow 0$. To see that $\gamma$ approaches $p_0$ non-tangentially, write

$$\gamma(t) = g^{-1}(it) = p_0 + ite + w' = p_0 + t(-N(p_0)) + it\psi(it),$$
using $\psi(\lambda) = w'/\lambda$. Since $-N(p_0)$ is the inward unit normal to $bD$ at $p_0$, we must show, for non-tangential approach, that $||w'||/t$ is bounded as $t \downarrow 0$. But this quotient equals $||\psi(it)||$, which is bounded by $M$. The theorem follows.

5.4 Proof of lemma 8. — We first show that $\psi$ is in the Hardy space $H^2(\Omega, \mathbb{C}^{N-1})$. For this, we need to show that $||\psi(\lambda)||^2$ has a harmonic majorant on $\Omega$. Since $D$ is strictly convex, there is a large ball containing $D$ whose boundary contains $p_0$ and such that this boundary is tangent to $bD$ at $p_0$. As $-N(p_0)$ is the inward normal to $D$ at $p_0$ this ball is of the form $B(p_0 - RN(p_0), R)$ for some $R > 0$. Hence

$$W = g^{-1}(\Omega) \cap V \subseteq D \subseteq B(p_0 - RN(p_0), R).$$

Writing $w = p_0 + \lambda e + w', \langle w', e \rangle = 0$, we get

$$R^2 > ||w - (p_0 - RN(p_0))||^2 = ||(\lambda - iR)e + w'||^2 = ||\lambda - iR|^2 + ||w'||^2,$$

since $N(p_0) = -ie$. Thus $||w'||^2 < R^2(1 - |\lambda/R - i|^2)$. Set

$$h(\zeta) = \frac{1 - |\zeta - i|^2}{|\zeta|^2}$$

for $\zeta \neq 0$ in $\mathbb{C}$. We have $||\psi(\lambda)||^2 = ||w'||^2/|\lambda|^2 \leq h(\lambda/R)$. A computation shows that $h(\zeta)$ is harmonic on the upper half plane and therefore $\lambda \mapsto h(\lambda/R)$ is harmonic on $\Omega$. Thus $\psi \in H^2$.

5.5. — Next we show that $\psi$ is bounded on $b\Omega \setminus \{0\}$.

Lemma 9. — There exists an $M > 0$ such that for all $\zeta \in b\Omega \setminus \{0\}$

$$\limsup_{\lambda \to \zeta, \lambda \in \Omega} ||\psi(\lambda)|| \leq M.$$

Proof. — We have $||\psi(\lambda)||^2 \leq u(\lambda) \equiv h(\lambda/R)$ for all $\lambda \in \Omega$, with $h(\zeta) = (1 - |\zeta - i|^2)/|\zeta|^2$. It suffices thus to show that $u$, which satisfies $u \geq 0$ on $\Omega$, is bounded above on $b\Omega \setminus \{0\}$. Suppose not! Then there exist $\lambda_n \in b\Omega$, $\lambda_n \neq 0$ such that $u(\lambda_n) \to \infty$ and $\lambda_n \to 0$. Since $g(bV)$ contains a neighborhood of 0 in $b\Omega$, there exist $w_n \in bV$ such that $g(w_n) = \lambda_n$ and then $w_n \to p_0$, $w_n \neq p_0$. 

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Since $bD$ is smooth, there exists an internally tangent sphere at $p_0 \in bD$; i.e., there exists a $\delta > 0$ such that $B(p_0 - \delta N(p_0), \delta) \subseteq D$. Then $bD$ is exterior to this ball and therefore $\|w_n - (p_0 - \delta N(p_0))\| \geq \delta^2$.

Writing $w_n = p_0 + \lambda_n e + w'_n$, $\langle e, w'_n \rangle = 0$, we get $\delta^2 \leq |\lambda_n - i\delta|^2 + \|w'_n\|^2$.

Setting $\lambda_n = s_n + it_n$ we get

$$t_n \leq \frac{1}{2\delta} (|\lambda_n|^2 + \|w'_n\|^2).$$

Since the unit tangent $\tau$ to $bV$ at $p_0$ is not complex tangential to $bD$, we have $\tau = ae + w'_0$ where $0 < a \leq 1$. Passing to a subsequence if necessary, we can assume that $(w_n - p_0)/\|w_n - p_0\| \to ae + w'_0$. Taking the inner product with $e$ yields

$$\frac{\lambda_n}{\sqrt{|\lambda_n|^2 + \|w'_n\|^2}} \to a.$$ 

Taking real parts gives $s_n/\sqrt{|\lambda_n|^2 + \|w'_n\|^2} \to a$. By (8) we get

$$t_n \leq \frac{1}{2\delta} \left[ \frac{|\lambda_n|^2 + \|w'_n\|^2}{s_n^2} \right] s_n^2.$$ 

Since the quotient in brackets converges to $1/a^2$, we get a $C > 0$ such that $t_n \leq Cs_n^2$ for all $n$. Now we have

$$u(\lambda_n) = \frac{R^2 - |\lambda_n - iR|^2}{|\lambda_n|^2} = \frac{2t_n R}{s_n^2 + t_n^2} - 1 \leq 2CR - 1.$$ 

This contradicts the fact that $u(\lambda_n) \to \infty$.

5.6. — To complete the proof of Lemma 8 we consider a Riemann map $\varphi : U \to \Omega$ and the pull-back $\Psi = \varphi^*(\psi) = \psi \circ \varphi$ of $\psi$. The harmonic majorant of $\|\psi\|^2$ pulls back to one for $\|\Psi\|^2$ and so $\Psi \in H^2(U, \mathbb{C}^{n-1})$, the usual Hardy space. By Lemma 9, the boundary function $\Psi^*$ of $\Psi$ satisfies $\|\Psi^*\|^2 \leq M$ a.e. on $bU$ and therefore $\Psi$ is a bounded holomorphic function on $U$. Hence $\psi$ is bounded on $\Omega$.

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