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RIESZ MEANS ON LIE GROUPS AND RIEMANNIAN
MANIFOLDS OF NONNEGATIVE CURVATURE

BY

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0. Introduction and statement of the results

The Riesz means have already been extensively studied in the case of $\mathbb{R}^n$ (cf. [7], [8], [27], [29] as well as the book [13]) and in the case of elliptic differential operators on compact manifolds (cf. [2], [9], [16], [18], [25], [26]). Some of these results have been generalised to the case of dilation invariant sub-Laplacians on stratified nilpotent Lie groups (cf. [19], [21], [22]), to the case of compact semisimple Lie groups (cf. [10]) and more recently to the case of noncompact symmetric spaces (cf. [16]).

The goal of this article is to study the Riesz means associated to left invariant sub-Laplacians on connected Lie groups of polynomial volume...
growth (connected nilpotent Lie groups are examples of such groups) and to the Laplace Beltrami operator on Riemannian manifolds of nonnegative curvature:

**a) Lie groups of polynomial growth.**

We consider a connected Lie group $G$ and we fix a left invariant Haar measure $dg$ on $G$. If $A$ is a Borel measurable subset of $G$, then we denote by $|A|$ its $dg$-measure.

We assume that $G$ has polynomial volume growth, that is, for every compact neighborhood $U$ of its identity element $e$ of $G$, there is a constant $c > 0$ such that

$$|U^n| \leq cn^c,$$

for $n \in \mathbb{N}$.

It is easy to see that this assumption makes $G$ unimodular. Furthermore, it can be proved (cf. [17]) that there is an integer $D \geq 0$, such that:

$$|U^n| \sim n^D, \quad (n \to \infty).$$

By $f(t) \sim h(t)$, as $t \to t_0$ we mean that there is a constant $c > 0$ such that:

$$c^{-1} \cdot h(t) \leq f(t) \leq c \cdot h(t) \quad \text{as} \quad t \to t_0.$$

Notice that every connected nilpotent Lie group has polynomial volume growth.

We consider left invariant vector fields $X_1, ..., X_n$ on $G$ that satisfy Hörmander’s condition, i.e. they generate together with their successive Lie brackets $[X_{i_1}, [X_{i_2}, [... , X_{i_k}]]$, at every point of $G$, the tangent space of $G$. To those vector fields is associated, in a canonical way, the control distance $d(\cdot, \cdot)$. This distance is left invariant and compatible with the topology of $G$. We put:

$$|x| = d(e, x) \quad \text{and} \quad B_r(x) = \{ y \in G : d(x,y) < r \}, \quad x \in G, \ r > 0.$$

Then, we know that there is $d \in \mathbb{N}$, not depending on $x$ (cf. [24], [30] and [33]), such that:

$$(1) \quad |B_r(x)| \sim r^d \quad (r \to 0), \quad |B_r(x)| \sim r^D \quad (r \to \infty)$$

We call $d$ the **local dimension** and $D$ the **dimension at infinity** of $G$.

**b) Riemannian manifolds of nonnegative curvature.**

We consider a complete non-compact $n$-dimensionnal Riemannian manifold $M$ with non-negative Ricci curvature. We denote by $L$ the Laplace-Beltrami operator on $M$. Let $d(\cdot, \cdot)$ be the Riemannian distance on $M$ and denote by

$$B_r(x) = \{ y \in M : d(x,y) < r \}$$
the geodesic ball of radius $r > 0$ and centered at $x \in M$.

Let also $|B_r(x)|$ denote the volume of $B_r(x)$. Then there is a constant $c_x > 0$ (depending on $x \in M$) such that

$$|B_r(x)| \geq c_x r^n, \quad 0 < r \leq 1.$$  

Although we have, by the Bishop comparison theorem (cf. [3]), that there is a constant $c > 0$ independent of $x \in M$ and $r > 0$ such that $|B_r(x)| \leq cr^n$, it may happen that $|B_r(x)|$ grows much slower as $r \to \infty$. For example if $M$ is a complete noncompact homogeneous space with nonnegative sectional curvature then $M = \mathbb{R}^k \times \overline{M}$, where $\overline{M}$ is a compact homogeneous space and $k \geq 1$ (cf. [4]). So in that case we have that $|B_r(x)| \sim r^k$ ($r \to \infty$). In general all we can say (cf. [5]) is that there is a constant $c_x > 0$ depending on $x \in M$ such that $|B_r(x)| \geq c_x r$, where $r \geq 1$. In this article we shall only use the following inequality, which also follows from the Bishop comparison theorem (cf. [3], [5]):

$$\frac{|B_r(x)|}{|B_t(x)|} \leq \left(\frac{r}{t}\right)^n, \quad r \geq t.$$  

We shall also put $d = D = n$.

In both of the above cases the operator $L$ admits a spectral resolution (cf. [34]), which we denote by:

$$L = \int_0^\infty \lambda dE_\lambda.$$  

For $\alpha > 0$, the Riesz means of order $\alpha$ are defined to be the operators

$$m_{\alpha,R}(L) = \int_0^\infty \left(1 - \frac{\lambda}{R}\right)_+^\alpha dE_\lambda, \quad R > 0,$$

and the corresponding maximal operators by:

$$m^*_\alpha(L)f(x) = \sup_{R > 0} |m_{\alpha,R}(L)f(x)|.$$  

That $m^*_\alpha(L)f(x)$ is well defined will be shown in the proof of Theorem 3 below.

We denote by $K_{\alpha,R}(x,y)$ the Schwartz kernel of the operator $m_{\alpha,R}(L)$.

**Theorem 1.** — *There is a constant $c > 0$ such that*

(a) *if $\alpha > \frac{1}{2} D$ then $\|K_{\alpha,R}(x,\cdot)\|_1 \leq c$, $0 < R \leq 1$;*

(b) *if $\alpha > \frac{1}{2} \max(d,D)$ then $\|K_{\alpha,R}(x,\cdot)\|_1 \leq c$, $R > 1$;*

(c) *if $\alpha = \frac{1}{2} d > \frac{1}{2} D$ then $\|K_{\alpha,R}(x,\cdot)\|_1 \leq c(1 + \log R)$, $R > 1$;*

(d) *if $\frac{1}{2} d > \alpha > \frac{1}{2} D$ then $\|K_{\alpha,R}(x,\cdot)\|_1 \leq c R^{d/4 - \alpha/2}$, $R > 1$.***
Theorem 2.
a) If \( \alpha > \frac{1}{2} D \) then \( m_{\alpha,R}(L) \) is bounded on \( L^p(G) \) for \( 1 \leq p \leq \infty \).
b) If \( 0 < \alpha < \frac{1}{2} D \) then \( m_{\alpha,R}(L) \) is bounded on \( L^p(G) \) for \( \alpha > D \left| \frac{1}{p} - \frac{1}{2} \right| \).
c) If \( 0 < \alpha < \frac{1}{2} D \) then the operators \( m_{\alpha,R}(L), R > 0 \) are uniformly bounded on \( L^p(G) \) for \( \alpha > \left| \frac{1}{p} - \frac{1}{2} \right| \max(d, D) \).

Theorem 3.
a) If \( \alpha > \frac{1}{2} \max(d, D) \) then \( m_\alpha^*(L) \) is bounded on \( L^p \), for \( 1 < p < \infty \).
b) If \( 0 < \alpha < \frac{1}{2} \max(d, D) \) then \( m_\alpha^*(L) \) is bounded on \( L^p \), for \( \alpha > \left| \frac{1}{p} - \frac{1}{2} \right| \max(d, D) \).

Theorem 4. — If \( \alpha \) and \( p \) are as in theorem 3 above and \( f \in L^p \), then:

\[
\left\| m_{\alpha,R}(L)f - f \right\|_p \to 0 \quad \text{as} \quad R \to \infty,
\]

\( m_{\alpha,R}(L)f(x) \to f(x) \quad \text{a.e. as} \quad R \to \infty. \)

We point out that for the Laplace operator on \( \mathbb{R}^n, n = d = D \) and the critical power in the above results is \( \frac{1}{2} (n-1) \) rather than \( \frac{1}{2} n \) (cf. [13], [29]).

The proof of the above results relies on the following two ideas: assume to simplify things that \( f \in C_0^\infty(\mathbb{R}^+) \) and that we want to obtain estimates of the kernel of the operator \( f(L) = \int_0^\infty f(\lambda) dE_\lambda \). Then the first idea which is due to M. Taylor (see for example [5]), consists of writing \( f(L) = h(\sqrt{L}) \) (with \( h(t) = f(t^2), t \in \mathbb{R} \)). Then, using the fact that \( h(t) \) is an even function, we have that:

\[
h(\sqrt{L}) = (2\pi)^{-1/2} \int h(t) \cos t \sqrt{L} \, dt.
\]

This expression allows us to take advantage of the fact that \( \cos t \sqrt{L} \) is an operator bounded on \( L^2 \) as well as the fact that its kernel \( G_t(x, y) \) being a fundamental solution for the wave equation

\[
\left( \frac{\partial^2}{\partial t^2} + L \right) u(t, x) = 0, \quad u(0, x) = f(x), \quad \left( \frac{\partial}{\partial t} \right) u(0, x) = 0
\]

propagates with finite speed, that is

\[
\text{supp} \ (G_t) \subseteq \{(x, y) : d(x, y) \leq |t|\}
\]

a result proved, in the case of subelliptic operators by Melrose [23].
The second idea, which is due to Hulanicki and Stein (cf. [14, p. 208–215]), and which has also been exploited by Christ [6] is to exploit the existence of very good estimates for the heat kernel \( p_t(x, y) \), i.e. the fundamental solution of the associated heat equation

\[
\left( \frac{\partial}{\partial t} + L \right) u(t, x) = 0, \quad u(0, x) = f(x).
\]

To do this we observe first that \( p_t(x, y) \) the Schwartz kernel of the operator \( e^{-tL} \), \( t > 0 \). So, if \( f \in C_0^\infty(\mathbb{R}^+) \) and we put \( h(t) = f(t) e^{t_0 t} \), with \( t_0 > 0 \) appropriately chosen we get \( f(L) = h(L) e^{-t_0 L} \). This in turn implies that the Schwartz kernel of \( f(L) \) is equal to \( h(L)p_{t_0}(x, y) \). This last remark is one of the basic ingredients of the proofs.

The estimate for \( p_t(x, y) \), we shall use in this article, is the following (cf. [12], [20], [30], [33])

\[
(4) \quad p_t(x, y) \leq \frac{c}{|B_{\sqrt{t}}(x)|} \exp\left(-\frac{d(x, y)^2}{ct}\right), \quad t > 0.
\]

1. Proof of theorems 1 and 2

We have that

\[
m_{\alpha, R}(\lambda) = \left(1 - \left| \frac{\lambda}{R} \right| \right)_+^\alpha = \left(1 - \left| \frac{\lambda}{R} \right| \right)_+^\alpha e^{\lambda/R} e^{-\lambda/R}.
\]

Hence if we put \( r = \sqrt{R} \) and

\[
h_{\alpha, r}(\lambda) = \left(1 - \left( \frac{\lambda}{r} \right)^2 \right)_+^\alpha e^{(\lambda/r)^2}
\]

then

\[
(5) \quad m_{\alpha, R}(L) = h_{\alpha, r}(\sqrt{L}) e^{-1/r^2 L}
\]

The function \( \psi(\lambda) = e^{-\lambda^2} \) is \( C^\infty \) and supported in \([0, \infty)\). Hence the function \( \psi_1(\lambda) = \psi(\lambda)\psi(1 - \lambda) \) is also \( C^\infty \) and supported in \([0, 1]\). We put:

\[
\varphi(\lambda) = \psi_1(\lambda + \frac{5}{4}), \quad \varphi_j(\lambda) = \varphi(2^j(|\lambda| - 1)).
\]

Then \( \varphi_j(\lambda) \) is a \( C^\infty \) function with support contained in \( J_j = I_j \cup -I_j \), where \( I_j = [1 - 5/2^{j+2}, 1 - 1/2^j] \). We put

\[
\chi_j(\lambda) = \frac{\varphi_j(\lambda)}{\sum_{i \geq 0} \varphi_i(\lambda)} \quad \text{and} \quad \chi_{j, r}(\lambda) = \chi_j\left(\frac{\lambda}{r}\right)^2.
\]
We also put:
\[ h_{j,r}(\lambda) = h_{\alpha,r}(\lambda) \chi_{j,r}(\lambda). \]

Notice that there is \( c > 0 \) such that
\[
|\text{supp} h_{j,r}| \leq cr^{-j}.
\]

Also, for all \( k \in \mathbb{N} \) there is \( c_k > 0 \) such that
\[
\|\chi_{j,r}^{(k)}\|_\infty \leq c_k r^{-k} 2^{kj}, \quad \|h_{j,r}^{(k)}\|_\infty \leq c_k r^{-k} 2^{(\alpha-k)j}.
\]

By a simple calculation we can deduce from the estimates (6) and (7) above that for all \( k \in \mathbb{N} \) there is \( c_k > 0 \) such that
\[
\int |\hat{h}_{j,r}(t)| dt \leq c_k s^{-k} r^{-k} 2^{(k-\alpha)j}, \quad s > 0.
\]

We consider the operator
\[ m_{j,r}(L) = h_{j,r}(\sqrt{L}) e^{-1/r^2 L} \]
and we denote by \( K_{j,r}(x, y) \) its Schwartz kernel. Since the operators \( h_{j,r}(\sqrt{L}) \) and \( e^{-1/r^2 L} \) are selfadjoint and commute, we have
\[
K_{j,r}(x, y) = h_{j,r}(\sqrt{L}) \Phi_{r-2}(x, y)
\]
with the operator \( h_{j,r}(\sqrt{L}) \) acting on the variable \( y \).

**Lemma 5.** — Let \( i \in \mathbb{Z} \) such that \( 2^{i-1} < r \leq 2^i \). Then there is a constant \( c > 0 \) such that
\[
\|K_{j,r}(x, \cdot)\|_1 \leq \begin{cases} 
  c \cdot 2^{(D/2-\alpha)j} & \text{if } i \leq 0, \ j \geq 0; \\
  c \cdot 2^{(d/2-\alpha)j} & \text{if } i > 0, \ 0 \leq j < i; \\
  c \cdot 2^{(d/2-D/2)j} 2^{(D/2-\alpha)j} & \text{if } i \leq 0, \ j \geq i. 
\end{cases}
\]

**Proof.** — It follows from (4) that
\[
\|P_t(x, \cdot)\|_2 \leq c \cdot |B_{2t}(x)|^{-1/2}.
\]

We also have
\[
\|h_{j,r}(\sqrt{L})\|_{2 \to 2} \leq \|h_{j,r}\|_\infty \leq 2^{-\alpha j}.
\]
Hence, it follows from (9) that
\[
\|K_{j,r}(x,\cdot)\|_{L^1(B_{2j-i}(x))} \\
\leq |B_{2j-i}|^{1/2} \|K_{j,r}(x,\cdot)\|_2 \\
\leq |B_{2j-i}(x)|^{1/2} \|h_{j,r}(\sqrt{L})\|_{L^2} \|p_{r-2}(x,\cdot)\|_2 \\
\leq c \frac{|B_{2j-i}(x)|^{1/2} \|h_{j,r}\|_{L^\infty} \|p_{2r-2}(x,\cdot)\|_2^{1/2}}{|B_{2j-i}(x)|^{1/2}} \\
\leq c \left( \frac{|B_{2j-i}(x)|}{|B_{2j-i}(x)|} \right)^{1/2} 2^{-\alpha j}
\]
and from this, by using either (1) or (2), we get:
\[
(12) \quad \|K_{j,r}(x,\cdot)\|_{L^1(B_{2j-i})} \leq \begin{cases} 
  c \cdot 2^{(D/2-\alpha)j} & \text{if } i < 0, \\
  c \cdot 2^{(d/2-\alpha)j} & \text{if } 0 \leq j \leq i, \\
  c \cdot 2^{(d/2-D/2)j} & \text{if } j > i \geq 0.
\end{cases}
\]
Let \( A_p(x) = \{ y : 2^p \leq d(x,y) < 2^{p+1} \} \), where \( p \geq j - i \). Then, it follows from (3) that, if \( z \in A_p(x) \), then
\[
K_{j,r}(x,z) \\
= [h_{j,r}(\sqrt{L})p_{r-2}(x,\cdot)](z) \\
= (2\pi)^{-1/2} \int_{-\infty}^{+\infty} \hat{h}_{j,r}(t) [\cos t \sqrt{L} p_{r-2}(x,\cdot)](z) \, dt \\
= (2\pi)^{-1/2} \int_{t \geq 2^{p-1}} \hat{h}_{j,r}(t) \{ \cos t \sqrt{L} [p_{r-2}(x,\cdot)1_{\{ y : d(x,y) \leq 2^{p-1} \}} + p_{r-2}(x,\cdot)1_{\{ y : d(x,y) > 2^{p-1} \}}] \}(z) \, dt \\
= (2\pi)^{-1/2} \int_{t \geq 2^{p-1}} \hat{h}_{j,r}(t) \{ \cos t \sqrt{L} [p_{r-2}(x,\cdot)1_{\{ y : d(x,y) \leq 2^{p-1} \}}] \}(z) \, dt \\
+ (2\pi)^{-1/2} \int_{-\infty}^{+\infty} \hat{h}_{j,r}(t) \{ \cos t \sqrt{L} [p_{r-2}(x,y)1_{\{ y : d(x,y) > 2^{p-1} \}}] \}(z) \, dt.
\]
Hence
\[
(13) \quad \|K_{j,r}(x,\cdot)\|_{L^1(A_p(x))} \\
\leq |A_p(x)|^{1/2} \int_{t \geq 2^{p-1}} |\hat{h}_{j,r}(t)| \cdot \|p_{r-2}(x,\cdot)\|_2 \\
+ |A_p(x)|^{1/2} \|h_{j,r}\|_{L^\infty} \|p_{2r-2}(x,\cdot)1_{\{ y : d(x,y) > 2^{p-1} \}}\|_2.
\]
Now it follows from (3) and (12) that there are constants $c$ and $C > 0$ such that

$$|A_p(x)|^{1/2} \|h_{j,r}\|_\infty \|p_{1/r^2}(x, \cdot)1_{\{y:d(x,y)>2^{p-1}\}}\|_2 \leq c \left\{ \frac{|B_{2^p}(x)|}{|B_{2^{-i}}(x)|} \right\}^{1/2} 2^{-\alpha j} e^{-C2^{i+p}}$$

and from this, by using either (1) or (2), we get that there is $c > 0$ such that

$$(14) \sum_{p \geq j-i} |A_p(x)|^{1/2} \|h_{j,r}\|_\infty \|p_{1/r^2}(x, \cdot)1_{\{y:d(x,y)>2^{p-1}\}}\|_2 \leq c \cdot 2^{-\alpha j}.$$  

On the other hand if we put

$$I_p(x) = |A_p(x)|^{1/2}(2\pi)^{-1/2} \int_{|t| \geq 2^{p-1}} |\hat{h}_{j,r}(t)| \|p_{r^{-2}}(x, \cdot)\|_2,$$

then it follows from (10) that there is $c > 0$ such that

$$I_p(x) \leq c \left\{ \frac{|B_{2^p}(x)|}{|B_{2^{-i}}(x)|} \right\}^{1/2} \int_{|t| \geq 2^{p-1}} |\hat{h}_{j,r}(t)| \, dt.$$

Hence, if we chose $k \in \mathbb{N}$, $k > \frac{1}{2} \max(d, D)$, then it follows from (8) (as well as either (1) or (2)) that there is $c > 0$ such that

$$I_p(x) \leq \begin{cases} c \cdot 2(D/2-k)p \cdot 2(d/2-k)i \cdot 2^{(k-\alpha)j} & \text{if } i \leq 0, \\ c \cdot 2(d/2-k)p \cdot 2(d/2-k)i \cdot 2^{(k-\alpha)j} & \text{if } i > 0, \ \min(0, j-i) \leq p \leq 0, \\ c \cdot 2(D/2-k)p \cdot 2(d/2-k)i \cdot 2^{(k-\alpha)j} & \text{if } i > 0, \ p \geq \max(0, j-i) \end{cases}$$

and from this

$$\sum_{p \geq j-i} I_p(x) \leq \begin{cases} c \cdot 2(D/2-\alpha)j & \text{if } i \leq 0, \\ c \cdot 2(d/2-\alpha)j & \text{if } i > 0, \ j < i, \\ c \cdot 2(D/2-d/2)i \cdot 2(D/2-\alpha)j & \text{if } i > 0, \ j \geq i, \end{cases}$$

which together with (12), (13) and (14) prove the lemma.

**Proof of theorem 1.** — This follows immediately from Lemma 5 and the inequality

$$\|K_{\alpha,R}(x, \cdot)\|_1 \leq \sum_{j \geq 0} \|K_{j,r}(x, \cdot)\|_1.$$
Proof of theorem 2. — We observe that (a) follows immediately from theorem 1 and that it is enough to prove (b) and (c) for those \( p \) for which we also have \( p < 2 \). Then, since \( m_{\alpha,R}(L) \) is self adjoint, by duality, we shall also have these results for those \( p \) for which we also have \( p > 2 \).

Now, if \( 0 < t < 1 \),
\[
\frac{1}{p} = \frac{t}{2} + \frac{1-t}{2}, \quad \text{i.e.} \quad t = \frac{2}{p} - 1,
\]
then, by interpolation, we have
\[
\|m_{j,r}(L)\|_{p \to p} \leq \|m_{j,r}(L)\|_{1 \to 1}^t \|m_{j,r}(L)\|_{2 \to 2}^{1-t} \leq (\sup_x \|K_{j,r}(x,\cdot)\|_1)^t \|h_{j,r}(\lambda)\|_\infty^{1-t}.
\]

Hence it follows from (11) and LEMMA 5 that there is \( c > 0 \) such that
\[
\|m_{j,r}(L)\|_{p \to p} \leq \begin{cases} 
  c \cdot 2^{-[\alpha-D(1/p-1/2)]j} & \text{if } 0 < R \leq 1, \\
  c \cdot 2^{-[\alpha-d(1-1/p)]j} & \text{if } R > 1, \ 0 \leq j < i, \\
  c \cdot 2^{-[\alpha-D(1-1/p)]j} 2^{(d-D)(1/p-1/2)i} & \text{if } R > 1, \ 0 < i \leq j.
\end{cases}
\]

Assertions (b) and (c) of THEOREM 1 follow from the above estimates, by taking the sums over \( j \).

2. Proof of theorem 3

We shall prove first the following

LEMMA 6. — If \( f \in L^p, 1 < p < \infty \), then \( \gamma \mapsto L^{i\gamma}f \) is a strongly continuous \( L^p \)-valued function.

Proof. — If \( \epsilon, \delta > 0 \) then
\[
\|L^{i(\gamma+\epsilon)}f - L^{i\gamma}f\|_p \leq \|L^{i(\gamma+\epsilon)}(f - e^{-\delta L}f)\|_p + \|(L^{i(\gamma+\epsilon)} - L^{i\gamma})e^{-\delta L}f\|_p + \|L^{i\gamma}(e^{-\delta L}f - f)\|_p.
\]

Now since, by the multiplier theorem of Stein [28], the operators \( L^{i(\gamma+\epsilon)}, 0 \leq \epsilon \leq 1 \) are uniformly bounded on \( L^p \) and since \( \|e^{-\epsilon L}f - f\|_p \to 0 \), as \( \delta \to 0 \) we have
\[
\|L^{i(\gamma+\epsilon)}(f - e^{-\delta L}f)\|_p + \|L^{i\gamma}(e^{-\delta L}f - f)\|_p \to 0, \quad (\delta \to 0).
\]
On the other hand, since
\[ \| (L^{i(\gamma+\epsilon)} - L^{i\gamma}) e^{-\delta L} \|_{2 \to 2} \leq \| (\lambda^{i(\gamma+\epsilon)} - \lambda^{i\gamma}) e^{-\delta \lambda} \|_\infty \to 0, \quad (\epsilon \to 0), \]
and since again by the multiplier theorem of Stein [28], the operators \((L^{i(\gamma+\epsilon)} - L^{i\gamma}) e^{-\delta L}\), for \(0 \leq \epsilon \leq 1\), are uniformly bounded on \(L^p\), it follows by interpolating with \(L^2\) that
\[ \| (L^{i(\gamma+\epsilon)} - L^{i\gamma}) e^{-\delta L} f \|_p \to 0, \quad (\epsilon \to 0) \]
and the lemma follows. 

Now, we continue with the proof of Theorem 3. Following [21] we write
\[ m_{\alpha,1}(\lambda) = M(\lambda) + e^{-\lambda} \quad \text{with} \quad M(\lambda) = m_{\alpha,1}(\lambda) - e^{-\lambda}. \]
Then we have that
\[ m_*(L)f(x) \leq \sup_{t>0} |M(tL)f(x)| + \sup_{t>0} |e^{-tL}f(x)|. \]
Now we know that the heat maximal operator \(\sup_{t>0} |e^{-tL}f(x)|\) is bounded on \(L^p\), \(1 < p < \infty\) (cf. [28]).

To deal with the maximal operator \(\sup_{t>0} |M(tL)f(x)|\), we proceed as in [11], that is we consider the Mellin inversion formula
\[ M(t\lambda) = \int_{-\infty}^{\infty} M(\gamma)(t\lambda)^{i\gamma} d\gamma, \]
where \(M(\gamma)\) is the Mellin transform of \(M(\lambda)\)
\[ M(\gamma) = (2\pi)^{-1} \int_0^{\infty} M(\lambda) \lambda^{-i\gamma} d\lambda. \]
This formula gives :
\[ M(tL)f = \int_{-\infty}^{\infty} M(\gamma) t^{i\gamma} L^{i\gamma} f d\gamma. \]
From this we have
\[
\sup_{t>0} |M(tL)f| = \sup_{t>0} \left| \int_{-\infty}^{\infty} M(\gamma) t^{i\gamma} L^{i\gamma} f d\gamma \right| \\
\leq \int_{-\infty}^{\infty} |M(\gamma)| \cdot |L^{i\gamma} f| d\gamma,
\]
which in turn implies:

\[
\|\sup_{t>0} M(tL)f\|_p \leq \int_{-\infty}^{\infty} |M(\gamma)| \cdot \|L^{i\gamma}\|_{p\to p} \|f\|_p \, d\gamma.
\]

The above formal calculations are justified by the fact that as was proved in Lemma 6, \( \gamma \mapsto L^{i\gamma}f \) is a strongly continuous, hence strongly measurable, \( L^p \)-valued function. So if

\[
(15) \quad \int_0^{\infty} |M(\gamma)| \cdot \|L^{i\gamma}\|_{p\to p} \, d\gamma < \infty,
\]

then

\[
\int_0^{\infty} M(\gamma) t^{i\gamma} L^{i\gamma} f \, d\gamma
\]

is a convergent \( L^p \)-valued integral. This integral defines a continuous function of \( t \), which implies that \( \sup_{t>0} |M(tL)f| \) is well defined in \( L^p \).

Now, it has been proved in [21] that

\[
(16) \quad |M(\gamma)| \leq c(1 + |\gamma|)^{-\alpha+1}.
\]

Furthermore, we have that \( \|L^{i\gamma}\|_{2\to 2} = 1 \) and it follows from the proof of the main result of [1] (that result is proved only for left invariant sub-Laplaceans on Lie groups of polynomial growth, but it is also true for the Laplace-Beltrami operator on a Riemannian manifold of non-negative curvature; the proof is exactly the same) that for every \( \epsilon > 0 \)

\[
\|L^{i\gamma}\|_{L^1\to \text{weak-}L^1} \leq c(1 + |\gamma|)^{(\max(d/2,D/2)+\epsilon)}^{\max(d/2,D/2)+\epsilon}.
\]

So, by interpolation and duality if necessary, we have that

\[
(17) \quad \|L^{i\gamma}\|_{p\to p} \leq c(1 + |\gamma|)^{(\max(d/2,D/2)+\epsilon)(2/p-1)}, \quad 1 < p < \infty.
\]

Now, it follows from (16) and (17) that when

\[
\alpha > \max\left(\frac{d}{2}, \frac{D}{2}\right)^{\frac{2}{p}} - 1 = \max(d, D)\left[\frac{1}{p} - \frac{1}{2}\right],
\]

then (15) holds and Theorem 3 follows.
3. Proof of theorem 4

It is enough to prove this theorem for functions \( f \) belonging to some space \( A \) which is dense to all spaces \( L^p \), \( 1 < p < \infty \). Then **Theorem 4** will follow from **Theorem 3** by well known measure theoretic arguments.

The space \( A \) we shall consider is

\[
A = \{ \varphi_t(L) e^{-sL} f ; \ f \in C^\infty_0, \ t \geq 1, \ 0 < s \leq 1 \},
\]

where \( \varphi_t(\lambda) = \varphi(\lambda/t) \) and \( \varphi \in C^\infty_0(\mathbb{R}) \) with \( \varphi(0) = 1 \).

That \( A \) is dense to all spaces \( L^p \), \( 1 < p < \infty \), follows from the fact that

\[
\| e^{-sL} f - f \|_p \to 0 \quad \text{as} \quad s \to 0 \quad \text{for all} \quad f \in C^\infty_0(G) \quad \text{and} \quad 1 < p < \infty
\]

and the observation that for all \( \lambda \in \mathbb{N} \)

\[
\sup_{\lambda > 0} \left| \frac{\lambda^k}{k!} \frac{d^k}{d\lambda^k} \left[ e^{-s\lambda} - \varphi_t(\lambda)e^{-s\lambda} \right] \right| \to 0, \quad (t \to \infty),
\]

which together with the proof of the main result of [1] (we repeat that the main result of [1], although is proved only for left invariant sub-Laplaceans on Lie groups of polynomial growth, but it is also true for the Laplace-Beltrami operator on a Riemannian manifold of non-negative curvature; the proof is exactly the same) imply that:

\[
\| e^{-sL} f - \varphi_t(L)e^{-sL} f \|_p \to 0, \quad \text{as} \quad t \to \infty.
\]

Let us now fix some \( h = \varphi_t(L)e^{-sL} f \in A \). Let us also consider a function \( \psi \in C^\infty(\mathbb{R}) \) such that

\[
\psi(\lambda) = \begin{cases} 1 & \text{for} \ |\lambda| \leq \frac{1}{4}, \\ 0 & \text{for} \ |\lambda| \geq \frac{1}{2}, \end{cases}
\]

and put \( \psi_R(\lambda) = \psi(\lambda/R) \), \( R > 0 \). Then for \( R \) large enough we have that

\[
m_{\alpha,R}(L)h = \psi_R(L)m_{\alpha,R}(L)\varphi_t(L)e^{-sL} f
\]

and therefore

\[
m_{\alpha,R}(L)h - h = [\psi_R(L)m_{\alpha,R}(L) - 1] \varphi_t(L)e^{-sL} f.
\]

Now since for all \( k \in \mathbb{N} \)

\[
\sup_{\lambda > 0} \left| \frac{\lambda^k}{k!} \frac{d^k}{d\lambda^k} \left[ [\psi_R(\lambda)m_{\alpha,R}(\lambda) - 1] \varphi_t(\lambda)e^{-s\lambda} \right] \right| \to 0, \quad (R \to \infty),
\]

it follows from the proof of the main result of [1] that

\[
\|m_{\alpha,R}(L)h - h\|_p = \left\| [\psi_R(L)m_{\alpha,R}(L) - 1] \varphi_t(L)e^{-sL} f \right\|_p \to 0, \quad (R \to 0),
\]

which proves the first part of Theorem 4.

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The second part of the theorem follows from the observation that

\[ |m_{\alpha,R}(L)h(x) - h(x)| = \left| \psi_R(L)m_{\alpha,R}(L) - 1 \right| \varphi_t(L)e^{-sL}f(x) \]

\[ \leq \left\| \psi_R(L)m_{\alpha,R}(L) - 1 \right\|_2 \cdot \|f\|_2, \]

\[ \leq \sup_{\lambda>0} \left[ \psi_R(\lambda)m_{\alpha,R}(\lambda) - 1 \right] \cdot |\varphi_t(\lambda)| \cdot \left\| p_s(x, \cdot) \right\|_2 \cdot \|f\|_2, \]

which together with the fact that

\[ \sup_{\lambda>0} \left[ \psi_R(\lambda)m_{\alpha,R}(\lambda) - 1 \right] \cdot |\varphi_t(\lambda)| \to 0, \quad (R \to \infty), \]

imply that

\[ |m_{\alpha,R}(L)h(x) - h(x)| \to 0, \quad (R \to \infty). \]

This completes the proof of Theorem 4.  

4. Final remarks

We point out that that our method also works when \( L \) is a self-adjoint non-negative real subelliptic differential operator on a compact manifold \( X \), since, in that case, the finite propagation speed (3) for the wave operator has already been proved in [23] and the gaussian estimates (4) for the associated heat kernel have been proved in [31], [32]. The results that we shall obtain are similar. The only change is that as dimension at infinity \( D \) we shall put \( D = 0 \) and as local dimension \( d \) we shall put the best constant \( b \) for which we have that

\[ |B_t(x)| \leq c \left( \frac{r}{t} \right)^b, \quad r \geq t \]

with the \( c > 0 \) independent of \( x \in X \) (cf. [24]). For example when \( L \) is a sum of squares of vector fields that satisfy Hörmanders condition in a uniform way, then there are constants \( c > 0 \) and \( k \in \mathbb{N} \), independent of \( x \in X \), such that (cf. [24], [30], [33])

\[ c^{-1}t^k \leq |B_t(x)| \leq ct^k, \quad 0 < t \leq 1 \]

and then, of course, we take \( d = k \).

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