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ON THE HAUSDORFF DIMENSION OF JULIA SETS OF MEROMORPHIC FUNCTIONS II

BY

JANINA KOTUS (*)

RÉSUMÉ. — On considère dans ce travail les estimations de la dimension de Hausdorff des ensembles de Julia pour certaines fonctions meromorphes transcendentes. Le résultat principal contient une borne inférieure pour la dimension de Hausdorff obtenue sous des hypothèses convenables portant sur la position des pôles et sur le comportement asymptotique de la fonction au voisinage des pôles. Des applications du théorème sont données pour des fonctions elliptiques et trigonométriques.

ABSTRACT. — The paper deals with estimates of the Hausdorff dimension of Julia sets of transcendental meromorphic functions. The main theorem gives a lower estimate under some regularity assumptions on the location of the poles and the behaviour of functions near their poles. Applications of this theorem include Julia sets of trigonometric and elliptic functions.

1. Introduction

Let $f : \mathbb{C} \to \widehat{\mathbb{C}}$ denote a transcendental meromorphic function. For $n \in \mathbb{N}$, denote by f^n the n-th iterate of f, and $f^{-n} = (f^n)^{-1}$. The *Fatou set* $F(f)$ is the set of all the points $z \in \mathbb{C}$ such that $(f^n), n \in \mathbb{N}$, is defined, meromorphic, and forms a normal family in some neighbourhood of z. The complement of $F(f)$ in $\widehat{\mathbb{C}}$ is called the *Julia set* $J(f)$ of f.

Every transcendental meromorphic function belongs to one of the following classes :

(i) f is entire;

(ii) f is a self-map of the punctured plane, i.e. $f = f_0$, where

$$f_0(z) = z_0 + (z - z_0)^{-k} \exp\big(g(z)\big)$$

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with $k \in \mathbb{N}$ and a non-constant entire function g;

(iii) f is neither entire nor a self-map of the punctured plane (i.e. f satisfies *assumption A* of [2], [3], [4], [5]).

We will consider functions belonging to the third class.

It was shown by GARBER [10, thm. 1] that for each rational function f the *Hausdorff dimension* of the Julia set $\mathrm{HD}(J(f))$ satisfies

$$0 < \mathrm{HD}\big(J(f)\big) \leq 2.$$

The estimate of BROLIN [6, thm. 12.2] that for $f_c(z) = z^2 - c$ with $c \geq 2 + 2^{1/2}$ one has

$$\mathrm{HD}\big(J(f_c)\big) < \frac{\log 2}{\log(2q(c))},$$

where

$$q(c) = \big(c - \tfrac{1}{2} - (\tfrac{1}{4}c)^{1/2}\big)^{1/2}$$

shows that the lower bound of Garber estimate is sharp. The upper bound of this estimate can be also attained as there are rational functions f such that $J(f) = \widehat{\mathbb{C}}$, e.g. $f(z) = ((z-2)/z)^2$.

For transcendental entire functions it is known that $\mathrm{HD}(J(f)) \geq 1$. This follows from Baker's result [1, cor. to thm. 1] : *if f is transcendental entire, then $J(f)$ contains a non-degenerate continuum.* However, it remains an open question whether there exists a transcendental entire function whose Julia set has dimension exactly one. The upper bound for the Hausdorff dimension of Julia set for these functions is 2, and it is attained e.g. for $f(z) = \exp z$, since $J(f) = \widehat{\mathbb{C}}$. Thus $1 \leq \mathrm{HD}(J(f)) \leq 2$.

For a transcendental meromorphic function satisfying *assumption A* the estimate of $\mathrm{HD}(J(f))$ is the same as for rational functions i.e.

$$0 < \mathrm{HD}\big(J(f)\big) \leq 2.$$

The lower bound is sharp (a result announced by STALLARD) as well as the upper bound since for $f(z) = \pi i \tan z$, $J(f) = \widehat{\mathbb{C}}$.

In this paper we give a lower estimate for $\mathrm{HD}(J(f))$ for meromorphic functions with infinitely many poles. Let

$$S = \big\{z \in \mathbb{C} : \text{ some branch of } f^{-1} \text{has a singularity at } z\big\},$$
$$P = \bigcup_{n=0}^{\infty} f^n(S).$$

Equivalently one may write

$$P = \{z : \text{ for some } n \in \mathbb{N} \text{ some branch of } f^{-n} \text{ has a singularity at } z\}.$$

Denote the closure of P by $\overline{P}$. Let :

- $A = \{a_n : f(a_n) = \infty,\ n \in \mathbb{N}\}$ be the set of poles of f ;
- b_n denote the first non-vanishing coefficient of the principal part of f of the Laurent expansion at the pole a_n ;
- $D(a, r) = \{z : |z - a| < r\}$.

THEOREM. — *Let f be a meromorphic function satisfying :*

(i) *A is infinite and* $\operatorname{dist}(A, \overline{P}) = d > 0$;

(ii) *there exist $\beta > 0$ and $\xi \geq 0$ such that for $a_n \in A$ one has $|a_n| \sim n^{\beta}$ and $|b_n| \sim n^{-\xi}$;*

(iii) *there exist $m \in \mathbb{N}, \gamma < (\beta + \xi)/m$, and r_n such that $r_n \sim n^{-\gamma}$, and*

$$|f'(z)| \sim \frac{m\,|b_n|}{|z - a_n|^{m+1}}, \qquad |f(z)| \sim \frac{|b_n|}{|z - a_n|^{m}}$$

for $z \in D(a_n, r_n)$.

Then $\operatorname{HD}(J(f)) \geq m/(\xi + (m+1)\beta)$.

Above and in the sequel $|a_n| \sim n^{\beta}$ (analogously $|b_n| \sim n^{-\xi}$, etc.) means

$$C^{-1} \leq |a_n| n^{-\beta} \leq C$$

for some constant $C > 0$ and all $n \geq n_0$.

Roughly speaking, the assertion (iii) enables us to replace f' by its principal part in a r_n-neighbourhood of the pole a_n *uniformly* with respect to n. Observe that the condition on f in (iii) implies

$$f(z) \sim c_n + b_n (z - a_n)^{-m} + \cdots$$

in $D_n = D(a_n, r_n)$ with c_n controlled uniformly in n. In other words, when we reconstruct f from f' in D_n this condition says that the « constants of integration » c_n are not too large.

Of course, the hypothesis (iii) is satisfied for periodic functions when we can, however, substantially simplify the formulation of this theorem. On the other hand, the assumption (iii) is satisfied for some non-periodic functions (see examples 4 and 5). In the proof we use this to control the geometry of a hyperbolic Cantor set of points escaping to infinity through neighbourhoods of poles.

In section 2 we prove the theorem, while in section 3 we indicate its applications to certain families of functions.

2. Proof of the theorem

Let

$$L(f,D)=\sup_{z_1,z_2}\frac{|f'(z_1)|}{|f'(z_2)|}$$

be the *distortion* of f on a domain D. It is clear that if f is a homeomorphism of a domain D onto $f(D)$, then $L(f,D)=L(f^{-1},f(D))$. In the proof we apply the following propositions.

PROPOSITION 2.1 (cf. [9, p. 32]). — *For every* $0<s<r$ *and for every univalent map* $g: D(z,r)\to\mathbb{C}$, *the distortion of* g *in the disc* $D=D(z,s)$ *is bounded by the constant* L

$$L(g,D)\le L=L\Big(\frac{s}{r}\Big)=\Big(\frac{r+s}{r-s}\Big)^4.$$

PROPOSITION 2.2 (cf. [13]). — *Let* E *be a compact subset of* $\mathbb{C}$ *and* α *be a positive number. Suppose that there exist a probability measure* μ *supported on* E, *and constants* K, $r_0>0$, *such that for each* $z\in E$ *and* $r<r_0$ *we have* $\mu(D(z,r))\le Kr^\alpha$. *Then* $\mathrm{HD}(E)\ge\alpha$.

For $a_n\in A$ let D_n be the disc $D(a_n,r_n)$, where r_n is given in THEOREM (iii). There exists $0<t<1$ such that $\overline{D}_k\subset f(D_n)$ if $k\ge n^t$ and n is very large. Indeed, from (iii)

$$|f|\sim\frac{|b_n|}{|z-a_n|^m}\quad\text{for}\quad z\in D_n$$

uniformly with respect to n. As $|a_n|\sim n^\beta$, $|b_n|\sim n^{-\xi}$ and $r_n\sim n^{-\gamma}$ we have that

$$|b_n|r_n^{-m}=\mathcal{O}(n^{m\gamma-\xi}).$$

Thus if $\gamma<(\beta+\xi)/m$, then

$$|b_n|r_n^{-m}=o(n^\beta).$$

It follows that $\overline{D}_k\subset f(D_n)$ since $|a_k|\sim k^\beta\ge n^{t\beta}=o(n^\beta)$. Let $0<t<1$, $n_1\in\mathbb{N}$ be very large and

$$\mathcal{N}=\{n\in\mathbb{N}: n_1^t\le n\le n_1\}.$$

Then $\overline{D}_k\subset f(D_n)$ if $k,n\in\mathcal{N}$. Define the sets :

$$T=\bigcup_{n=1}^\infty D_n,\qquad E=\{z:\ f^n(z)\in T\text{ for all }n\in\mathbb{N}\}.$$

PROPOSITION 2.3. — *We have* $E \subset J(f)$.

Proof. — First we prove $K = \inf_T |f'| > 1$. By assumption (iii) if $z \in D_n$ then

$$|f'| \sim \frac{m\,|b_n|}{|z-a_n|^{m+1}} \geq \frac{m|b_n|}{r_n^{m+1}} \geq Cn^{(m+1)\gamma-\xi}.$$

Shrinking, if necessary, the neighbourhoods of poles in (iii) we may assume that $r_n \geq (Cn^\gamma)^{-1}$ with some $\gamma \in (\xi/(m+1), (\beta+\xi)/m)$ and $C > 0$. Then $K = \inf_{D_n} |f'| > 1$ for all $n \geq n_2$ and we may assume $n_2 = n_1^t$ for suitably chosen $C > 0$. Suppose $z_0 \in E \cap F(f)$. By the definition of the Fatou set there exists disc $D = D(z_0, r)$ such that $(f^n{}_{|D})$, $n \in \mathbb{N}$, is defined, meromorphic and forms a normal family. Thus there exists a subsequence of iterates (f^{n_k}) holomorphic on D and converging to a holomorphic function g. Hence $g'(z) \neq \infty$ in D. On the other hand $g'(z_0) = \lim_{k\to\infty}(f^{n_k})'(z_0) = \infty$, as $K^n = \inf_T |(f^n)'| \to \infty$, so we arrive at a contradiction. Thus, $z_0 \in J(f)$. ☐

Take a disc D_t for some $t \in \mathcal{N}$. We introduce the following collection of sets :

$\mathcal{B}_0 = \{B_0 = D_t\}$,

$\mathcal{B}_1 = \{B_{1,n} : B_{1,n}$ is a component of $f^{-1}(D_n)$ for some $n \in \mathcal{N}, B_{1,n} \subset B_0\}$,

. . .

$\mathcal{B}_k = \{B_{k,n} : B_{k,n}$ is a component of $f^{-k}(D_n)$ for some $n \in \mathcal{N},\ B_{k,n} \subset B_{k-1,\ell} \in \mathcal{B}_{k-1}\}$.

. . .

Moreover, we define

$$\mathcal{A}_k = \bigcup_{B_{k,n}\in\mathcal{B}_k} B_{k,n} \quad \text{and} \quad \mathcal{B} = \bigcap_{k=1}^{\infty} \mathcal{A}_k.$$

Of course, $\mathcal{B}$ is contained in E.

PROPOSITION 2.4. — *There are constants* $\kappa = \gamma + \beta(m+1)/m$ *and* $\nu = \gamma - \xi/m > 0$ *such that for each* $k \in \mathbb{N}, n \in \mathcal{N}$ *and* $B_{k,n} \in \mathcal{B}_k$

$$\frac{\operatorname{diam} B_{k,n}}{\operatorname{diam} B_{k-1,\ell}} \sim n^{-\kappa}\ell^{\nu}$$

with $B_{k-1,\ell} \in \mathcal{B}_{k-1}$ *and* $B_{k,n} \subset B_{k-1,\ell}$.

Proof. — We may again increase slightly γ to have $\gamma > \xi/m$, and assume that $r_n < d = \operatorname{dist}(A, \overline{P})$ for all $n \in \mathcal{N}$. Thus f^{k-1} is a homeomorphism of $B_{k-1,\ell}$ onto $f^{k-1}(B_{k-1,\ell}) = D_\ell$. Let $L(f^{k-1}, B_{k-1,\ell})$ be the distortion of the function f^{k-1} in $B_{k-1,\ell}$. Clearly

$$L(f^{k-1}, B_{k-1,\ell}) = L(f^{-(k-1)}, D_\ell),$$

and by PROPOSITION 2.1,

$$L(f^{-(k-1)}, D_\ell) \le L\Big(\frac{r_\ell}{d}\Big) = \Big(\frac{d + r_\ell}{d - r_\ell}\Big)^4 = L.$$

Thus

$$\begin{aligned}\frac{\operatorname{diam} B_{k,n}}{\operatorname{diam} B_{k-1,\ell}} &\ge L^{-1}\,\frac{\operatorname{diam} f^{k-1}(B_{k,n})}{\operatorname{diam} f^{k-1}(B_{k-1,\ell})}\\ &= L^{-1}\,\frac{\operatorname{diam}(f^{-1}(D_n)\cap D_\ell)}{\operatorname{diam} D_\ell}\\ &\ge \big(LC\, r_\ell\, n^\gamma \sup\nolimits_{f^{-1}(D_n)\cap D_l} |f'|\big)^{-1}.\end{aligned}$$

For $z \in D_\ell$ we have $|f'| \sim m\,|b_\ell|/|z - a_\ell|^{m+1}$, so

$$\sup\big|f'(z)\big| \sim n^{\beta(m+1)m}\,\ell^{\xi/m}$$

if $z \in f^{-1}(D_n)\cap D_\ell$. This implies that for some $\delta > 0$, $\kappa = \gamma + \beta(m+1)/m$, $\nu = \gamma - \xi/m > 0$

$$\frac{\operatorname{diam} B_{k,n}}{\operatorname{diam} B_{k-1,\ell}} \ge \delta n^{-(\gamma+\beta(m+1)/m)}\,\ell^{\gamma-\xi/m} = \delta\, n^{-\kappa}\,\ell^{\nu}.$$

Similarly we have

$$\frac{\operatorname{diam} B_{k,n}}{\operatorname{diam} B_{k-1,\ell}} \le LC\big(r_\ell\, n^\gamma \inf\nolimits_{f^{-1}(D_n)\cap D_\ell} |f'|\big)^{-1} \le \eta\, n^{-\kappa}\,\ell^{\nu}$$

for some universal $\eta > 0$. $\square$

Proof of the theorem. — Let $\alpha \le \kappa^{-1}$, N be an integer such that $n_1 \ge N \ge n_2 > n_1^t + 1$ and

$$\sum_{n=n_1^t}^{N} n^{-\kappa\alpha} \ge \max\big\{\delta^{-\alpha}, (\delta\ \operatorname{diam} B_0)^{-\alpha}\big\}.$$

Now, inspired by an idea in [11], we define a probability measure μ on $\mathcal{B}$ such that for $r < r_0$ and $z \in \mathcal{B}$, $\mu(D(z,r)) \le Kr^\alpha$ with some constant $K > 0$. The measure μ is the limit of probability measures μ_k, as $k \to \infty$. Let μ_0 be the suitably scaled Lebesgue measure on B_0, i.e. $\mu_0(B_0) = 1$. We put $\mu_1 = 0$ on $B_0 \setminus \mathcal{A}_1$ and $\mu_1(B_{1,n}) = 0$ for $n \in \mathcal{N}$ and $n > N$. If $n \in \mathcal{N}$ and $n \le N$

$$\mu_1(B_{1,n}) = (\operatorname{diam} B_{1,n})^\alpha \Big/ \Big(\sum_{n=n_1^t}^{N} (\operatorname{diam} B_{1,n})^\alpha\Big).$$

Thus $\mu_1(B_{1,n}) > 0$ and $\sum_{n=n_1^t}^{N} \mu_1(B_{1,n}) = \mu_0(B_0)$. By PROPOSITION 2.4 and the choice of N,

$$\sum_{n=n_1^t}^{N} (\operatorname{diam} B_{1,n})^\alpha \ge \sum_{n=n_1^t}^{N} (\delta \operatorname{diam} B_0 n^{-\kappa})^\alpha \ge (\delta \operatorname{diam} B_0)^\alpha \sum_{n=n_1^t}^{N} n^{-\kappa\alpha} \ge 1,$$

so $\mu_1(B_{1,n}) \le (\operatorname{diam} B_{1,n})^\alpha$ for $n \in \mathcal{N}, n \le N$. Assume that we have defined μ_{k-1} on $B_{k-1,\ell} \in \mathcal{B}_{k-1}$. We define μ_k by letting $\mu_k = 0$ on $B_{k-1,\ell} \setminus \mathcal{A}_k$, $\mu_k(B_{k,n}) = 0$ if $n \in \mathcal{N}$ and $n > N$. For $n \in \mathcal{N}$ and $n \le N$

$$\mu_k(B_{k,n}) = (\operatorname{diam} B_{k,n})^\alpha \mu_{k-1}(B_{k-1,\ell}) \Big/ \Big(\sum_{n=n_1^t}^{N} (\operatorname{diam} B_{k,n})^\alpha\Big).$$

Now, we apply PROPOSITION 2.4 to $B_{k,n} \in \mathcal{B}_k$, $n \in \mathcal{N}, n \le N$, that is,

$$\begin{aligned}\sum_{n=n_1^t}^{N} (\operatorname{diam} B_{k,n})^\alpha &\ge \sum_{n=n_1^t}^{N} (\operatorname{diam} B_{k-1,\ell}\, \delta n^{-\kappa} \ell^\nu)^\alpha \\ &= (\operatorname{diam} B_{k-1,\ell})^\alpha \ell^{\nu\alpha} \sum_{n=n_1^t}^{N} (\delta n^{-\kappa})^\alpha \\ &\ge (\operatorname{diam} B_{k-1,\ell})^\alpha\end{aligned}$$

by the choice of N. This, together with the assumption $\mu_{k-1}(B_{k-1,\ell}) \le (\operatorname{diam} B_{k-1,\ell})^\alpha$, implies

$$\mu_{k-1}(B_{k-1,\ell}) \Big/ \sum_{n=n_1^t}^{N} (\operatorname{diam} B_{k,n})^\alpha \le 1,$$

hence $\mu_k(B_{k,n}) \leq (\operatorname{diam} B_{k,n})^\alpha$. The sequence μ_k converges to the limit probability measure μ supported on a set $\mathcal{B}'$ contained in $\mathcal{B}$. By the construction $\mu(B_{k,n}) = \mu_k(B_{k,n})$ for all $k \in \mathbb{N}$ and $n \in \mathcal{N}$, so $\mu(B_{k,n}) \leq (\operatorname{diam} B_{k,n})^\alpha$.

Take $z \in \mathcal{B}'$ and $r > 0$. Define

$$\varepsilon_{k,n} = \inf\{\operatorname{dist}(z', \partial B_{k,n}) : z' \in \mathcal{B}' \cap B_{k,n}\},$$

and observe that if $B_{k,n} \subset B_{k-1,\ell}$ then $\varepsilon_{k,n} < \varepsilon_{k-1,\ell}$. Moreover $\varepsilon_{k,n} \to 0$ when $k \to \infty$. Hence there exists $k \in \mathbb{N}$ such that $\varepsilon_{k,n} \leq r < \varepsilon_{k-1,\ell}$, so

$$D(z,r) \subset D(z, \varepsilon_{k-1,\ell}) \subset B_{k-1,\ell}.$$

But the diameter of $B_{k-1,\ell}$ is also comparable with r. To see this observe that

$$\operatorname{diam} B_{k-1,\ell} \leq \sigma \sup_{D_\ell} \left|(f^{-(k-1)})'\right| \leq \sigma \sup_{B_{k,\ell}} \left|(f^{k-1})'\right|^{-1}$$

with

$$\sigma = \max\{\operatorname{diam} D_\ell : n_1^t \leq \ell \leq N\}.$$

On the other hand, for some $z' \in \mathcal{B}' \cap B_{k,n}$

$$\begin{aligned} r \geq \varepsilon_{k,n} &= \operatorname{dist}(z', \partial B_{k,n}) \\ &\geq \operatorname{dist}\big(f^{k-1}(z'), f^{k-1}(\partial B_{k,n})\big) \inf_{D_\ell} \left|(f^{-(k-1)})'\right| \\ &\geq \tau \inf_{B_{k-1,\ell}} \left|(f^{k-1})'\right|^{-1} \end{aligned}$$

with

$$\tau = \min\{\operatorname{dist}(\partial B_{1,m}, \partial B_{2,p}) : n_1^t \leq m,\ p \leq N\}.$$

Finally we obtain

$$\frac{\operatorname{diam} B_{k-1,\ell}}{r} \leq \frac{\sigma}{\tau} \frac{(\sup_{B_{k-1,\ell}} |(f^{k-1})'|)}{(\inf_{B_{k-1,\ell}} |(f^{k-1})'|)} \leq \frac{\sigma}{\tau} \sup_n L(f^{-k}, D_n) \leq \frac{\sigma}{\tau} L$$

by PROPOSITION 2.1 and $\operatorname{dist}(A, \overline{P}) = d > 0$. Consequently we get

$$\begin{aligned} \mu(D(z,r)) &\leq \sum_n \mu(B_{k,n}) = \sum_n \mu_k(B_{k,n}) \leq \sum_n (\operatorname{diam} B_{k,n})^\alpha \\ &\leq N(N^\nu \eta)^\alpha (\operatorname{diam} B_{k-1,\ell})^\alpha \\ &\leq N(N^\nu \eta\, L\sigma/\tau)^\alpha r^\alpha. \end{aligned}$$

By PROPOSITION 2.2 (with the constant $K = N^{1+\nu\alpha}(L\eta\sigma/\tau)^\alpha)$ we have $\operatorname{HD}(\mathcal{B}') \geq \alpha$. As we have chosen arbitrary $\gamma \in (\xi/m,\ (\beta+\xi)/m)$ we may write

$$\operatorname{HD}\big(J(f)\big) \geq \operatorname{HD}(\mathcal{B}') \geq m/(\xi + (m+1)\beta). \quad \square$$

3. Examples

Let f be a meromorphic map on $\mathbb{C}$. For $\lambda \in \mathbb{C}$ let

$$f_\lambda(z) = \lambda f(z).$$

We consider further only these λ for which $\operatorname{dist}(A(f_\lambda), \overline{P}(f_\lambda)) > 0$. Recall that for meromorphic functions f with finitely many singular values (i.e. $\operatorname{card} S(f) < \infty$) the Fatou-Sullivan classification of periodic components of $F(f)$ holds. This is a consequence of the general version of the Sullivan theorem proved in [5].

EXAMPLE 1.
Let $f_\lambda(z) = \lambda(\tan z)^m$, $m \in \mathbb{N}$, $\lambda \in \mathbb{C}$. Then

$$A(f_\lambda) = \{a_n = (n + \tfrac{1}{2})\pi : n \in \mathbb{Z}\}$$

is the set of poles. It is easy to see that $\beta = 1$, $\xi = 0$, so

$$\operatorname{HD}\big(J(f_\lambda)\big) \geq \frac{m}{m+1}\cdot$$

In particular, $\operatorname{HD}(J(f_\lambda)) \geq \frac{1}{2}$ for $m = 1$.

Now, we compare the above estimate with an upper bound for the Hausdorff dimension of Julia set for $f_\lambda(z) = \lambda \tan z$.

PROPOSITION 3.1. — *Let $f_\lambda(z) = \lambda \tan z$, with $\lambda \in \mathbf{R}$ and $0 < |\lambda| \ll 1$. Then*

$$\operatorname{HD}\big(J(f_\lambda)\big) \leq \tfrac{1}{2} + |\lambda|^{1/2}/\pi + \mathcal{O}\big(|\lambda|\big).$$

Proof. — For $|\lambda| < 1$ we have $f_\lambda'(0) = \lambda$, so 0 is an attracting fixed point. If additionally $\lambda > 0$, then there is a repelling fixed point $z_\lambda \in (0, \frac{1}{2}\pi) \subset \mathbb{R}$, which belongs to the boundary of the basin of attraction of 0. Analogously, for $-1 < \lambda < 0$ there is a repelling periodic point of order 2 with the same property. The functions f_λ have poles at the points of the set

$$A = \big\{a_n = (n + \tfrac{1}{2})\pi : n \in \mathbb{Z}\big\}.$$

Let $\delta_\lambda = |a_0 - z_\lambda|$, and $I_n = (a_n - \delta_\lambda, a_n + \delta_\lambda) \subset \mathbb{R}$. Then $J(f_\lambda)$ is a Cantor subset of $\mathbb{R}$ and $J(f_\lambda) \subset \bigcup_{n=-\infty}^{\infty} I_n$, see [8]. Take I_t, $t \in \mathbb{Z}$. As in the proof of THEOREM we define the following family of sets.

$$\mathcal{B}_0 = \{B_0 = I_t\},$$
$$\mathcal{B}_1 = \big\{B_{1,n} : B_{1,n} \text{ is a component of } f^{-1}(I_n) \text{ for some } n \in \mathbb{Z},\ B_{1,n} \subset B_0\big\},$$
$$\dots$$
$$\mathcal{B}_k = \big\{B_{k,n} : B_{k,n} \text{ is a component of } f^{-k}(I_n) \text{ for some } n \in \mathbb{Z},\ B_{k,n} \subset B_{k-1,\ell} \in \mathcal{B}_{k-1}\big\},$$
$$\dots$$

Thus
$$\mathcal{A}_k = \bigcup_{B_{k,n}\in\mathcal{B}_k} B_{k,n}$$
is a covering of $J(f_\lambda)$ and $\operatorname{diam} B_{k,n} \le d_k \to 0$ if $k \to \infty$. To prove that $\mathrm{HD}(J(f_\lambda)) \le \alpha$ it is sufficient to show that, for each $k \in \mathbb{N}$,
$$\sum_{B_{k,n}\in\mathcal{B}_k} (\operatorname{diam} B_{k,n})^\alpha \ \le \sum_{B_{k-1,\ell}\in\mathcal{B}_{k-1}} (\operatorname{diam} B_{k-1,\ell})^\alpha,$$
or equivalently that for each $k \in \mathbb{N}$ and each $B_{k-1,\ell} \in \mathcal{B}_{k-1}$
$$\sum_{B_{k,n}\in G(B_{k-1,\ell})} (\operatorname{diam} B_{k,n})^\alpha \le (\operatorname{diam} B_{k-1,\ell})^\alpha,$$
where
$$G(B_{k-1,\ell}) = \big\{B_{k,n} \in \mathcal{B}_k : B_{k,n} \subset B_{k-1,\ell}\big\}.$$
By PROPOSITION 2.1 the distortion $L(f^{-(k-1)}, I_\ell)$ is bounded by
$$L(\lambda) = \Big(\frac{\frac12\pi + \delta_\lambda}{\frac12\pi - \delta_\lambda}\Big)^4,$$
so
$$\begin{aligned}
&\frac{1}{(\operatorname{diam} B_{k-1,\ell})^\alpha} \sum_{B_{k,n}\in G(B_{k-1,\ell})} (\operatorname{diam} B_{k,n})^\alpha \\
&\le L^\alpha\big(f^{k-1}, B_{k-1,\ell}\big)\,\frac{\sum_{B_{k,n}\in G(B_{k-1,\ell})}(\operatorname{diam} f^{k-1}(B_{k,n}))^\alpha}{(\operatorname{diam} f^{k-1}(B_{k-1,\ell}))^\alpha} \\
&= L^\alpha\big(f^{-(k-1)}, I_\ell\big)\,\frac{\sum_{n=-\infty}^{+\infty}(\operatorname{diam}(f^{-1}(I_n)\cap I_\ell))^\alpha}{(\operatorname{diam} I_\ell)^\alpha} \\
&\le 2L^\alpha(\lambda)(2\delta_\lambda)^{-\alpha}\sum_{n=0}^{\infty}\big(2\delta_\lambda \sup_{I_n}\big|(f^{-1})'\big|\big)^\alpha \\
&= 2L^\alpha(\lambda)\sum_{n=0}^{\infty}\Big(\frac{|\lambda|}{\lambda^2 + ((n+\frac12)\pi - \delta_\lambda)^2}\Big)^\alpha
\end{aligned}$$

since $(f_\lambda^{-1}(w))' = (\arctan w/\lambda)' = \lambda/[\lambda^2+w^2]$ and $\sup_{I_n}(f_\lambda^{-1})'$ is attained at the left edge of I_n, i.e. at $w = a_n - \delta_\lambda$. Thus

$$\begin{aligned}\frac{1}{(\operatorname{diam} B_{k-1,\ell})^\alpha} &\sum_{B_{k,n}\in G(B_{k-1,\ell})} (\operatorname{diam} B_{k,n})^\alpha \\ &\le 2\big(|\lambda|L(\lambda)\big)^\alpha\Big[\big(\lambda^2+(\tfrac12\pi-\delta_\lambda)^2\big)^{-\alpha} \\ &\qquad + \sum_{n=1}^\infty\{\lambda^2+\big((n+\tfrac12)\pi-\delta_\lambda\big)^2\}^{-\alpha}\Big] \\ &\le 2\big(|\lambda|L(\lambda)\big)^\alpha\Big[(\tfrac12\pi-\delta_\lambda)^{-2\alpha}+\sum_{n=1}^\infty(n\pi)^{-2\alpha}\Big] \\ &= 2\big(|\lambda|L(\lambda)\big)^\alpha\Big[(\tfrac12\pi-\delta_\lambda)^{-2\alpha}+\pi^{-2\alpha}\zeta(2\alpha)\Big],\end{aligned}$$

where ζ is the Riemann zeta function. If $\lambda > \frac12$ then $\zeta(2\alpha) < \infty$ and for $|\lambda|^\alpha \le \pi^{2\alpha}(2\zeta(2\alpha))^{-1}$

$$\sum_{B_{k,n}\in G(B_{k-1,\ell})} (\operatorname{diam} B_{k,n})^\alpha \le (\operatorname{diam} B_{k-1,\ell})^\alpha.$$

The inequality $|\lambda|^\alpha \le \pi^{2\alpha}(2\zeta(2\alpha))^{-1}$ implies that

$$\alpha \le \tfrac12 + |\lambda|^{1/2}/\pi + 2|\lambda|/\pi^2 = \tfrac12 + |\lambda|^{1/2}/\pi + \mathcal{O}(|\lambda|). \quad \square$$

Thus for $f_\lambda(z) = \lambda\tan z$, $\lambda\in\mathbb{R}$, the estimate in THEOREM is sharp when $\lambda\to 0$.

EXAMPLE 2.

Let f_λ be a simply periodic function with finitely many poles in each strip of periodicity. Then $\beta = 1$, and $\xi = 0$, thus

$$\mathrm{HD}\big(J(f_\lambda)\big) \ge \frac{m}{m+1} \ge \frac12,$$

and $\mathrm{HD}(J(f_\lambda)) \to 1^-$ as $m\to\infty$.

The assumptions concerning β, ξ and m in THEOREM are fairly general. In some cases PROPOSITION 3.1 shows that the conclusion of THEOREM is precise. However, for

$$g_\lambda(z) = \frac{1}{\lambda+\exp(-2z)}$$

with $\lambda > 0$, $\beta = 1$, $\xi = 0$, $m = 1$ (like for f_λ in Example 1, $m = 1$) we have proved in [12, thm. 2] that

$$\mathrm{HD}\big(J(g_\lambda)\big) \ge 1 - C\big(\log|\log\lambda|\big)^{-1}$$

for some $C > 0$ and $\lambda\to 0^+$. Hence the lower bound for $\mathrm{HD}(J(g_\lambda))$ tends to 1 when $\lambda \to 0$. This estimate has been shown by studying the dynamics of these maps. So, the estimate in THEOREM is not optimal for these g_λ's.

EXAMPLE 3.

Let f_λ be a doubly periodic function of order $m \in \mathbb{N}$ of the poles (e.g. an elliptic function). Clearly, $\xi = 0$ and we show that for all these functions $\beta = \frac{1}{2}$. Let $a_1, \dots, a_N$ be all the poles belonging to $D(0,r)$. The number of poles in $D(0,r)$ is proportional to the number of fundamental parallelograms contained in $D(0,r)$, i.e. $N \approx \pi r^2/a^2$, where a^2 is the area of a fundamental parallelogram. Thus $r \sim N^{1/2}$ and $|a_N| \sim r \sim N^{1/2}$. By THEOREM

$$\mathrm{HD}\big(J(f_\lambda)\big) \geq \frac{2m}{m+1} \geq 1,$$

and $\mathrm{HD}(J(f_\lambda)) \to 2^-$ if $m \to \infty$.

Now we apply THEOREM to non-periodic functions.

EXAMPLE 4.

Let

$$f_\lambda(z) = \big(\lambda + \exp(-z^{2k})\big)^{-1}, \quad k \in \mathbb{N} \cup \{\tfrac{1}{2}\}.$$

For $\lambda > 0$ we have $\mathrm{dist}(A(f_\lambda), \overline{P}(f_\lambda)) > 0$ and $J(f_\lambda)$ is a Cantor set. If $k = \frac{1}{2}$, f_λ is periodic, so $\beta = 1$, $\xi = 0$, and $\mathrm{HD}(J(f_\lambda)) \geq \frac{1}{2}$. For $k \geq 1$ we have $\beta = 1/(2k)$, $\xi = 1 - 1/(2k)$, $m = 1$, which implies that

$$\mathrm{HD}(J(f_\lambda)) \geq \frac{2k}{2k+1} \cdot$$

For second order differential equations of the form $f'' + h(z)f = 0$, where $h(z)$ is an entire function, it is known from the elementary theory of differential equations in complex domain that all solutions f of this equation are entire functions, and that the zeros of any $f \not\equiv 0$ are simple. Let $h(z)$ be a polynomial of degree k. Obviously, for $k = 0$ the equation possesses two linearly independent solutions, each of which has no zeros. In the case $k \geq 1$, it follows from the Wiman-Valiron theory (see [14, p. 281]) that the order of growth of $f \not\equiv 0$ is $\frac{1}{2}(k+2)$. Hence, when k is an odd integer, the Hadamard factorization theorem and the Borel theorem on roots of entire functions (e.g. [7]) imply that the exponent of convergence ρ of the sequence $\{a_n : f(a_n) = 0,\ n \in \mathbb{N}\}$ is equal to $\frac{1}{2}(k+2)$. Let $g = 1/f$, then

$$A(g) = \{a_n : n \in \mathbb{N}\}, \quad |a_n| = \mathcal{O}(n^{1/\rho + \varepsilon})$$

for any $\varepsilon > 0$ and $|b_n| = |f'(a_n)|^{-1}$.

EXAMPLE 5.
If $h(z) = -z$, then one of solutions of $f'' - zf = 0$ is the Airy function

$$\mathrm{Ai}(z) = \frac{1}{2\pi} \int_{\mathrm{Im}\,\zeta=\eta>0} \exp\big(\tfrac{1}{3} i\zeta^3 + i\zeta z\big)\,\mathrm{d}\zeta.$$

The zeros of Ai are at the points

$$a_n = \big(\tfrac{3}{2}\pi(n - \tfrac{1}{4})\big)^{2/3} + \mathcal{O}(n^{-4/3}).$$

Set $f_\lambda = \lambda/\,\mathrm{Ai}$, then $m = 1$, $\beta = \frac{2}{3}$, $\xi = \frac{1}{6}$, and $\mathrm{HD}(J(f_\lambda)) \geq \frac{2}{3}$.

BIBLIOGRAPHY

[1] BAKER (I.N.). — *The domains of normality of an entire function*, Ann. Acad. Sci. Fenn. Ser. A, I. Math., t. **1**, 1975, p. 277–283.

[2] BAKER (I.N.), KOTUS (J.) and LÜ (Y.). — *Iterates of meromorphic functions I*, Ergod. Th. Dynam. Sys., t. **11**, 1991, p. 241–248.

[3] BAKER (I.N.), KOTUS (J.) and LÜ (Y.). — *Iterates of meromorphic functions II : Examples of wandering domains*, J. London Math. Soc., t. **42**, 2, 1990, p. 267–278.

[4] BAKER (I.N.), KOTUS (J.) and LÜ (Y.). — *Iterates of meromorphic functions III : Preperiodic domains*, Ergod. Th. Dynam. Sys., t. **11**, 1991, p. 603–618.

[5] BAKER (I.N.), KOTUS (J.) and LÜ (Y.). — *Iterates of meromorphic functions IV : Critically finite functions*, Results in Mathematics, t. **22**, 1992, p. 651–656.

[6] BROLIN (H.). — *Invariant sets under iteration of rational functions*, Arkiv für Mathematik, t. **6**, 1965, p. 103–144.

[7] BOAS (R.P.). — *Entire Functions.* — Academic Press, New York, 1954.

[8] DEVANEY (R.L.) and KEEN (L.). — *Dynamics of meromorphic maps with polynomial Schwarzian derivative*, Ann. Sci. École Norm. Sup., t. **22**, 1989, p. 55–81..

[9] DUREN (P.L.). — *Univalent Functions.* — Springer, New York, 1983.

[10] GARBER (V.). — *On the iteration of rational functions*, Math. Proc. Camb. Phil. Soc., t. **84**, 1978, p. 497–505.

[11] GRACZYK (J.) and ŚWIĄTEK (G.). — *Critical circle maps near bifurcation*, preprint SUNY, Stony Brook, IMS n° 1991/8.

[12] KOTUS (J.). — *On the Hausdorff dimension of Julia sets of meromorphic functions I*, Bull. Soc. Math. France, t. **122**, 1994, p. 305–331.

[13] STALLARD (G.). — *The Hausdorff dimension of Julia sets of entire functions*, Ergod. Th. Dynam. Sys., t. **11**, 1991, p. 769–777.

[14] WITTICH (H.). — *Eindeutige Lösungen der Differentialgleichung* $w' = R(z, w)$, Math. Z., t. **74**, 1960, p. 278–288.