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FAILURE OF CONVERGENCE OF THE LAX-OLEINIK SEMI-GROUP IN THE TIME-PERIODIC CASE

BY ALBERT FATHI AND JOHN N. MATHER (*)

ABSTRACT. — For a time-independent Lagrangian, the so-called Lax-Oleinik semi-group converges with an arbitrary continuous function as initial condition. Using twist maps, we show that there is no such convergence for time-periodic Lagrangians.

RÉSUMÉ. — NON CONVERGENCE DU SEMI-GROUPE DE LAX-OLEINIK DANS LE CAS PÉRIODIQUE EN LE TEMPS. — Pour un lagrangien dépendant du temps le semi-groupe de Lax-Oleinik converge pour toute condition initiale continue. En utilisant des applications déviant la verticale, nous montrons que ce n'est pas le cas pour des lagrangiens dépendant périodiquement du temps.

Introduction

Let $L:\mathbb{T} \times TM \to \mathbb{R}$, $(t,x,v) \mapsto L(t,x,v)$ be a time-periodic Lagrangian satisfying the assumptions of [8], *i.e.*, the manifold M whose tangent bundle is TM is compact and smooth, the Lagrangian L is twice continuously differentiable, the fiberwise Hessian of L is positive definite, L has uniformly superlinear growth along the fibers, and the Euler-Lagrange flow is complete. Here, as is usual $\mathbb{T} = \mathbb{R}/\mathbb{Z}$.

Associated to this Lagrangian, there is a Hamiltonian $H:\mathbb{T}\times T^*M \to \mathbb{R}$, where T^*M is the cotangent bundle. For $p \in T_x^*M$, a cotangent vector at $x \in M$, the Hamiltonian $H(t,x,p)$ is defined by:

$$H(t,x,p) = \max_{v\in T_x M} p(v) - L(t,x,v).$$

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In several domains (PDE, Dynamical Systems, Optimization and Control Theory) it is important to understand the solutions of the Hamilton-Jacobi Equation:

$$\frac{\partial U}{\partial t} + H\Big(t, x, \frac{\partial U}{\partial x}\Big) = 0,$$

where U is a function defined on an open set of $\mathbb{R} \times M$. The well-known method of characteristics allows to find $\mathcal{C}^2$ solutions U with $U_{|\{0\}\times M}$ a given $\mathcal{C}^2$ function on M, the domain of definition of U is some (rather small) neighborhood of $\{0\} \times M$. Usually it is impossible to find a $\mathcal{C}^2$ solution U defined on $\mathbb{R} \times M$. There is however a way to define weak (viscosity) global solutions, using $T_t^- : \mathcal{C}^0(M, \mathbb{R}) \circlearrowleft$ defined for $t \geq 0$ by

$$T_t^- u(x) = \inf_\gamma \Big\{ u(\gamma(0)) + \int_0^t L\big(s, \gamma(s), \dot\gamma(s)\big)\, \mathrm{d}s \Big\},$$

where the infimum is taken over all continuous piecewise $\mathcal{C}^1$ paths $\gamma: [0, t] \to M$ with $\gamma(t) = x$. It is not difficult to check that the function $U: [0, +\infty[\times M \to \mathbb{R}$ defined by $U(t, x) = T_t^- u(x)$ is a solution of the Hamilton-Jacobi Equation on each open set where it is smooth.

Since L is time-periodic, $T_{t+1}^- = T_t^- \circ T_1^-$. Hence, $\{T_n^-\}_{n=0,1,\dots}$ is a semi-group, called the *Lax-Oleinik semi-group.* One would like to understand the behavior of this non-linear semi-group as $n \to +\infty$.

The first author proved [4] the convergence of the full Lax-Oleinik semi-group (*i.e.*, $\{T_t^-\}_{t\geq 0}$) in the time-independent case. For previous work by Namah and Roquejoffre see [10], [11], [12], for different proofs and extensions of the result contained in [4], see the work by Barles and Souganidis [3] and the work of Roquejoffre [13].

In [4], the first author raised the question as to whether the analogous result holds in the time-periodic case. This would be the convergence of $T_n^- u + n\alpha_0$, as $n \to +\infty$, $n \in \mathbb{N}$. Here, $\alpha_0 \in \mathbb{R}$ is Mañé critical value [6] which equals $\alpha(0)$, defined earlier in [8]. It depends only on L.

In this paper, we provide examples with $M = \mathbb{T}$ where there is no such convergence, thus answering this question negatively. In fact, there is no convergence of the Lax-Oleinik semi-group for a generic Lagrangian $L: \mathbb{T} \times T\mathbb{T} \to \mathbb{R}$.

1. The Function h_L

Our construction depends on results in [9]. In order to explain it, we need to recall some definitions from [8].

We view a one form on M as a function on TM, linear along the fibers, and furthermore as a function on $\mathbb{T} \times TM$, by ignoring the $\mathbb{T}$-factor. If η is a closed one form, $L - \eta: \mathbb{T} \times TM \to \mathbb{R}$ is a Lagrangian, still satisfying the assumptions of [8]. Moreover, it has the same Euler-Lagrange flow as L. Following [8], we set:

$$\alpha_L(c) = -\inf_{\mu} \Big\{ \int (L - \eta)\, \mathrm{d}\mu \Big\},$$

where c is the de Rham cohomology class of η, and μ ranges over all probability measures invariant under the Euler-Lagrange flow associated L. This is independent of the closed one form representing c. The case $c = 0$ gives Mañé's critical value, *i.e.*, $\alpha_0 = \alpha_L(0)$.

Following [9], we define a function $h_L: M \times M \to \mathbb{R}$, as follows:

$$h_L(x, x') = \alpha_0 + \inf_{\gamma} \Big\{ \int_0^1 L\big(t, \gamma(t), \dot\gamma(t)\big)\, \mathrm{d}t \Big\}, \quad x, x' \in M.$$

The infimum is taken over all continuous piecewise $\mathcal{C}^1$ curves $\gamma: [0,1] \to M$ such that $\gamma(0) = x$ and $\gamma(1) = x'$. In addition, we set

$$h_L^n(x, x') = \inf\big\{ h_L(x_0, x_1) + \cdots + h_L(x_{n-1}, x_n) \big\},$$

where the infimum is taken over all $(x_0, \ldots, x_n) \in M^{n+1}$ such that $x_0 = x$ and $x_n = x'$;

$$h_L^{q\infty + r}(x, x') = \liminf_{n \to \infty} h_L^{qn+r}(x, x');$$

$h_L^{q\infty} = h_L^{q\infty+0}$; and $h_L^{\infty} = h_L^{1\infty}$. (We have changed the notation of [9]: if η is a closed 1-form on M whose cohomology class is c, then $h_{L-\eta}$, $h_{L-\eta}^n$, and $h_{L-\eta}^{\infty}$ were denoted h_c, h_c^n, and h_c^{∞} there. Note that these depend on η, not just c.) The function $h_L^{q\infty}$ is always finite and continuous [9].

Can lim inf always be replaced by lim in the definition of h_L^{∞}? This is related to the question of the convergence of the Lax-Oleinik semi-group in the time-periodic case. For, it is clear that $T_1^- u(x) + \alpha_0 = \inf_{y \in M} \{u(y) + h_L(y, x)\}$ and $T_n^- u(x) + n\alpha_0 = \inf_{y \in M} \{u(y) + h_L^n(y, x)\}$. Moreover, if we set $u_y(x) = h_L(y, x)$, we have $h_L^{n+1}(y, x) = T_n^-(u_y)(x) + n\alpha_0$. Thus, the convergence of $T_n^- u + n\alpha_0$, for all $u \in \mathcal{C}^0(M, \mathbb{R})$, would imply the convergence of $h_L^n(y, x)$, as $n \to +\infty$, for all $y, x \in M$.

We will construct examples where the convergence of $h_L^n(x, x)$ fails, and hence the Lax-Oleinik semi-group does not converge.

2. The Examples

We will construct examples of non-convergence in the case $M = \mathbb{T}$.

We let $L: \mathbb{T} \times T\mathbb{T} \to \mathbb{R}$ be a time-periodic Lagrangian satisfying the hypotheses of [8]. For $p/q \in \mathbb{Q}$, expressed in lowest terms, we let $M_{p/q} \subset \mathbb{T} \times T\mathbb{T}$ denote the union of all action minimizing periodic orbits which are periodic of period q and rotation number p/q. Then $M_{p/q}$ is a closed, non-void subset, invariant under the Euler-Lagrange flow. Let π denote the canonical projection of $\mathbb{T} \times T\mathbb{T}$ onto $\mathbb{T} \times \mathbb{T}$. The restriction of π to $M_{p/q}$ is injective. See [1], [5] or [8].

Following [8], we let

$$\beta_L : H_1(M, \mathbb{R}) \longrightarrow \mathbb{R}$$

denote the conjugate of $\alpha_L : H^1(M, \mathbb{R}) \to \mathbb{R}$ in the sense of convex analysis. Thus,

$$\beta_L(h) = -\min\left\{\alpha_L(c) - \langle c, h\rangle : c \in H^1(M, \mathbb{R})\right\}.$$

Note that both α_L and β_L are convex functions with super-linear growth. We let $\mathcal{L}_\beta$ denote the Legendre transform associated to $\beta = \beta_L$. Thus, for $h \in H_1(M, \mathbb{R})$, we have that $\mathcal{L}_\beta(h)$ is the non-empty, convex, compact subset of $H^1(M, \mathbb{R})$ defined by

$$\mathcal{L}_\beta(h) = \left\{c \in H^1(M, \mathbb{R}) : \beta_L(c) + \alpha_L(c) = \langle c, h\rangle\right\}.$$

Note that $\beta_L(h) + \alpha_L(c) \geq \langle c, h\rangle$, for all c, h.

In the case $M = \mathbb{T}$, we have canonical identifications $H_1(\mathbb{T}, \mathbb{R}) = \mathbb{R}$ and $H^1(\mathbb{T}, \mathbb{R}) = \mathbb{R}$. Bangert [2] proved:

- if ω is irrational, then $\mathcal{L}_\beta(\omega)$ is one point;
- if $\omega = p/q$, then $\mathcal{L}_\beta(p/q)$ is reduced to one point if and only if $\Sigma_{p/q} := \pi(M_{p,q}) = \mathbb{T} \times \mathbb{T}$.

(See [7] for another proof, which the second author found after Bangert told him his result.)

For a generic L, the invariant set $M_{p/q}$ is a single orbit. In this case, $\Sigma_{p/q}$ is homeomorphic to a circle and by Bangert's theorem, $\mathcal{L}_\beta(p/q)$ is an interval $[c_-, c_+]$ (with $c_- < c_+$).

In what follows, we will suppose that $q \geq 2$, $\Sigma_{p/q} \neq \mathbb{T} \times \mathbb{T}$, and η is a closed one form on $\mathbb{T}$ such that the de Rham cohomology class $[\eta]$ satisfies $c_- < [\eta] < c_+$, where $\mathcal{L}_\beta(p/q) = [c_-, c_+]$. As we have just observed, there exist examples satisfying these conditions: the condition $\Sigma_{p/q} \neq \mathbb{T} \times \mathbb{T}$ holds for generic L, and then $c_- < c_+$, by Bangert's theorem. Since $L - \eta$ is a Lagrangian satisfying the assumptions of [8], this will provide the required example.

Under these conditions, we will show that the Lax-Oleinik semi-group associated to $L - \eta$ does not converge. More precisely, we will show that there exists $x \in \mathbb{T}$ such that $h^n_{L-\eta}(x, x)$ does not converge, as $n \to \infty$.

Let

$$\Sigma^0_{p/q} = \Sigma_{p/q} \cap (0 \times \mathbb{T}) \subset \mathbb{T}.$$

We recall from [9] that $h^\infty_{L-\eta}(x, x) \geq 0$ and $h^\infty_{L-\eta}(x, x) = 0$ if and only if $x \in \Sigma^0_{p/q}$. Given $x \in \Sigma^0_{p/q}$, we let $\{(t \text{ mod. } 1, x_t, \dot{x}_t) : t \in \mathbb{R}\}$ be the unique orbit of the Euler-Lagrange flow in $M_{p/q}$ such that $x_0 = x$.

THEOREM. — *If $x \in \Sigma^0_{p/q}$ and $0 < r < q$, then*

$$\sum_{i=1}^{q/\mu} h^{q\infty+r}_{L-\eta}(x_{ir}, x_{ir}) > 0,$$

where μ is the greatest common divisor of r and q.

This theorem implies that for some $1 \leq i \leq q/\mu$, we have that $h^{q\infty+r}_{L-\eta}(x_{ir}, x_{ir}) > 0$. On the other hand, we have that $h^\infty_{L-\eta}(x_{ir}, x_{ir}) = 0$ since $x_{ir} \in \Sigma^0_{p/q}$. Thus,

$$\limsup_{n\to\infty} h^n_{L-\eta}(x_{ir}, x_{ir}) \geq h^{q\infty+r}_{L-\eta}(x_{ir}, x_{ir}) > 0 = \liminf_{n\to\infty} h^n_{L-\eta}(x_{ir}, x_{ir}).$$

It follows that $\lim_{n\to\infty} h^n_{L-\eta}(x_{ir}, x_{ir})$ does not exist. This provides the required example.

The proof of this theorem is given in the following sections.

3. The Metric d^q_c

We retain the notations of the theorem. Thus, we suppose that $x \in \Sigma^0_{p/q}$ and let $\{(t \text{ mod. } 1, x_t, \dot{x}_t) : t \in \mathbb{R}\}$ be the unique orbit of the Euler-Lagrange flow in $M_{p/q}$ such that $x_0 = x$. We suppose that $c_- < c < c_+$. For $0 \leq i, j \leq q - 1$, we set

$$d^q_c(x_i, x_j) = h^{q\infty}_{L-\eta}(x_i, x_j) + h^{q\infty}_{L-\eta}(x_j, x_i),$$

where η is a closed one form on $\mathbb{T}$ whose de Rham cohomology class is c. This depends only on L and c, not on the choice of one form η within the cohomology class c.

We will prove the following lemma in §4:

LEMMA. — *If $c_- < c < c_+$, then d^q_c is a metric on the set $\{x_0, \dots, x_{q-1}\}$, i.e.*

$$d^q_c(x_i, x_j) \geq 0, \quad d^q_c(x_i, x_j) = d^q_c(x_j, x_i),$$

$$d^q_c(x_i, x_k) \leq d^q_c(x_i, x_j) + d^q_c(x_j, x_k),$$

and $d^q_c(x_i, x_j) = 0$ if and only if $i = j$.

More generally, for $c_- \le c \le c_+$, we may present d_c^q as a special case of a pseudo-metric introduced in [9]. Recall that in [9, §6], we associated to a Lagrangian $L\colon \mathbb{T}\times TM \to \mathbb{R}$ and a cohomology class $c \in H^1(M,\mathbb{R})$ a function $d_c\colon M\times M \to \mathbb{R}$. For what we do next, we need to make the dependence on L explicit in the notation: we write d_c^L for d_c. We let

$$L^q(t,x,\dot{x}) = qL(qt,x,q^{-1}\dot{x}).$$

It is easily checked that $d_c^q = d_c^{L^q}$.

In [9, §6], we observed that the restriction of d_c to a set $\Sigma_c^{0'}$ (defined there) is a pseudo-metric. Applied to $d_c^q = d_c^{L^q}$, this observation shows that d_c^q is a pseudo-metric on $\{x_0,\ldots,x_q\}$ (which is a subset of $\Sigma_c^{0'}$). In other words, all the conditions for d_c^q to be a metric hold, except possibly the condition that $d_c^q(x_i,x_j)=0$ implies $i=j$.

4. Proof that d_c^q is a Metric

In this section, we finish the proof of the lemma, by showing that $d_c^q(x_i,x_j)>0$ when $i\neq j$ and $c_- < c < c_+$. In fact, we will show

$$d_c^q(x_i,x_j) = \min\bigl(c_+ - c, c - c_-, \|\{pi/q\} - \{pj/q\}\|(c_+ - c_-)\bigr).$$

Here, $\{x\} \in \mathbb{T} = \mathbb{R}/\mathbb{Z}$ denotes the residue class of $x\in\mathbb{R}$, and $\|\{x\}\|$ denotes $\min\{|x-n|, n\in\mathbb{Z}\}$.

Let us write $\rho(x_i,x_j)$ for the right side above. To show that $d_c^q(x_i,x_j) \ge \rho(x_i,x_j)$, it is enough to show that for any continuous piecewise $\mathcal{C}^1$ curve $\gamma\colon[0,nq]\to\mathbb{T}$ such that $\gamma(0)=\gamma(nq)=x_i$ and $\gamma(mq)=x_j$ for some $0<m<n$, we have

$$(*)\qquad nq\alpha_L(c) + \int_0^{nq} (L-\eta)\bigl(t,\gamma(t),\dot\gamma(t)\bigr)\,\mathrm{d}t \ge \rho(x_i,x_j).$$

Note that the left side of (*) is unchanged if we change η within a cohomology class. Thus, it depends only on γ, L and $c=[\eta]$. Now we fix γ and L and consider the left side of (*) as a function of $c\in[c_-,c_+]$. Since $[c_-,c_+] = \mathcal{L}_\beta(p/q)$, the function α_L has constant slope p/q on $[c_-,c_+]$, by convex duality. Hence, the left side of (*) has constant slope $np-[\gamma]$ on $[c_-,c_+]$, where $[\gamma]\in H_1(\mathbb{T},\mathbb{Z})=\mathbb{Z}$ denotes the homology class of γ. Moreover, in view of the definition of α_L, the left side of (*) is non-negative on $[c_-,c_+]$. Hence, it is

$$\ge \bigl(np-[\gamma]\bigr)(c-c_-), \quad \text{if } [\gamma]<np,$$
$$\ge \bigl([\gamma]-np\bigr)(c_+-c), \quad \text{if } [\gamma]>np.$$

In either of these two cases, the left side above is $\ge \min(c_+ - c, c - c_-) \ge \rho(x_i,x_j)$.

The only case which remains is $[\gamma] = np$. In this case, the slope of the left side above (as a function of $c \in [c_-, c_+]$) vanishes, by what we showed in the previous paragraph. Thus, the left side of (*) is independent of η, as long as $c_- \le c = [\eta] \le c_+$. To analyze this case, we fix L and let $\sigma(i,j)$ be the infimum of the left side above over all continuous, piecewise $\mathcal{C}^1$ curves $\gamma\colon [0, nq] \to \mathbb{T}$ such that $\gamma(0) = \gamma(nq) = x_i$, $\gamma(mq) = x_j$ for some $0 < m < n$ and $[\gamma] = np$. Obviously, $\sigma(i+q, j) = \sigma(i, j+q) = \sigma(i,j)$ and $\sigma(i,k) \le \sigma(i,j) + \sigma(j,k)$.

In addition, σ is symmetric: $\sigma(i,j) = \sigma(j,i)$. For, if γ is a curve as in the definition of $\sigma(i,j)$, then γ_1 is a curve as in the definition of $\sigma(j,i)$, where $\gamma_1(t) = \gamma(mq+t)$, for $0 \le t \le (n-m)q$, and $\gamma_1(t) = \gamma(t-(n-m)q)$, for $(n-m)q \le t \le nq$. Moreover, the left side of (*) is the same whether the integral is taken over γ or γ_1.

Note that $\sigma(i,j)$ depends only on $\|\{pi/q\} - \{pj/q\}\|$, which permits us to introduce the notation

$$\widetilde{\sigma}\big(q\|\{pi/q\} - \{pj/q\}\|\big) = \sigma(i,j).$$

For, by the symmetry just proved, it is enough to suppose that $\{pi/q\} - \{pj/q\} = \{pk/q\} - \{p\ell/q\}$. We may also suppose that $i - q < k \le i$ and $j \le \ell < j+q$. Then $(i-k) + (\ell - j) = q$. If γ is a curve as in the definition of $\sigma(i,j)$, we define γ_1 to be the curve consisting of five pieces, as follows:

1) $\gamma_{1\,|[0,i-k]}$ is action minimizing for $L - \eta$ with $\gamma_1(0) = x_k$, $\gamma_1(i-k) = x_i$;
2) $\gamma_{1\,|[i-k,i-k+mq]}$ is defined by $\gamma_1(t+i-k) = \gamma(t)$;
3) $\gamma_{1\,|[i-k+mq,i-k+(m+1)q]}$ is action minimizing for $L - \eta$ with $\gamma_1(i-k+mq) = \gamma_1(i-k+(m+1)q) = x_j$;
4) $\gamma_{1\,|[i-k+(m+1)q,i-k+(n+1)q]}$ is defined by $\gamma_1(t+i-k+q) = \gamma(t)$; and
5) $\gamma_{1\,|[i-k+(n+1)q,(n+2)q]}$ is action minimizing with $\gamma_1(i-k+(n+1)q) = x_i$ and $\gamma_1((n+2)q) = x_k$.

Here, action minimizing means relative to curves on $\mathbb{T}^2$ with the same endpoints.

Note that the sum of the left side of (*) over the first and fifth pieces vanishes, since together they constitute an action minimizing periodic curve of rotation number p/q. Likewise, the left side of (*) over the third piece vanishes. Finally, the left side of (*) over the second and fourth piece is the same as over γ. To summarize,

$$(n+2)q\alpha_L(c) + \int_0^{(n+2)q} (L-\eta)\big(t, \gamma_1(t), \dot\gamma_1(t)\big)\,\mathrm{d}t$$

equals the left side of (*). Moreover, $\gamma_1((m+1)q) = x_\ell$, in view of the fact that the third piece is action minimizing and $(i-k) + (\ell - j) = q$. Thus,

$\gamma_1(0) = \gamma_1((n+2)q) = x_k$ and $\gamma_1((m+1)q) = x_\ell$. Moreover, $[\gamma_1] = (n+2)p$. This proves $\sigma(k,\ell) \le \sigma(i,j)$, and, of course, this inequality can be reversed by the same argument with the variables interchanged.

Thus, $\widetilde{\sigma}(i)$ is defined for integers $0 \le i \le \frac{1}{2}q$. We have $\widetilde{\sigma}(i) \ge 0$ and $\widetilde{\sigma}(0) = 0$. We set $\widetilde{\sigma}(q-i) = \widetilde{\sigma}(i)$, for $0 \le i \le \frac{1}{2}q$. Then $\widetilde{\sigma}(i)$ is defined for integers $0 \le i \le q$. It is clear that $\widetilde{\sigma}(i+j) \le \widetilde{\sigma}(i) + \widetilde{\sigma}(j)$ for $0 \le i, j, i+j \le q$.

We will show next that $\widetilde{\sigma}(k) = k\widetilde{\sigma}(1)$, for $0 \le k \le \frac{1}{2}q$. This is obvious for $k \le 1$, so we assume that $1 < k \le \frac{1}{2}q$.

For $0 \le i < q$, we let $0 \le \widehat{\imath} < q$ be such that $p\widehat{\imath} \equiv i \pmod{q}$. We let $\mu_i : [0,q] \to \mathbb{T}$ be the action minimizing curve of period q and rotation number p/q such that $\mu_i(0) = \mu_i(q) = x_{\widehat{\imath}}$. We let $\gamma : [0, nq] \to \mathbb{T}$ be as in the definition of $\sigma(0, \hat{k})(= \widetilde{\sigma}(k))$ with the further property that $[\gamma] = np$. For topological reasons, there are two possibilities:

- either $t \mapsto (t \bmod q, \gamma(t))$ crosses $t \to (t \bmod q, \mu_i(t))$ twice (once in $[0, mq]$ and once in $[mq, nq]$), for each $0 < i < k$;
- or this happens for each $k < i < q$.

Suppose that the first of these two possibilities holds: We choose $0 < s_1 < \cdots < s_{k-1} < mq$ such that $\gamma(s_i) = \mu_i(s'_i)$ with $0 < s'_i < q$ and $s_i - s'_i \in q\mathbb{Z}$, and $mq < t_{k-1} < \cdots < t_1 < nq$ such that $\gamma(t_i) = \mu_i(t'_i)$ with $0 < t'_i < q$ such that $t'_i - t_i \in q\mathbb{Z}$. We define

$$\gamma_0(t) = \begin{cases} \gamma(t) & 0 \le t \le s_1, \\ \mu_1(t + s'_1 - s_1) & s_1 \le t \le m_0 q, \\ \mu_1(t - m_0 q) & m_0 q \le t \le m_0 q + t'_1, \\ \gamma(t + t_1 - t'_1 - m_0 q) & m_0 q + t'_1 \le t \le n_0 q, \end{cases}$$

where $m_0 = 1 + (s_1 - s'_1)/q$ and $n_0 = m_0 + n + (t'_1 - t_1)/q$.

For $0 < i < k - 1$, we define

$$\gamma_i(t) = \begin{cases} \mu_i(t) & 0 \le t \le s'_i, \\ \gamma(t + s_i - s'_i) & s'_i \le t \le s_{i+1} - s_i + s'_i, \\ \mu_{i+1}(t + s'_{i+1} - s_{i+1} + s_i - s'_i) & s_{i+1} - s_i + s'_i \le t \le m_i q, \\ \mu_{i+1}(t - m_i q) & m_i q \le t \le m_i q + t'_{i+1}, \\ \gamma(t + t_{i+1} - m_i q - t'_{i+1}) & m_i q + t'_{i+1} \le t, \\ & \quad t \le m_i q + t'_{i+1} + t_i - t_{i+1}, \\ \mu_i(t + t'_i - m_i q - t'_{i+1} - t_i + t_{i+1}) & m_i q + t'_{i+1} + t_i - t_{i+1} \le t \le n_i q, \end{cases}$$

where $m_i = 1-(s'_{i+1}-s_{i+1}+s_i-s'_i)/q$, and $n_i = m_i+1-(t'_i-t_i+t_{i+1}-t'_{i+1})/q$, for $0 < i < k-1$. Note that m_i is the least integer $> (s_{i+1}-s_i+s'_i)/q$ and n_i is the least integer $> m_i + (t'_{i+1}-t_{i+1}+t_i)/q$. Finally, we define

$$\gamma_{k-1}(t) = \begin{cases} \mu_{k-1}(t) & 0 \le t \le s'_{k-1}, \\ \gamma(t+s_{k-1}-s'_{k-1}) & s'_{k-1} \le t, \\ & t \le m_{k-1}q + (t_{k-1}-mq), \\ \mu_{k-1}(t+t'_{k-1}-m_{k-1}q-t_{k-1}+mq) & m_{k-1}q+(t_{k-1}-mq) \le t, \\ & t \le n_{k-1}q, \end{cases}$$

where $m_{k-1} = m-(s_{k-1}-s'_{k-1})/q$ and $n_{k-1} = m_{k-1}-m+(t_{k-1}-t'_{k-1})/q+1$.

The sum over $0 \le i < k$ of the left side of (*) over γ_i is the left side of (*) over γ. Moreover, γ_i satisfies the conditions of the definition of $\sigma(\widehat{\imath}, \widehat{\imath+1})$. Now assume that for every $\epsilon > 0$, there exists a curve γ as in the definition of $\sigma(0,\hat{k})$ with $[\gamma] = np$, the left side of (*) is $< \sigma(0,\hat{k})+\epsilon$, and the first possibility holds. In this case, the argument we have just given shows that

$$\widetilde{\sigma}(k)+\epsilon = \sigma(0,\hat{k})+\epsilon \ge \sum_{i=0}^{k-1} \sigma(\hat{\imath}, \widehat{\imath+1}) = k\widetilde{\sigma}(1).$$

Thus, $\widetilde{\sigma}(k)+\epsilon \ge k\widetilde{\sigma}(1)$. Since this holds for every $\epsilon > 0$, it follows that $\widetilde{\sigma}(k) \ge k\widetilde{\sigma}(1)$. Since the opposite inequality holds, we obtain $\widetilde{\sigma}(k) = k\widetilde{\sigma}(1)$, in this case.

If this case does not hold, then for every $\epsilon > 0$, there exists a curve γ as in the definition of $\sigma(0,\hat{k})$ with $[\gamma] = np$, the left side of (*) is $< \sigma(0,\hat{k})+\epsilon$, and the second possibility holds. When this happens, the argument we have just given shows that

$$\widetilde{\sigma}(k) \ge (q-k)\widetilde{\sigma}(1).$$

However, since $k \le \frac{1}{2}q$ and $\widetilde{\sigma}(k) \le k\widetilde{\sigma}(1)$, we see that this is impossible unless $q = \frac{1}{2}k$, when we again have $\widehat{\sigma}(k) = k\widehat{\sigma}(1)$.

This concludes the proof that $\widetilde{\sigma}(k) = k\widetilde{\sigma}(1)$, when $0 \le k \le \frac{1}{2}q$.

Next, we show that $c_+ - c_- = q\widetilde{\sigma}(1)$. We let $\gamma\colon [0,nq] \to \mathbb{T}$ be a continuous, piecewise $\mathcal{C}^1$ curve such that $\gamma(0) = \gamma(mq) = \gamma(nq) = x_0$, for some $0 < m < n$. Thus, γ is the concatenation $\gamma = \gamma_0 * \gamma_1$, where $\gamma_0 = \gamma|[0,mq]$ and $\gamma_1 = \gamma|[mq,nq]$. We impose the further condition on γ that $[\gamma_0] = mp+1$ and $[\gamma_1] = (n-m)p-1$. By what we showed in the beginning of this section, the left side of (*), taken over γ_0 is $\ge c_+ - c$; taken over γ_1 it is $\ge c - c_-$. Thus, taken over γ, it is $\ge (c_+-c)+(c-c_-) = c_+ - c_-$. Moreover, it is easily seen that the infimum of the left side of (*), taken over such γ, is $c_+ - c_-$.

On the other hand, the argument above which shows that $\widetilde{\sigma}(k) = k\widetilde{\sigma}(1)$ also shows that this infimum is $q\widetilde{\sigma}(1)$. Thus, $c_+ - c_- = q\widetilde{\sigma}(1)$.

Thus, we have shown $d_c^q(x_i, x_j) \geq \rho(x_i, x_j)$, in all cases. The opposite inequality follows easily from what we did above.

5. End of the Proof of the Theorem

Clearly,

$$h_{L-\eta}^{nq+r}(x_{i(q-r)}, x_{i(q-r)}) + h_{L-\eta}^{q-r}(x_{i(q-r)}, x_{(i+1)(q-r)}) \geq h_{L-\eta}^{(n+1)q}(x_{i(q-r)}, x_{(i+1)(q-r)}).$$

Summing, we get

$$\sum_{i=1}^{q/\mu} h_{L-\eta}^{nq+r}(x_{i(q-r)}, x_{i(q-r)}) \geq \sum_{i=1}^{q/\mu} h_{L-\eta}^{(n+1)q}(x_{i(q-r)}, x_{(i+1)(q-r)}) \geq d_c^q(x_0, x_{q-r}) = \rho(x_0, x_{q-r}) > 0.$$

Here, we have used $\sum_{i=1}^{q/\mu} h_{L-\eta}^{q-r}(x_{i(q-r)}, x_{(i+1)(q-r)}) = 0$, which holds because $x_0, x_1, \ldots, x_q = x_0$ is minimizing and periodic.

Since n is an arbitrary positive integer, it follows that

$$\sum_{i=1}^{q/\mu} h_{L-\eta}^{q\infty+r}(x_{ir}, x_{ir}) = \sum_{i=1}^{q/\mu} h_{L-\eta}^{q\infty+r}(x_{i(q-r)}, x_{i(q-r)}) \geq \rho(x_0, x_{q-r}) > 0.$$

as required. ☐

BIBLIOGRAPHY

[1] BANGERT (V.). — *Mather Sets for Twist Maps and Geodesics on Tori*, Dyn. Rep., t. **1**, 1988, p. 1–56.

[2] BANGERT (V.). — *Geodesic Rays, Busemann Functions, and Monotone Twist Maps*, Calc. Var., t. **2**, 1994, p. 49–63.

[3] BARLES (G.), SOUGANIDIS (P.E.). — *On the Large Time Behavior of Solutions of Hamilton-Jacobi Equations.* — Preprint, 1998.

[4] FATHI (A.). — *Sur la convergence du semi-groupe de Lax-Oleinik*, C. R. Acad. Sci. Paris, Série I, t. **327**, 1998, p. 267–270.

[5] DENZLER (J.). — *Mather Sets for Plane Hamiltonian Systems*, J. Applied Math. Phys. (ZAMP), t. **38**, 1987, p. 791–812.

[6] MAÑÉ (R.). — *On the Minimizing Measures of Lagrangian Dynamical Systems*, Nonlinearity, t. **5**, 1992, p. 623–638.

[7] MATHER (J.). — *Differentiability of the Minimal Average Action as a Function of the Rotation Number*, Bol. Soc. Bras. Mat., t. **21**, 1990, p. 59–70.

[8] MATHER (J.). — *Action Minimizing Measures for Positive Definite Lagrangian Systems*, Math. Z., t. **207**, 1991, p. 169–207.

[9] MATHER (J.). — *Variational Construction of Connecting Orbits*, Ann. Inst. Fourier, Grenoble, t. **43**, 1993, p. 1349–1386.

[10] NAMAH, (G.), ROQUEJOFFRE, (J.-M.). — *Comportement asymptotique des solutions d'une classe d'équations paraboliques et de Hamilton-Jacobi*, C. R. Acad. Sci. Paris, Série I, t. **324**, 1997, p. 1367–1370.

[11] NAMAH, (G.), ROQUEJOFFRE, (J.-M.). — *Remarks on the Long Time Behavior of the Solutions of Hamilton-Jacobi Equations*, Comm. Partial Diff. Eq., t. **5**, 1999, p. 883–894..

[12] ROQUEJOFFRE (J.-M.). — *Comportement asymptotique des solutions d'équations de Hamilton-Jacobi monodimensionnelles*, C. R. Acad. Sci. Paris, Série I, t. **326**, 1998, p. 185–189.

[13] ROQUEJOFFRE (J.-M.). — *Convergence to Steady States or Periodic Solutions in a Class of Hamilton-Jacobi Equations.* — Preprint, 1998.