Bull. Soc. math. France
130 (1), 2002, p. 35–48

AN APPROXIMATION PROPERTY OF QUADRATIC IRRATIONALS

BY TAKAO KOMATSU

Dedicated to Professor Iekata Shiokawa
on the occasion of his 60th birthday

ABSTRACT. — Let $\alpha > 1$ be irrational. Several authors studied the numbers

$$\ell^m(\alpha) = \inf\{\,|y| : y \in \Lambda_m,\ y \neq 0\},$$

where m is a positive integer and Λ_m denotes the set of all real numbers of the form $y = \epsilon_0\alpha^n + \epsilon_1\alpha^{n-1} + \cdots + \epsilon_{n-1}\alpha + \epsilon_n$ with restricted integer coefficients $|\epsilon_i| \leq m$. The value of $\ell^1(\alpha)$ was determined for many particular Pisot numbers and $\ell^m(\alpha)$ for the golden number. In this paper the value of $\ell^m(\alpha)$ is determined for irrational numbers α, satisfying $\alpha^2 = a\alpha \pm 1$ with a positive integer a.

RÉSUMÉ (*Une approximation des irrationnels quadratiques*). — Soit $\alpha > 1$ un irrationnel. Plusieurs auteurs ont étudié les nombres

$$\ell^m(\alpha) = \inf\{\,|y| : y \in \Lambda_m,\ y \neq 0\},$$

où m est un entier positif et Λ_m est l'ensemble de tous les réels de la forme $y = \epsilon_0\alpha^n + \epsilon_1\alpha^{n-1} + \cdots + \epsilon_{n-1}\alpha + \epsilon_n$ avec des $|\epsilon_i| \leq m$ entiers. La valeur de $\ell^1(\alpha)$ a été précisée pour beaucoup de nombres de Pisot et $\ell^m(\alpha)$ pour le nombre d'or. Dans cet article, on détermine $\ell^m(\alpha)$ lorsque α est un irrationnel qui satisfait $\alpha^2 = a\alpha \pm 1$ avec a entier positif.

Texte reçu le 16 juin 2000, révisé le 18 décembre 2000, accepté le 11 avril 2001

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2000 Mathematics Subject Classification. — 11A63, 11J04, 11J70.

Key words and phrases. — Approximation property, quadratic irrationals, continued fractions.

1. Introduction

Let α be a positive real number and an integer $m \geq 1$. Denote by $\Lambda = \Lambda_m$ the set of all real numbers y having at least one representation of the form

$$y = \epsilon_0\alpha^n + \epsilon_1\alpha^{n-1} + \cdots + \epsilon_{n-1}\alpha + \epsilon_n$$

with some positive integer n and $|\epsilon_i| \leq m$, $\epsilon_i \in \mathbb{Z}$ $(i = 0, 1, \ldots, n)$. Set

$$\ell^m(\alpha) = \inf\bigl\{|y| : y \in \Lambda_m,\ y \neq 0\bigr\}.$$

Several authors studied the numbers $\ell^m(\alpha)$. The value of $\ell^1(\alpha)$ was determined for many particular Pisot numbers (see [1], [3], [4], [5], [6], [7], [8]) and $\ell^m(\alpha)$ for the golden number (see [8]). In this paper the value of $\ell^m(\alpha)$ is determined for two kinds of irrational numbers:

$$\alpha = \frac{1}{2}\bigl(a + \sqrt{a^2+4}\,\bigr) = [a; a, a, \ldots] \quad (a \geq 1) \quad \text{and}$$

$$\alpha = \frac{1}{2}\bigl(a + \sqrt{a^2-4}\,\bigr) = [a-1; 1, a-2, 1, a-2, \ldots] \quad (a \geq 3).$$

We shall prove the following two theorems.

THEOREM 1.1. — *Let $\alpha = \frac{1}{2}(a + \sqrt{a^2+4}\,)$ $(a \geq 1)$. If*

$$\alpha^k(\alpha - 1) < m \leq \alpha^{k+1}(\alpha - 1)$$

for some integer $k \geq -1$, then

$$\ell^m(\alpha) = \bigl|q_k\alpha - p_k\bigr|.$$

THEOREM 1.2. — *Let $\alpha = \frac{1}{2}(a + \sqrt{a^2-4})$ $\quad (a \geq 3)$.*

- *If for some non-negative integer i,*

$$\alpha^i(\alpha - a + 1) < m \leq \alpha^i(a - 2)$$

then

$$\ell^m(\alpha) = \bigl|q_{2i-1}\alpha - p_{2i-1}\bigr|.$$

- *If for some non-negative integer i,*

$$\alpha^i(a - 2) < m < \alpha^{i+1}(\alpha - a + 1)$$

then

$$\ell^m(\alpha) = |q_{2i}\alpha - p_{2i}|.$$

In addition to prove these two theorems, we shall show how to find a representation form $y = \epsilon_0\alpha^n + \epsilon_1\alpha^{n-1} + \cdots + \epsilon_{n-1}\alpha + \epsilon_n$ which gives $\ell^m(\alpha) = |q_k\alpha - p_k|$.

2. General sketch

If α is a root of the quadratic equation, then any of the form

$$\epsilon_0\alpha^n + \epsilon_1\alpha^{n-1} + \cdots + \epsilon_{n-1}\alpha + \epsilon_n \quad (n \ge 2)$$

can be reduced to the form $q\alpha - p$ for some integers q and p. We can set $\alpha > 1$. For, $\ell^m(\alpha) = 0$ if $0 < \alpha < 1$; $\ell^m(\alpha) = 1$ if $\alpha = 1$. Concerning the linear form $q\alpha - p$, the following approximation theorem is well-known (see Thm. 5E (ii) in [10], *e.g.*).

THEOREM A. — *If $k \ge 1$, $0 < q \le q_k$ and $p/q \ne p_k/q_k$, $p/q \ne p_{k-1}/q_{k-1}$, then*

$$|q_{k-1}\alpha - p_{k-1}| < |q\alpha - p|.$$

Here, $p_k/q_k = [a_0; a_1, \ldots, a_k]$ denotes the k-th convergent of the continued fraction expansion of α, $\alpha = [a_0; a_1, a_2, \ldots]$. Namely,

$$\alpha = a_0 + \frac{1}{\alpha_1}, \qquad a_0 = \lfloor \alpha \rfloor,$$

$$\alpha_n = a_n + \frac{1}{\alpha_{n+1}}, \qquad a_n = \lfloor \alpha_n \rfloor \quad (n \ge 1)$$

and

$$p_k = a_k p_{k-1} + p_{k-2} \quad (k \ge 0), \qquad p_{-1} = 1, \qquad p_{-2} = 0,$$

$$q_k = a_k q_{k-1} + q_{k-2} \quad (k \ge 0), \qquad q_{-1} = 0, \qquad q_{-2} = 1.$$

Therefore, this kind of problems is equivalent to how to find the least m with $|\epsilon_i| \le m$, $\epsilon_i \in \mathbb{Z}$ $(0 \le i \le n)$, satisfying

$$\epsilon_0\alpha^n + \epsilon_1\alpha^{n-1} + \cdots + \epsilon_{n-1}\alpha + \epsilon_n = q\alpha - p = \pm(q_k\alpha - p_k)$$

for a fixed integer k. In other words, $\ell^m(\alpha)$ is equal a priori to some $|q_k\alpha - p_k|$ for every m. Namely, for any positive integer m there exists an integer k such that for some $y = y_{-1}, y_0, \ldots, y_k$ we have

$$y_{-1} = |q_{-1}\alpha - p_{-1}|, \quad y_0 = |q_0\alpha - p_0|, \ \ldots, \quad y_k = |q_k\alpha - p_k|,$$

but $y \ne |q_{k+1}\alpha - p_{k+1}|$ for any y. From the result of van Ravenstein [9], the integer q satisfying $\ell^m(\alpha) = q\alpha - p$ should be one of the first values

$$\big\{(-1)^n j q_n/q_{n+1}\big\} q_{n+1} \quad (j = 1, 2, \ldots),$$

and the integer p be its counterpart. Notice that larger m becomes, more choices each ϵ_i can have. So, always $\ell^m(\alpha) \le \ell^{m+1}(\alpha)$ for every m. If $q > q_{k+1}$ then from our decision of m (see (7) below) we could choose some y so that $y = |q_{k+1}\alpha - p_{k+1}|$ because

$$|q|s_{n-1} + |p|s_n > q_{k+1}s_{n-1} + p_{k+1}s_n.$$

Hence, it is sufficient to consider the integers q with $q_k < |q| < q_{k+1}$. But, by Theorem A always $|q\alpha - p| > |q_k\alpha - p_k|$ holds for such q's.

Suppose that α is the larger root of the quadratic equation $x^2 = ax + b$. Here, $a, b \in \mathbb{Z}$ because both q and p are integers. Notice that $x^2 - ax - b = (x - \alpha)(x - \beta)$, where

$$\alpha = \frac{a + \sqrt{a^2 + 4b}}{2} \quad \text{and} \quad \beta = \frac{a - \sqrt{a^2 + 4b}}{2},$$

satisfying $\alpha + \beta = a$, $\alpha\beta = -b$. By $\alpha > 1$, we have $a + b > 1$, satisfying $a^2 + 4b > 0$. Put

$$\alpha^n = s_n\alpha + t_n \quad \text{for} \quad n \geq 0.$$

Lemma 2.1. — *One has*

$$s_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad t_n = b\frac{\alpha^{n-1} - \beta^{n-1}}{\alpha - \beta} = bs_{n-1}. \tag{1}$$

$$s_{n-1}s_i - s_n s_{i-1} = (-b)^{i-1}s_{n-i} \quad (i = 1, 2, \ldots, n). \tag{2}$$

Proof. — (1) Since the recurrence relation $r_n = ar_{n-1} + br_{n-2}$ has the general solution

$$r_n = \frac{(\alpha^{n-1} - \beta^{n-1})r_2 - \alpha\beta(\alpha^{n-2} - \beta^{n-2})r_1}{\alpha - \beta},$$

by using $s_2 = a$, $s_1 = 1$, $t_2 = b$ and $t_1 = 0$ we have

$$s_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad t_n = b\frac{\alpha^{n-1} - \beta^{n-1}}{\alpha - \beta} = bs_{n-1}.$$

(2) For $i = 1, 2, \ldots, n$

$$\begin{aligned} s_{n-1}s_i - s_n s_{i-1} &= s_{n-1}(as_{i-1} + bs_{i-2}) - (as_{n-1} + bs_{n-2})s_{i-1} \\ &= -b(s_{n-2}s_{i-1} - s_{n-1}s_{i-2}) \\ &= (-b)^2(s_{n-3}s_{i-2} - s_{n-2}s_{i-3}) = \cdots \\ &= (-b)^{i-1}(s_{n-i}s_1 - s_{n-i+1}s_0) \\ &= (-b)^{i-1}s_{n-i}. \end{aligned}$$

□

By using s_n, y can be written as a linear form:

$$\begin{aligned} y &= \epsilon_0\alpha^n + \epsilon_1\alpha^{n-1} + \cdots + \epsilon_{n-1}\alpha + \epsilon_n \\ &= (\epsilon_0 s_n + \epsilon_1 s_{n-1} + \cdots + \epsilon_{n-2}s_2 + \epsilon_{n-1}s_1)\alpha \\ &\qquad + b(\epsilon_0 s_{n-1} + \epsilon_1 s_{n-2} + \cdots + \epsilon_{n-2}s_1) + \epsilon_n. \end{aligned}$$

Suppose that

$$\epsilon_0 s_n + \epsilon_1 s_{n-1} + \cdots + \epsilon_{n-2} s_2 + \epsilon_{n-1} s_1 = q_k, \tag{3}$$

$$b(\epsilon_0 s_{n-1} + \epsilon_1 s_{n-2} + \cdots + \epsilon_{n-2} s_1) + \epsilon_n = -p_k \tag{4}$$

for some integer k. Otherwise, we interchange each sign of ϵ_i $(i = 0, 1, \ldots, n)$. We shall find the least integer m (say, m') which satisfies $|\epsilon_i| \le m$ for all $i = 0, 1, \ldots, n$. By eliminating ϵ_0 we have

$$\begin{aligned} &b(s_{n-1}^2 - s_n s_{n-2})\epsilon_1 + b(s_{n-1}s_{n-2} - s_n s_{n-3})\epsilon_2 \\ &\quad + \cdots + b(s_{n-1}s_2 - s_n s_1)\epsilon_{n-2} + b s_{n-1}\epsilon_{n-1} - s_n \epsilon_n = q_k b s_{n-1} + p_k s_n. \end{aligned}$$

By Lemma 2.1(2), we obtain

$$\begin{aligned} &-(-b)^{n-1} s_1 \epsilon_1 - (-b)^{n-2} s_2 \epsilon_2 - \cdots - (-b)^2 s_{n-2}\epsilon_{n-2} \\ &\qquad + b s_{n-1}\epsilon_{n-1} - s_n \epsilon_n = q_k b s_{n-1} + p_k s_n. \end{aligned} \tag{5}$$

If $\gcd(a, b) = 1$, then we have $\gcd(s_{i+1}, b s_i) = 1$ $(i \ge 1)$, yielding

$$\gcd(b^{n-1}s_1, b^{n-2}s_2, \ldots, b s_{n-1}, s_n) = 1.$$

In fact, $\gcd(s_2, b s_1) = \gcd(a, b) = 1$. Assume that $\gcd(s_n, b s_{n-1}) = 1$ for some n. Suppose that, however,

$$\gcd(s_{n+1}, b s_n) = \gcd(a s_n + b s_{n-1}, b s_n) = c$$

with $c \ge 2$. Since $c \mid b s_n$, we have for some divisor of c, say $c_1 > 1$, $c_1 \mid b$ or $c_1 \mid s_n$. If $c_1 \mid b$ then by $c_1 \mid s_{n+1}$ and $c_1 \nmid a$ we have $c_1 \mid s_n$, yielding $\gcd(s_n, b s_{n-1}) = c_1$. If $c_1 \mid s_n$ then by $c_1 \mid s_{n+1}$ we have $c_1 \mid b s_{n-1}$, yielding $\gcd(s_n, b s_{n-1}) = c_1$, which is the contradiction again.

Therefore, the linear equation (5) is solvable in integers. We shall show the concrete step to obtain one of solutions in (5) in the following sections. $|\epsilon_n|$, $|\epsilon_{n-1}|$, …, $|\epsilon_1|$ can be chosen as lexicographically minimal among those giving $\ell^m(\alpha) = |q_k \alpha - p_k|$.

After choosing the integers from ϵ_n to ϵ_1, ϵ_0 can be naturally determined as an integer if $\gcd(a, b) = 1$. For, by (3) and (4)

$$\epsilon_0 = \frac{q_k - (\epsilon_1 s_{n-1} + \cdots + \epsilon_{n-1} s_1)}{s_n} = \frac{-p_k - b(\epsilon_1 s_{n-2} + \cdots + \epsilon_{n-2} s_1) - \epsilon_n}{b s_{n-1}}.$$

Since both of two fractions are integral and $\gcd(s_n, b s_{n-1}) = 1$, both of fractions cannot be the same unless ϵ_0 becomes integral.

We assume that $b = \pm 1$ in stating two theorems. This assumption guarantees that ϵ_1 becomes integral after deciding ϵ_n, ϵ_{n-1}, …, ϵ_2. Otherwise, ϵ_1 may not become integral by the method in this paper.

3. Proof of Theorem 1.1

Let $b=1$. Since the continued fraction expansion of α is $\alpha=[a;a,a,\dots]$, the sequence $(q_n)_n$ also satisfies the recurrence relation $r_n=ar_{n-1}+1\cdot r_{n-2}$, which appeared in the proof of Lemma 2.1(1), with $q_1=a$ and $q_0=1$. Hence, $s_n=q_{n-1}$ $(n\ge 0)$. Thus, we obtain

$$\begin{aligned}
q_ks_{n-1}+p_ks_n &= q_kq_{n-2}+q_{k+1}q_{n-1}\\
&= q_kq_{n-2}+q_{k+1}(aq_{n-2}+q_{n-3})\\
&= (aq_{k+1}+q_k)q_{n-2}+q_{k+1}q_{n-3}\\
&= q_{k+1}q_{n-3}+q_{k+2}q_{n-2}=\cdots\\
&= q_{n+k-2}q_0+q_{n+k-1}q_1=aq_{n+k-1}+q_{n+k-2}=q_{n+k}
\end{aligned}$$

and

$$\begin{aligned}
s_1+s_2+\cdots+s_{n-1}+s_n &= q_0+q_1+\cdots+q_{n-1}\\
&= \frac{1}{\alpha-\beta}\Big(\alpha\frac{\alpha^n-1}{\alpha-1}-\beta\frac{\beta^n-1}{\beta-1}\Big)\\
&= \frac{1}{a}\,\frac{\alpha^{n+1}-\beta^{n+1}+\alpha^n-\beta^n-(\alpha-\beta)}{\alpha-\beta}\\
&= \frac{1}{a}(s_{n+1}+s_n-1)=\frac{1}{a}(q_n+q_{n-1}-1).
\end{aligned}$$

Therefore, the equation (5) can be written as

$$-(-1)^{n-1}s_1\epsilon_1-(-1)^{n-2}s_2\epsilon_2-\cdots+s_{n-1}\epsilon_{n-1}-s_n\epsilon_n=q_{n+k}.$$

Since we would like to choose ϵ_n, ϵ_{n-1}, …, ϵ_1 so that $\max_i|\epsilon_i|$ is as small as possible, it is sufficient to consider the equation

$$s_1|\epsilon_1|+s_2|\epsilon_2|+\cdots+s_{n-1}|\epsilon_{n-1}|+s_n|\epsilon_n|=q_{n+k}. \tag{6}$$

We shall choose m' so that

$$m\ge m'=\inf_n\Big\lceil\frac{q_ks_{n-1}+p_ks_n}{s_1+s_2+\cdots+s_{n-1}+s_n}\Big\rceil=\inf_n\Big\lceil\frac{aq_{n+k}}{q_n+q_{n-1}-1}\Big\rceil \tag{7}$$

and take $|\epsilon_n|=m'$. This m' is the lower bound of m because if we take $|\epsilon_n|\le m'-1$, then at least one of $|\epsilon_1|,\dots,|\epsilon_{n-1}|$ must exceed m'.

We need the following lemma to complete the proof of Theorem 1.1.

LEMMA 3.1. — *The sequence*

$$\frac{aq_{n+k}}{q_n+q_{n-1}-1}$$

is a monotone decreasing function in n, tending to $\alpha^k(\alpha-1)$.

REMARK 3.2. — In general, if $a+b>1$ and $a+1>b>0$ then

$$\frac{q_k b s_{n-1}+p_k s_n}{s_{n+1}+s_n-b^n} \downarrow \frac{bq_k+\alpha_k}{\alpha(\alpha+1)} \qquad (n\to\infty).$$

Proof of Lemma 3.1. — Since $|\beta|<1<|\alpha|$, we have

$$\frac{aq_{n+k}}{q_n+q_{n-1}-1} = \frac{a(\alpha^k-(\beta/\alpha)^{n+1}\beta^k)}{1-(\beta/\alpha)^{n+1}+1/\alpha-\beta/\alpha^{n+1}-(\alpha-\beta)/\alpha^{n+1}}$$
$$\longrightarrow \frac{a\alpha^k}{1+1/\alpha} = \alpha^k(\alpha-1) \qquad (n\to\infty).$$

Putting

$$f(n)=\frac{q_k b s_{n-1}+p_k s_n}{s_{n+1}+s_n-b^n} \qquad (n=1,2,\dots),$$

show $f(n)>f(n+1)$, which is equivalent to

$$(8)\quad (q_{n+k}q_{n+1}-q_{n+k+1}q_n)+(q_{n+k}q_n-q_{n+k+1}q_{n-1})-q_{n+k}+q_{n+k+1}>0.$$

By Lemma 2.1(2) with $q_{n-1}=s_n$, for $k=0,1,2,\dots$

$$(q_{n+k}q_{n+1}-q_{n+k+1}q_n)+(q_{n+k}q_n-q_{n+k+1}q_{n-1}) = (-1)^{n-1}q_{k-1}+(-1)^n q_k.$$

If n is even, then the left-hand side of (8) $=q_k-q_{k-1}+q_{n+k+1}-q_{n+k}>0$.
If n is odd, then the left-hand side of (8) $\ge q_{k+2}-q_{k+1}-q_k+q_{k-1}\ge q_{k-1}>0$.
When $k=-1$, the left-hand side of (8) $=q_n-q_{n-1}+(-1)^{n-1}\ge 0$. The equality sign holds only when $a=b=1$ and $n=2$. □

We can find the sequence $\epsilon_{n-1},\dots,\epsilon_1$ and ϵ_0 one after another, as follows. Concerning the equation (5) or (6) we choose n' as the least integer satisfying

$$-\epsilon_{n'}=|\epsilon_{n'}|=\left\lceil\frac{q_{n'+k}}{s_1+s_2+\cdots+s_{n'}}\right\rceil = m' \quad \Big(=\inf_n\left\lceil\frac{aq_{n+k}}{q_n+q_{n-1}-1}\right\rceil\Big).$$

We can take any integer n with $n\ge n'$. It is, however, simple and easy to take $n=n'$ in the practical applications (see examples in section 5). Concerning the new equation

$$-(-1)^{n-1}s_1\epsilon_1-(-1)^{n-2}s_2\epsilon_2-\cdots-s_{n-2}\epsilon_{n-2}+s_{n-1}\epsilon_{n-1}=q_{n+k}+s_n\epsilon_n$$

or

$$|\epsilon_1|+a|\epsilon_2|+\cdots+s_{n-1}|\epsilon_{n-1}|=q_{n+k}-s_n|\epsilon_n|,$$

we take

$$\epsilon_{n-1}=\left\lceil\frac{q_{n+k}-s_n|\epsilon_n|}{s_1+s_2+\cdots+s_{n-1}}\right\rceil.$$

We repeat the similar step. For $i=n-1,n-2,\dots,2$ we take

$$(-1)^{n-i+1}\epsilon_i=|\epsilon_i|=\left\lceil\frac{q_{n+k}-s_n|\epsilon_n|-\cdots-s_{i+1}|\epsilon_{i+1}|}{s_1+s_2+\cdots+s_i}\right\rceil.$$

Since $|(-1)^{n-1}s_1| = 1$, ϵ_1 is also chosen as an integer with $|\epsilon_1| \le |\epsilon_2| \le \cdots \le |\epsilon_n| \le m$. Notice that if q_{n+k}, the right-hand side of (6), is small, $|\epsilon_1|$ may become negative in this algorithm (see [2], p. 21). But since $q_{n+k} > \bar{d}_n$, where

$$\bar{d}_1 = -1, \quad \bar{d}_i = \left\lceil \frac{\bar{d}_{i-1} + s_i}{\sum_{j=1}^{i-1} s_j} \right\rceil s_i + \bar{d}_{i-1} \quad (i = 2, \dots, n),$$

such a troublesome case does never happen and this algorithm works properly (see [2], Theorem 1). We omit this proof because it is clear if k is large; it is manually checked if k is small.

Finally, by substituting these $\epsilon_1, \epsilon_2, \dots, \epsilon_n$ into one of the equations (3) and (4) one finds ϵ_0, which is an integer because $\gcd(s_n, s_{n-1}) = \gcd(a, 1) = 1$. Note that $|\epsilon_0| \le m'$ because from (3) and (5)

$$\epsilon_0 s_n = q_k - |\epsilon_{n-1}|s_1 + |\epsilon_{n-2}|s_2 - \cdots + (-1)^{n-1}|\epsilon_1|s_{n-1}.$$

Thus, when n is odd,

$$\begin{aligned}
\epsilon_0 &\le \frac{1}{s_n}\Big(q_k + \frac{aq_{n+k}}{q_n + q_{n-1} - 1}(-s_1 + s_2 - \cdots - s_{n-2} + s_{n-1})\Big) \\
&= \frac{1}{q_{n-1}}\Big(q_k + \frac{aq_{n+k}}{q_n + q_{n-1} - 1}\,\frac{s_n - s_{n-1} - 1}{a}\Big) \\
&= \frac{q_{n+k} - q_{n+k-1} + q_k - q_{k-1}}{q_n + q_{n-1} - 1} \le \frac{aq_{n+k}}{q_n + q_{n-1} - 1}.
\end{aligned}$$

When n is even,

$$\begin{aligned}
\epsilon_0 &\ge \frac{1}{s_n}\Big(q_k + \frac{aq_{n+k}}{q_n + q_{n-1} - 1}(-s_1 + s_2 - \cdots + s_{n-2} - s_{n-1})\Big) \\
&= \frac{1}{q_{n-1}}\Big(q_k + \frac{aq_{n+k}}{q_n + q_{n-1} - 1}\,\frac{-s_n + s_{n-1} - 1}{a}\Big) \\
&= \frac{-q_{n+k} + q_{n+k-1} + q_k - q_{k-1}}{q_n + q_{n-1} - 1} \ge -\frac{aq_{n+k}}{q_n + q_{n-1} - 1}.
\end{aligned}$$

Theorem 1.1 includes the case for the golden number $\alpha = [1; 1, 1, \dots] = \frac{1}{2}(1 + \sqrt{5})$ (see [8], Theorem 3). Put $a = 1$. Since $q_{k-1} = F_k$ and $\alpha^k(\alpha - 1) = \alpha^{k-1}$, where (F_k) is the Fibonacci sequence defined by

$$F_0 = 0, \quad F_1 = 1, \quad F_k = F_{k-1} + F_{k-2} \qquad (k = 2, 3, \dots),$$

we have

$$\ell^m(\alpha) = |q_{k-1}\alpha - q_k| = |F_k\alpha - F_{k+1}|$$

if $\alpha^{k-2} < m \le \alpha^{k-1}$.

Notice that $\alpha^k(\alpha - 1) = 1$ and

$$\left\lceil \frac{aq_{n+k}}{q_n + q_{n-1} - 1} \right\rceil = 2 \qquad (n = 1, 2, \dots)$$

when $\alpha = \frac{1}{2}(1+\sqrt{5})$ and $k=1$. Therefore, the equality sign on the right-hand side is necessary in Theorem 1.1.

4. Proof of Theorem 1.2

Let $b<0$. Put $b' = -b > 0$ for convenience. By $\alpha\beta = b'$ and $\alpha+\beta = a$ we have

$$\begin{aligned} b'^{n-1}s_1 + b'^{n-2}s_2 + \cdots + b's_{n-1} + s_n &= \frac{1}{\alpha-\beta}\Big(\alpha\frac{\alpha^n - b'^n}{\alpha-b'} - \beta\frac{\beta^n - b'^n}{\beta-b'}\Big) \\ &= \frac{1}{a-b'-1}\,\frac{\alpha^{n+1}-\beta^{n+1}-\alpha^n+\beta^n-b'^n(\alpha-\beta)}{\alpha-\beta} \\ &= \frac{1}{a+b-1}\big(s_{n+1}-s_n-(-b)^n\big). \end{aligned}$$

Hence, the equation (5) becomes

$$-b'^{n-1}s_1\epsilon_1 - b'^{n-2}s_2\epsilon_2 - \cdots - b's_{n-1}\epsilon_{n-1} - s_n\epsilon_n = -q_kb's_{n-1} + p_ks_n$$

or

$$b'^{n-1}|\epsilon_1| + b'^{n-2}a|\epsilon_2| + \cdots + b's_{n-1}|\epsilon_{n-1}| + s_n|\epsilon_n| = -q_kb's_{n-1} + p_ks_n.$$

We shall choose m' so that

$$\begin{aligned} m \ge m' = |\epsilon_n| &= \inf_n \Big\lceil \frac{q_kbs_{n-1}+p_ks_n}{b'^{n-1}s_1 + b'^{n-2}s_2 + \cdots + b's_{n-1} + s_n} \Big\rceil \\ &= \inf_n \Big\lceil (a+b-1)\frac{q_kbs_{n-1}+p_ks_n}{s_{n+1}-s_n-(-b)^n} \Big\rceil. \end{aligned}$$

We need Lemma 4.1 and Lemma 4.3 to complete the proof of Theorem 1.2.

Lemma 4.1. — *If $a \ge 3$ and $b=-1$, then*

$$\frac{-q_ks_{n-1}+p_ks_n}{s_{n+1}-s_n-1}$$

is a monotone decreasing function in n, tending to $(-q_k+\alpha p_k)/(\alpha(\alpha-1))$.

Remark 4.2. — In general, if $a+b>1$ and $b<0$ then

$$\frac{q_kbs_{n-1}+p_ks_n}{s_{n+1}-s_n-(-b)^n} \downarrow \frac{bq_k+\alpha p_k}{\alpha(\alpha-1)} \qquad (n\to\infty).$$

Lemma 4.3. — *When $b=-1$, for $i=0,1,2,\dots$*

$$\begin{aligned} p_{2i-1} &= s_{i+1}, & p_{2i-2} &= s_{i+1}-s_i, \\ q_{2i-1} &= s_i, & q_{2i-2} &= s_i - s_{i-1}. \end{aligned}$$

Moreover,

$$-q_k s_{n-1} + p_k s_n = \begin{cases} q_{2n+k}, & \textit{if } k \textit{ is odd;} \\ q_{2n+k+1} - q_{2n+k-1}, & \textit{if } k \textit{ is even.} \end{cases}$$

Proof of Lemma 4.1. — One has

$$\begin{aligned} \frac{-q_k s_{n-1} + p_k s_n}{s_{n+1} - s_n - 1} &= \frac{-q_k(\alpha^{n-1} - \beta^{n-1}) + p_k(\alpha^n - \beta^n)}{\alpha^{n+1} - \beta^{n+1} - \alpha^n + \beta^n - (\alpha - \beta)} \\ &\longrightarrow \frac{-q_k/\alpha + p_k}{\alpha - 1} = \frac{-q_k + \alpha p_k}{\alpha(\alpha - 1)} \qquad (n \to \infty). \end{aligned}$$

Putting

$$g(n) = \frac{-q_k s_{n-1} + p_k s_n}{s_{n+1} - s_n - 1} \qquad (n = 1, 2, \dots),$$

show $g(n) > g(n+1)$, which is equivalent to

$$\begin{aligned} &(-q_k s_{n-1} + p_k s_n)(s_{n+2} - s_{n+1} - 1) - (-q_k s_n + p_k s_{n+1})(s_{n+1} - s_n - 1) \\ &\qquad = p_k(s_{n+1} - s_n - 1) - q_k\big((s_n - s_{n-1}) - (a - 1)\big) > 0 \end{aligned}$$

by Lemma 2.1(2).

Since the continued fraction expansion of α is $\alpha = [a - 1; 1, \dots]$, we get

$$a - 1 = \frac{p_0}{q_0} \le \frac{p_k}{q_k} \le \frac{p_1}{q_1} = a \qquad (k = 0, 1, 2, \dots).$$

Thus, we have

$$\begin{aligned} p_k(s_{n+1} - s_n - 1) &- q_k\big((s_n - s_{n-1}) - (a - 1)\big) \\ &\ge q_k\big((a - 1)(s_{n+1} - s_n - 1) - (s_n - s_{n-1}) + (a - 1)\big) \\ &> q_k\big((s_{n+1} - s_n) - (s_n - s_{n-1})\big) \\ &= q_k(\alpha^{n+1} - \alpha^n) > 0. \end{aligned}$$

When $k = -1$, it is clear that $g(n) > g(n+1)$ by $p_{-1} = 1$ and $q_{-1} = 0$. □

Proof of Lemma 4.3. — When $b = -1$, $\alpha = [a-1; 1, a-2, 1, a-2, 1, a-2, \dots]$. Thus, $s_0 = 0$, $s_1 = 1$, $s_2 = a$, $s_3 = a^2 - 1$, $s_4 = a^3 - 2a$, … and

$$\frac{p_0}{q_0} = \frac{a - 1}{1}, \quad \frac{p_1}{q_1} = \frac{a}{1}, \quad \frac{p_2}{q_2} = \frac{a^2 - a - 1}{a - 1}, \quad \frac{p_3}{q_3} = \frac{a^2 - 1}{a}, \quad \dots.$$

Then, the first part is followed by the induction.

By using the first part we obtain

$$\begin{aligned}-q_{2i-1}s_{n-1}+p_{2i-1}s_n &= -s_is_{n-1}+s_{i+1}s_n\\ &= (as_i-s_{i-1})s_n-s_is_{n-1}=s_i(as_n-s_{n-1})-s_{i-1}s_n\\ &= s_is_{n+1}-s_{i-1}s_n=s_{i-1}s_{n+2}-s_{i-2}s_{n+1}=\cdots\\ &= s_1s_{n+i}-s_0s_{n+i-1}=s_{n+1}=q_{2n+k}\end{aligned}$$

if $k=2i-1$, and

$$\begin{aligned}-q_{2i}s_{n-1}+p_{2i}s_n &= (-s_{i+1}s_{n-1}+s_{i+2}s_n)-(s_{i+1}s_n-s_is_{n-1})\\ &= s_{n+i+1}-s_{n+i}=q_{2n+k+1}-q_{2n+k-1}\end{aligned}$$

if $k=2i$. □

Put $b=-1$. If $k=2i-1$, then

$$m'=\inf_n\left\lceil (a-2)\frac{q_{2n+k}}{q_{2n+1}-q_{2n-1}-1}\right\rceil$$

and

$$\begin{aligned}\frac{a+b-1}{\alpha(\alpha-1)}(bq_k+\alpha p_k) &= \frac{a-2}{\alpha(\alpha-1)}(-s_i+\alpha s_{i+1})\\ &= \frac{(a-2)\alpha^{i+1}}{\alpha(\alpha-1)}=(\alpha-a+1)\alpha^i.\end{aligned}$$

If $k=2i$, then

$$m'=\inf_n\left\lceil (a-2)\frac{q_{2n+k+1}-q_{2n+k-1}}{q_{2n+1}-q_{2n-1}-1}\right\rceil$$

and

$$\begin{aligned}\frac{a+b-1}{\alpha(\alpha-1)}(bq_k+\alpha p_k) &= \frac{a-2}{\alpha(\alpha-1)}\left((\alpha s_{i+2}-s_{i+1})-(\alpha s_{i+1}-s_i)\right)\\ &= (a-2)\alpha^i.\end{aligned}$$

The process to determine $|\epsilon_{n-1}|$, $|\epsilon_{n-2}|$, …, $|\epsilon_2|$, $|\epsilon_1|$ and ϵ_0 is similar to the case where $b=1$.

Notice that $m'=\lceil a-2\rceil=a-2$ and

$$\left\lceil (a-2)\frac{q_{2n+1}-q_{2n-1}}{q_{2n+1}-q_{2n-1}-1}\right\rceil=a-1\qquad(n=1,2,\dots)$$

when $k=0$. Therefore, the equality sign is necessary in Theorem 1.2.

5. Examples

EXAMPLE 5.1. — Take $\alpha = \sqrt{2} + 1 = [2; 2, 2, \ldots]$. Then by Theorem 1.1 we obtain that

$$\ell^m(\alpha) = \begin{cases} 1 & \text{if } m = 1; \\ |q_0\alpha - p_0| = \alpha - 2 & \text{if } m = 2, 3; \\ |q_1\alpha - p_1| = 5 - 2\alpha & \text{if } 4 \le m \le 8; \\ |q_2\alpha - p_2| = 5\alpha - 12 & \text{if } 9 \le m \le 19; \\ |q_3\alpha - p_3| = 29 - 12\alpha & \text{if } 20 \le m \le 48; \\ |q_4\alpha - p_4| = 29\alpha - 70 & \text{if } 49 \le m \le 115. \end{cases}$$

We shall check $\ell^{49}(\alpha) = 29\alpha - 70$. Find the least integer n satisfying

$$\left\lceil \frac{q_{n+4}}{q_0 + \cdots + q_{n-1}} \right\rceil = 49.$$

Since

$$\left\lceil \frac{q_8}{q_0 + \cdots + q_3} \right\rceil = 50 \quad \text{and} \quad \left\lceil \frac{q_9}{q_0 + \cdots + q_4} \right\rceil = 49,$$

we can take $n = 5$ (of course, it is possible to take $n = 6, 7, \ldots$ too). Thus, substituting $q_1 = 2$, $q_2 = 5$, $q_3 = 12$, $q_4 = 29$ and $q_9 = 2378$ into $-\epsilon_1 + q_1\epsilon_2 - q_2\epsilon_3 + q_3\epsilon_4 - q_4\epsilon_5 = q_9$, we get the equation

$$-\epsilon_1 + 2\epsilon_2 - 5\epsilon_3 + 12\epsilon_4 - 29\epsilon_5 = 2378.$$

Thus, $-\epsilon_5 = \lceil 2378/(1 + 2 + 5 + 12 + 29) \rceil = \lceil 48.53 \ldots \rceil = 49$.

By $-\epsilon_1 + 2\epsilon_2 - 5\epsilon_3 + 12\epsilon_4 = 957$ we take

$$\epsilon_4 = \left\lceil \frac{957}{1 + 2 + 5 + 12} \right\rceil = \lceil 47.85 \rceil = 48.$$

By $-\epsilon_1 + 2\epsilon_2 - 5\epsilon_3 = 381$ we take

$$-\epsilon_3 = \left\lceil \frac{381}{1 + 2 + 5} \right\rceil = \lceil 47.625 \rceil = 48.$$

By $-\epsilon_1 + 2\epsilon_2 = 141$ we take

$$\epsilon_2 = \left\lceil \frac{141}{1 + 2} \right\rceil = 47.$$

Hence, $-\epsilon_1 = 47$. Now, substitute the known values into $\epsilon_0 q_3 + \epsilon_1 q_2 + \epsilon_2 q_1 + \epsilon_3 q_0 + \epsilon_5 = -q_5$ (or $\epsilon_0 q_4 + \epsilon_1 q_3 + \epsilon_2 q_2 + \epsilon_3 q_1 + \epsilon_4 = q_4$) we get

$$12\epsilon_0 - 47 \cdot 5 + 47 \cdot 2 - 48 \cdot 1 - 49 = -70,$$

yielding $\epsilon_0 = 14$. In fact,

$$\begin{aligned} y &= 14\alpha^5 - 47\alpha^4 + 47\alpha^3 - 48\alpha^2 + 48\alpha - 49 \\ &= 14(29\alpha + 12) - 47(12\alpha + 5) + 47(5\alpha + 2) - 48(2\alpha + 1) + 48\alpha - 49 \\ &= 29\alpha - 70. \end{aligned}$$

EXAMPLE 5.2. — Let $a = 3$ and $b = -1$. Then $\alpha = \frac{1}{2}(3 + \sqrt{5}) = [2; 1, 1, \dots]$, satisfying $\alpha^2 = 3\alpha - 1$. Notice that

$$\frac{p_0}{q_0} = \frac{2}{1}, \quad \frac{p_1}{q_1} = \frac{3}{1}, \quad \frac{p_2}{q_2} = \frac{5}{2}, \quad \frac{p_3}{q_3} = \frac{8}{3}, \quad \frac{p_4}{q_4} = \frac{13}{5}, \quad \frac{p_5}{q_5} = \frac{21}{8}, \dots$$

and

$$s_0 = 0,\ s_1 = 1,\ s_2 = 3,\ s_3 = 8,\ s_4 = 21,\ s_5 = 55,\ s_6 = 144,\ s_7 = 377, \dots.$$

Then by Theorem 1.2 the least integer m giving $|q_4\alpha - p_4| = 5\alpha - 13$ is

$$m \ge m' = \lceil (3-2)\alpha^2 \rceil = \left\lceil \frac{3\sqrt{5} + 7}{2} \right\rceil = \lceil 6.854\dots \rceil = 7.$$

Find the least integer n satisfying

$$\left\lceil (a-2) \frac{-q_k s_{n-1} + p_k s_n}{s_{n+1} - s_n - 1} \right\rceil = 49$$

though we can take an arbitrary large n. Since

$$\left\lceil (3-2) \frac{-5 \cdot 8 + 13 \cdot 21}{55 - 21 - 1} \right\rceil = \lceil 7.06\dots \rceil = 8 \quad \text{for} \quad n = 4$$

and

$$\left\lceil (3-2) \frac{-5 \cdot 21 + 13 \cdot 55}{144 - 55 - 1} \right\rceil = \lceil 6.93\dots \rceil = 7 \quad \text{for} \quad n = 5,$$

we can take $n = 5$. Thus, substituting the known quantities into $-\epsilon_1 - q_1\epsilon_2 - q_2\epsilon_3 - q_3\epsilon_4 - q_4\epsilon_5 = -q_4 s_4 + p_4 s_5$, we get the equation

$$-\epsilon_1 - 3\epsilon_2 - 8\epsilon_3 - 21\epsilon_4 - 55\epsilon_5 = 610.$$

Thus, $-\epsilon_5 = \lceil 610/(1 + 3 + 8 + 21 + 55) \rceil = \lceil 6.9\dots \rceil = 7$.

By $-\epsilon_1 - 3\epsilon_2 - 8\epsilon_3 - 21\epsilon_4 = 225$ we take

$$-\epsilon_4 = \left\lceil \frac{225}{1 + 3 + 8 + 21} \right\rceil = \lceil 6.8\dots \rceil = 7.$$

By $-\epsilon_1 - 3\epsilon_2 - 8\epsilon_3 = 78$ we take

$$-\epsilon_3 = \left\lceil \frac{78}{1 + 3 + 8} \right\rceil = \lceil 6.5 \rceil = 7.$$

By $-\epsilon_1 - 3\epsilon_2 = 22$ we take

$$-\epsilon_2 = \left\lceil \frac{22}{1 + 3} \right\rceil = \lceil 5.5 \rceil = 6.$$

Hence, $-\epsilon_1 = 4$. Now, substitute the known values into $-(\epsilon_0 s_4 + \epsilon_1 s_3 + \epsilon_2 s_2 + \epsilon_3 s_1) + \epsilon_5 = -p_4$ (or $\epsilon_0 s_5 + \epsilon_1 s_4 + \epsilon_2 s_3 + \epsilon_3 s_2 + \epsilon_4 = q_4$) we get

$$-21\epsilon_0 + 4 \cdot 8 + 6 \cdot 3 + 7 \cdot 1 - 7 = -13,$$

yielding $\epsilon_0 = 3$. In fact,

$$\begin{aligned} y &= 3\alpha^5 - 4\alpha^4 - 6\alpha^3 - 7\alpha^2 - 7\alpha - 7 \\ &= 3(55\alpha - 21) - 4(21\alpha - 8) - 6(8\alpha - 3) - 7(3\alpha - 1) - 7\alpha - 7 \\ &= 5\alpha - 13 \quad (= 0.09016\ldots). \end{aligned}$$

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