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EQUIDISTRIBUTION TOWARDS THE GREEN CURRENT

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Abstract. — Let $f : \mathbb{P}^k \to \mathbb{P}^k$ be a dominating rational mapping of first algebraic degree $\lambda \geq 2$. If S is a positive closed current of bidegree $(1,1)$ on $\mathbb{P}^k$ with zero Lelong numbers, we show – under a natural dynamical assumption – that the pullbacks $\lambda^{-n}(f^n)^*S$ converge to the Green current T_f. For some families of mappings, we get finer convergence results which allow us to characterize all f^*-invariant currents.

Résumé (*Équidistribution vers le courant de Green*). — Soit $f : \mathbb{P}^k \to \mathbb{P}^k$ une application rationnelle dominante de premier degré algébrique $\lambda \geq 2$. Lorsque S est un courant positif fermé de bidegré $(1,1)$ sur $\mathbb{P}^k$ dont les nombres de Lelong sont tous nuls, nous montrons, sous une hypothèse dynamique naturelle, que les pull-backs $\lambda^{-n}(f^n)^*S$ convergent vers le courant de Green T_f. Pour certaines familles d'applications, des résultats de convergence raffinés nous permettent de caractériser tous les courants f^*-invariants.

Introduction

Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be a rational map of degree $\lambda \geq 2$. A celebrated result of Brolin, Lyubich, Freire-Lopez-Mañe asserts that the preimages $\lambda^{-n}(f^n)^*\sigma$ of any probability measure σ on $\mathbb{P}^1$ converge to an invariant measure μ_f as soon as $\sigma(\mathcal{E}_f) = 0$, where $\mathcal{E}_f$ is a (possibly empty) finite exceptional set. The purpose of this note is to prove similar results in higher dimension.

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Let $f : \mathbb{P}^k \to \mathbb{P}^k$ be a rational mapping. It can be written $f = [P_0 : \cdots : P_k]$ in homogeneous coordinates, where the P_j's are homogeneous polynomials of the same degree λ (the first algebraic degree of f) with no common factor $P_0 \wedge \cdots \wedge P_k = 1$. Note that when $k \geq 2$, f is not necessarily holomorphic: it is not well defined on the indeterminacy set $I_f = \bigcap_j P_j^{-1}(0)$ which is an algebraic subset of $\mathbb{P}^k$ of codimension ≥ 2. There are several ways one can try to generalize the one-dimensional result. Given Z an algebraic subset of $\mathbb{P}^k$ of pure codimension p, one can ask whether $f^{-n}(Z)$ (properly normalized) converges to an invariant current of bidegree (p,p). In this note we focus on the case $p = 1$.

Given S a positive closed current of bidegree $(1,1)$ on $\mathbb{P}^k$, we consider

$$S_n := \lambda^{-n}(f^n)^* S.$$

This is a bounded sequence of positive closed currents of bidegree $(1,1)$ on $\mathbb{P}^k$. When $S = \omega$ is the Fubini-Study Kähler form, it was proved by Sibony [19] that (ω_n) converges to an invariant Green current T_f. On the other hand Russakovskii and Shiffman have shown [18] that $[H]_n - \omega_n \to 0$ for almost every hyperplane H of $\mathbb{P}^k$. Our main result interpolates between these two extreme cases.

Theorem 0.1. — *Let $f : \mathbb{P}^k \to \mathbb{P}^k$ be a dominating rational mapping with $\lambda \geq 2$. Assume there exists an invariant probability measure μ such that $\log|J_{FS}(f)| \in L^1(\mu)$. Let S be a positive closed current of bidegree $(1,1)$ and unit mass on $\mathbb{P}^k$. If $\nu(S,p) = 0$ for all $p \in \mathbb{P}^k$, then*

$$\frac{1}{\lambda^n}(f^n)^* S \longrightarrow T_f \quad \textit{in the weak sense of currents.}$$

Here $\nu(S,p)$ denotes the Lelong number of S at point p and $J_{FS}(f)$ denotes the jacobian of f with respect to the Fubini-Study volume form ω^k. Similar (weaker) results were previously obtained for Hénon mappings [1], [8], birational mappings [6], and holomorphic endomorphisms of $\mathbb{P}^k$ [9], [19], [7].

Although Theorem 0.1 does not imply directly Russakovskii-Shiffman's result, the proof shows one essentially has to assume $\sup_{p \in \mathbb{P}^k \setminus E} \nu(S_n, p) \to 0$, where E is some (possibly empty) exceptional set (see Theorem 1.4). The key ingredients of the proof are: a pluripotential estimate of the volume of sublevel sets of a quasiplurisubharmonic function [16] and a dynamical estimate on the decreasing of volumes under iteration (Theorem 1.2). Note that all the volumes are computed with respect to the Fubini-Study volume form ω^k.

We prove the volume estimates and our main result in Section 1. We give refinements of the latter in Section 2 in case f is an holomorphic endomorphism of $\mathbb{P}^k$ (Section 2.1) or a special type of polynomial endomorphism of $\mathbb{C}^k$ (Section 2.2). This allows us to characterize every f^*-invariant current. Such equidistribution results should be understood as strong ergodic properties of the Green current T_f. In dimension 1 indeed this implies that T_f is strongly mixing (see Theorem VIII.22 in [2]). For the reader's convenience we recall in an Appendix compactness criteria for families of qpsh functions. They are the higher dimensional counterparts of Montel's Theorem.

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1. Equidistribution of pullbacks of currents

Let $f : \mathbb{P}^k \to \mathbb{P}^k$ be a rational mapping with first algebraic degree $\lambda \geq 2$. We always assume f is dominating, *i.e.* its jacobian does not vanish identically in any coordinate chart. It follows that a generic point has $d_t(f)$ well defined preimages by f. Note that $d_t(f) = \lambda$ when $k = 1$ but these two degrees differ in general when $k \geq 2$.

Let ω denote the Fubini-Study Kähler form on $\mathbb{P}^k$. The smooth form $f^*\omega$ is well defined in $\mathbb{P}^k \setminus I_f$ and extends trivially through I_f as a positive closed current of bidegree $(1,1)$ and mass $\|f^*\omega\| = \int_{\mathbb{P}^k} f^*\omega \wedge \omega^{k-1} = \lambda$. So $\lambda^{-1} f^*\omega$ is cohomologous to ω. Since $\mathbb{P}^k$ is Kähler, this can be written

$$\lambda^{-1} f^*\omega = \omega + \mathrm{dd}^c G,$$

where $G \in L^1(\mathbb{P}^k)$ (see [11, p. 149]). The function G is "quasiplurisubharmonic" (qpsh): it is locally given as the sum of a psh function (a local potential of $\lambda^{-1} f^*\omega \geq 0$) and a smooth function (a local potential of $-\omega$). In particular it is bounded from above on $\mathbb{P}^k$: replacing G by $G - C_0$, we can therefore assume $G \leq 0$. Sibony [19] has shown that the decreasing sequence of qpsh functions

$$(*) \qquad G_n := \sum_{j=0}^{n-1} \frac{1}{\lambda^j} G \circ f^j$$

converges in $L^1(\mathbb{P}^k)$ to a qpsh function $G_\infty \in L^1(\mathbb{P}^k)$. This shows that $\lambda^{-n}(f^n)^*\omega$ converges in the weak sense of (positive) currents to the so called Green current $T_f \geq 0$ which satisfies $f^*T_f = \lambda T_f$.

A natural question is then to look at the convergence of $S_n := \lambda^{-n}(f^n)^*S$, where S is now any positive closed current of bidegree $(1,1)$ and unit mass on $\mathbb{P}^k$. When $S = [H]$ is the current of integration along an hyperplane of $\mathbb{P}^k$, it was shown by Russakovskii and Shiffman [18] that $[H]_n \to T_f$ for every H outside some pluripolar set $\mathcal{E} \subset (\mathbb{P}^k)^*$. In order to prove convergence of S_n for more general currents S, we first need to get control on the decreasing of volumes under iteration.

1.1. Volume estimates. — Let $f : \mathbb{P}^k \to \mathbb{P}^k$ be a dominating rational mapping with $\lambda \geq 2$. Let $J_{FS}(f)$ denote its jacobian with respect to the Fubini-Study Kähler volume form. It is defined by

$$f^*\omega^k = \bigl|J_{FS}(f)\bigr|^2\omega^k.$$

PROPOSITION 1.1. — *Fix B an open subset of $\mathbb{P}^k$ and $\delta_0 > 0$. There exists $C_0 > 0$ such that for every open subset Ω of $\mathbb{P}^k$ with* $\operatorname{vol}(\Omega) \geq \delta_0$*,*

$$\operatorname{vol}\bigl(f^n(\Omega)\bigr) \geq (C_0)^{\lambda^n} \exp\Bigl(\frac{1}{\operatorname{vol}(B)\operatorname{vol}(\Omega)} \int_B \log\bigl|J_{FS}(f^n)\bigr|^2\omega^k\Bigr)$$

for all $n \in \mathbb{N}$.

Proof. — Fix $\mathbb{C}^k$ an affine chart of $\mathbb{P}^k$. We have

$$f = (P_1/P_0, \ldots, P_k/P_0)$$

in $\mathbb{C}^k$ where the P_j's are polynomials of degree $\leq \lambda = \delta_1(f)$. Since $\omega = \mathrm{dd}^c\frac{1}{2}\log[1 + \|z\|^2]$ in $\mathbb{C}^k$, we get

$$\omega^k(z) = \bigl(1 + \|z\|^2\bigr)^{-(k+1)}\mathrm{d}V,$$

where $\mathrm{d}V$ denotes the euclidean volume form in $\mathbb{C}^k$. Therefore

$$\bigl|J_{FS}(f)\bigr|^2 = \bigl|J_{\mathrm{eucl}}(f)\bigr|^2\Bigl(\frac{1 + \|z\|^2}{1 + \|f(z)\|^2}\Bigr)^{k+1}.$$

We infer

$$\log|J_{FS}(f)| = u - v,$$

where u, v are qpsh functions such that $\mathrm{dd}^c u, \mathrm{dd}^c v \geq -2\lambda k\omega$. Let Ω be an open subset of $\mathbb{P}^k$. We have

$$\operatorname{vol}\bigl(f^n(\Omega)\bigr) = \int_{f^n(\Omega)} \omega^k \geq \frac{1}{d_t(f)^n} \int_\Omega \bigl|J_{FS}(f^n)\bigr|^2\omega^k,$$

where the inequality follows from the change of variable formula. The concavity of the logarithm yields

$$\operatorname{vol}\bigl(f^n(\Omega)\bigr) \geq \frac{\operatorname{vol}(\Omega)}{d_t(f)^n} \exp\Bigl[\frac{2D_n}{\operatorname{vol}(\Omega)} \int_\Omega \frac{1}{D_n}(u_n - v_n)\omega^k\Bigr],$$

where

$$\log|J_{FS}(f^n)| = u_n - v_n$$

with $\mathrm{dd}^c u_n, \mathrm{dd}^c v_n \geq -D_n\omega$, $D_n \leq 2\lambda^n k$. Observe that we can decompose

$$\frac{1}{D_n}\log\bigl|J_{FS}(f^n)\bigr| = \varphi_n - \psi_n + \frac{1}{\operatorname{vol}(B)} \int_B \frac{1}{D_n}\log\bigl|J_{FS}(f^n)\bigr|\omega^k,$$

where

$$\varphi_n = D_n^{-1}u_n - \log\|z\| - \frac{1}{\mathrm{vol}(B)}\int_B \big(D_n^{-1}u_n - \log\|z\|\big)\omega^k,$$
$$\psi_n = D_n^{-1}v_n - \log\|z\| - \frac{1}{\mathrm{vol}(B)}\int_B \big(D_n^{-1}v_n - \log\|z\|\big)\omega^k.$$

The functions φ_n, ψ_n are quasiplurisubharmonic on $\mathbb{P}^k$ ($\mathrm{dd}^c\varphi_n, \mathrm{dd}^c\psi_n \geq -\omega$) with $\int_B \varphi_n = \int_B \psi_n = 0$. It follows therefore from Proposition 3.2 (Appendix) that they belong to a compact family of qpsh functions, so there exists $C_\Omega \in \mathbb{R}$ such that $\int_\Omega (\varphi_n - \psi_n)\omega^k \geq C_\Omega$, for all $n \in \mathbb{N}$. Since $D_n \leq 2k\lambda^n$, this yields the desired inequality. □

It remains to get a lower bound on $\int_B \log|J_{FS}(f^n)|\omega^k$, where B is an open subset which we may fix as we like.

Theorem 1.2. — *Let $f : \mathbb{P}^k \to \mathbb{P}^k$ be a dominating rational mapping with $\lambda \geq 2$. Assume there exists an invariant probability measure μ such that $\log|J_{FS}(f)| \in L^1(\mu)$. Fix $\delta_0 > 0$. Then there exists $C_0 > 0$ such that for every open subset Ω of $\mathbb{P}^k$ with* $\mathrm{vol}(\Omega) \geq \delta_0$,

$$\mathrm{vol}\big(f^n(\Omega)\big) \geq C_0^{\lambda^n}, \quad \forall n \in \mathbb{N}.$$

Proof. — Using Proposition 1.1, it is sufficient to find $M > 0$ such that for all n, $\int_B \log|J_{FS}(f^n)|\omega^k \geq -M\lambda^n$. We take here $B = \mathbb{P}^k$ (but other normalisations could be useful, see Remark 1.3 below).

We decompose $\lambda^{-1}\log|J_{FS}(f)| = u - v + C$, where u, v are qpsh functions ($\mathrm{dd}^c u, \mathrm{dd}^c v \geq -\omega$) such that $\sup_{\mathbb{P}^k} u = \sup_{\mathbb{P}^k} v = 0$ and $C \in \mathbb{R}$. Thus we get

$$\frac{1}{\lambda^n}\log\big|J_{FS}(f^n)\big| = \frac{1}{\lambda^n}\sum_{j=0}^{n-1}\log\big|J_{FS}(f)\circ f^j\big| \geq \sum_{j=0}^{n-1}\frac{1}{\lambda^{n-j}}u_j + \frac{n}{\lambda^n}C,$$

where $u_j := \lambda^{-j}u \circ f^j$. It is therefore sufficient to get a uniform lower bound on $\int_{\mathbb{P}^k} u_j\omega^k$. This is a consequence of the fact that (u_j) is relatively compact in $L^1(\mathbb{P}^k)$. Indeed $\mathrm{dd}^c u_j \geq -\lambda^{-j}(f^j)^*\omega$, so $u_j + G_j$ is qpsh. By Lemma 3.1 (Appendix), the sequence $(u_j + G_j)$ is either relatively compact or uniformly converges to $-\infty$. Since $G_j \to G_\infty \in L^1(\mathbb{P}^k)$, the sequence (u_j) is either relatively compact or converges to $-\infty$. But the latter can not happen since $u \in L^1(\mu)$ and $\int u_j\,\mathrm{d}\mu = \lambda^{-j}\int u\,\mathrm{d}\mu \to 0$. The desired control on $\int_{\mathbb{P}^k}\log|J_{FS}(f^n)|\omega^k$ follows. □

Remark 1.3. — The assumption on the existence of μ is natural in our dynamical context. Observe that it is satisfied if *e.g.* there exists a non critical periodic point.

Other assumptions could be made to obtain the final lower bound on $\int_B \log|J_{FS}(f^n)|\omega^k$. If $f_{|\mathbb{C}^k}$ is polynomial, it is enough to assume that

$\sup_B |J_{\mathrm{eucl}}(f^n)| \geq \alpha^{\lambda^n}$ for some relatively compact open subset B of $\mathbb{C}^k$. This holds in particular when f is a polynomial automorphism of $\mathbb{C}^k$.

1.2. Equidistribution of pullbacks. — A major difficulty in higher dimensional complex dynamics lies in the presence of points of indeterminacy and in the difficulty of analyzing the dynamics near

$$I_f^\infty := \bigcup_{n\geq 0} I_{f^n}.$$

Following Fornaess and Sibony [9], a rational mapping $f : \mathbb{P}^k \to \mathbb{P}^k$ is said to be *normal* if for every $p \in \mathbb{P}^k \setminus \overline{I_f^\infty}$, there exists a neighborhood W of p and V of $\overline{I_f^\infty}$ such that $f^n(W) \cap V = \varnothing$ for all $n \geq 0$. Note that the condition is empty if $\overline{I_f^\infty} = \mathbb{P}^k$, so one usually assumes further that $\overline{I_f^\infty}$ is "small", *e.g.* $\mathrm{vol}(\overline{I_f^\infty}) = 0$. Examples of such mappings include holomorphic endomorphisms of $\mathbb{P}^k$ (for which $I_f = I_f^\infty = \varnothing$) or some polynomial endomorphisms of $\mathbb{C}^k$ with small topological degree (*e.g.* Hénon mappings in $\mathbb{C}^2$, see other examples in [12], [13] and Section 2.2 below).

THEOREM 1.4. — *Let $f : \mathbb{P}^k \to \mathbb{P}^k$ be a dominating rational mapping with $\lambda \geq 2$. Assume there exists an invariant probability measure μ on $\mathbb{P}^k$ such that $\log|J_{FS}(f)| \in L^1(\mu)$. Let S be a positive closed current of bidegree $(1,1)$ and unit mass on $\mathbb{P}^k$. If $\nu(S,p) = 0$ for all $p \in \mathbb{P}^k$, then*

$$S_n := \frac{1}{\lambda^n}(f^n)^* S \longrightarrow T_f.$$

Moreover if f is normal with $\mathrm{vol}(\overline{I_f^\infty}) = 0$ then $S_n \to T_f$ if and only if $\nu(S_n, p) \to 0$ uniformly on compact subsets of $\mathbb{P}^k \setminus \overline{I_f^\infty}$.

Proof. — Write $S = \omega + \mathrm{dd}^c u$, where $u \leq 0$ is quasiplurisubharmonic on $\mathbb{P}^k$. Then $\lambda^{-n}(f^n)^*(S - \omega) = \mathrm{dd}^c u_n$, where $u_n = \lambda^{-n} u \circ f^n \leq 0$. So we need to show that $u_n \to 0$ in $L^1(\mathbb{P}^k)$.

Observe that $\mathrm{dd}^c(u_n + G_n) \geq -\omega$, so (u_n) is either relatively compact or uniformly converges to $-\infty$ (Lemma 3.1). It is therefore sufficient to prove that for all $\varepsilon > 0$, $\mathrm{vol}(\Omega_n^\varepsilon) \to 0$, where

$$\Omega_n^\varepsilon := \Big\{ p \in \mathbb{P}^k \,;\, \frac{1}{\lambda^n} u \circ f^n(p) < -\varepsilon \Big\}.$$

Assume on the contrary that $\mathrm{vol}(\Omega_{n_i}^\varepsilon) \geq \delta_0$ for some fixed ε, $\delta_0 > 0$ and $n_i \to +\infty$. Observe that

$$f^{n_i}(\Omega_{n_i}^\varepsilon) \subset \big\{ p \in \mathbb{P}^k \,;\, u(p) < -\varepsilon\lambda^{n_i} \big\}.$$

If $\nu(S,p) = 0$ for all $p \in \mathbb{P}^k$, it follows from Skoda's integrability Theorem (see Theorem 3.1 in [16]) that for every $A > 0$, there exists $C_A > 0$ such that

$$\mathrm{vol}\big(f^{n_i}(\Omega_{n_i}^\varepsilon)\big) \leq C_A \exp(-A\varepsilon\lambda^{n_i}).$$

On the other hand, since $\mathrm{vol}(\Omega_{n_i}^{\varepsilon}) \geq \delta_0 > 0$, there follows from Theorem 1.2 that there exists $C_0 > 0$ such that

$$\mathrm{vol}(f^{n_i}(\Omega_{n_i}^{\varepsilon})) \geq C_0^{\lambda^{n_i}}, \text{ for all } i \in \mathbb{N}.$$

Taking $A > -\log C_0$ yields a contradiction.

Assume now f is normal. It was proved by Favre [5] that $\nu(T_f, p) = 0$ for all $p \in \mathbb{P}^k \setminus I_f^\infty$. Therefore it is necessary, for S_n to converge to T_f, that for every open neighborhood V of $\overline{I_f^\infty}$, $\sup_{p\in\mathbb{P}^k\setminus V} \nu(S_n, p) \to 0$. This is because $(S, p) \mapsto \nu(S, p)$ is upper semi-continuous (u.s.c.). Assume it is the case. Fix W a relatively compact open subset of $\mathbb{P}^k \setminus \overline{I_f^\infty}$. Since $\mathrm{Vol}(\overline{I_f^\infty}) = 0$, it is sufficient to prove that $u_n \to 0$ on every such W. Since f is normal, we can fix V an open neighborhood of $\overline{I_f^\infty}$ such that $V \cap f^n(W) = \varnothing$, for all $n \geq 0$. We need to prove that $\mathrm{vol}(W \cap \Omega_n^{\varepsilon}) \to 0$. Now

$$f^n(W \cap \Omega_n^{\varepsilon}) \subset \{p \in \mathbb{P}^k \setminus V\,; u(p) < -\varepsilon\lambda^n\},$$

so the previous proof applies if $\sup_{p\in\mathbb{P}^k\setminus V} \nu(S, p)$ is small enough. When $\sup_{p\in\mathbb{P}^k\setminus V} \nu(S, p)$ is not small enough, we replace S by S_{N_0}, $N_0 \gg 1$. □

2. Invariant currents

It is an interesting problem to characterize all positive closed currents S of bidegree $(1, 1)$ on $\mathbb{P}^k$ such that $f^*S = \lambda S$. This can be done by using our equidistribution result (Theorem 1.4). We illustrate this on two families of mappings.

2.1. Holomorphic endomorphisms of $\mathbb{P}^k$. — We assume here $f : \mathbb{P}^k \to \mathbb{P}^k$ is holomorphic, *i.e.* $I_f = \varnothing$. In this case the construction of the Green current T_f is due to Hubbard and Papadopol [15]: G is smooth on $\mathbb{P}^k$, so (G_n) uniformly converges to G_∞ which is henceforth continuous (see $(*)$).

Since the Green current T_f has continuous potential, all its Lelong numbers are 0. Moreover, it follows from the work of Bedford and Taylor that the measure $\mu_f := T_f^k$ is well defined. The measure μ_f is invariant and every qpsh function is μ_f integrable (as follows from the Chern-Levine-Nirenberg inequalities, see the Appendix in [19]). Therefore f satisfies the assumptions of Theorem 1.4. Given S a positive closed current of bidegree $(1, 1)$ and unit mass on $\mathbb{P}^k$, Theorem 1.4 reads here

$$\frac{1}{\lambda^n}(f^n)^*S \to T_f \iff \sup_{p\in\mathbb{P}^k} \nu(\lambda^{-n}(f^n)^*S, p) \to 0.$$

It remains to understand the behavior of Lelong numbers under iteration. Since f is proper, one easily gets

$$\nu((f^n)^*S, p) \leq d(f^n, p)\nu(S, f^n(p)),$$

where $d(f^n,p)$ denotes the local topological degree of f^n at p, $d(f^n,p) = \Pi_{j=0}^{n-1} d(f,f^j(p))$. So we are done if $d(f^n,p) = o(\lambda^n)$.

Analyzing the behavior of $d(f^n,p)$ is quite easy in dimension 1 as shows the following elementary lemma whose proof is left to the reader.

LEMMA 2.1. — *Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be a rational map of degree $\lambda \geq 2$. Set*

$$\mathcal{E}_f := \{p \in \mathbb{P}^1 \,;\, d(f,p) = d(f,f(p)) = d(f,f^2(p)) = \lambda\}.$$

Then either

- *$\mathcal{E}_f$ is empty, or*
- *$\mathcal{E}_f = 1$ point, f is conjugate to a polynomial, or else*
- *$\mathcal{E}_f = 2$ points, f is conjugate to z^λ or $z^{-\lambda}$.*

Combining this with Theorem 1.4 yields the following celebrated result of Brolin [3], Lyubich [17] and Freire-Lopez-Mañe [10]. Note that positive closed currents of bidegree $(1,1)$ and unit mass are simply probability measures on $\mathbb{P}^1$.

THEOREM 2.2. — *Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be a rational map of degree $\lambda \geq 2$. Let σ be a probability measure on $\mathbb{P}^1$. Then*

$$\frac{1}{\lambda^n}(f^n)^*\sigma \to T_f \iff \sigma(\mathcal{E}_f) = 0.$$

When $k \geq 2$ the "crude" estimate $d(f^n,p) \leq d_t(f^n) = \lambda^{nk}$ becomes worse as the dimension grows. Nevertheless, one still has that $d(f^n,p) = O((\lambda-1)^n)$ for a "very generic" choice of f (*i.e.* for f outside a countable union of hypersurfaces), so we get the following result of Fornaess and Sibony [9].

COROLLARY 2.3. — *Let $f : \mathbb{P}^k \to \mathbb{P}^k$ be a "very generic" holomorphic mapping with $\lambda = \delta_1(f) \geq 2$. Then*

$$\lambda^{-n}(f^n)^*S \longrightarrow T_f$$

for every positive closed current S of bidegree $(1,1)$ and unit mass on $\mathbb{P}^k$. In particular T_f is the only f^*-invariant current.

It turns out that looking at local topological degrees is not sufficient to settle the problem of convergence to T_f when $k \geq 2$. Our volume estimates (Theorem 1.2) nevertheless allow us to complete the recent work of Favre and Jonsson [7] in dimension 2.

THEOREM 2.4. — *Let $f : \mathbb{P}^2 \to \mathbb{P}^2$ be a holomophic mapping with $\lambda \geq 2$. There exists a (possibly empty) totally invariant algebraic subset $\mathcal{E}_f$ of $\mathbb{P}^2$ such that if S is a positive closed current of bidegree $(1,1)$ and unit mass on $\mathbb{P}^k$, then*

$$\nu(S,\mathcal{E}_f) = 0 \implies \frac{1}{\lambda^n}(f^n)^*S \to T_f.$$

The set $\mathcal{E}_f$ can be decomposed as $\mathcal{E}_1 \cup \mathcal{E}_2$, where $\mathcal{E}_1$ is a union of at most three lines and $\mathcal{E}_2$ is a finite set. The condition $\nu(S, \mathcal{E}_f) = 0$ has to be understood as $\nu(S, p) = 0$ for every point $p \in \mathcal{E}_2$ and almost every point p of $\mathcal{E}_1$. We refer the reader to [7] for a proof. The only new information we provide are sufficient volume estimates near points of $\mathcal{E}_2$, without the extra assumption made by Favre and Jonsson that $\mathcal{E}_2$ consists of "homogeneous points".

2.2. Some polynomial endomorphisms of $\mathbb{C}^k$. — Let $f : \mathbb{C}^k \to \mathbb{C}^k$ be a proper polynomial mapping, $f = (P_1, \dots, P_k)$, where the P_j's are polynomials with $\lambda = \max(\deg P_j) \geq 2$. We let $d_t(f)$ denote the topological degree of f and shall assume here that $d_t(f) < \lambda$. Given S a positive closed current of bidegree $(1,1)$ and unit mass on $\mathbb{P}^k$, we thus get

$$\sup_{p \in \mathbb{C}^k} \nu(S_n, p) \leq \frac{\sup_{p \in \mathbb{C}^k} d(f^n, p)}{\lambda^n} \sup_{p \in \mathbb{C}^k} \nu\big(S, f^n(p)\big) \leq \Big(\frac{d_t(f)}{\lambda}\Big)^n \to 0,$$

where $S_n := \lambda^{-n}(f^n)^* S$.

We still denote by f the meromorphic extension of f to $\mathbb{P}^k = \mathbb{C}^k \cup (t = 0)$, where $(t = 0)$ denotes the hyperplane at infinity. Since f is polynomial (hence holomorphic) in $\mathbb{C}^k$, the indeterminacy set I_f is located within $(t = 0)$. Define by induction

$$X_1 := \overline{f\big((t = 0) \setminus I_f\big)}, \quad X_{j+1} := \overline{f(X_j \setminus I_f)}.$$

This is a decreasing sequence of irreducible analytic subsets of $(t = 0)$. We denote by $X_f := X_\ell$ the limit set, which we assume is non empty (this is equivalent to saying that f is *algebraically stable*, see [19]).

THEOREM 2.5. — *Let $f : \mathbb{C}^k \to \mathbb{C}^k$ be an algebraically stable polynomial endomorphism with $d_t(f) < \lambda = \delta_1(f)$. Assume I_f is an f^{-1}-attracting set and there exists an invariant probability measure μ such that $\log |J_{FS}(f)| \in L^1(\mu)$. Let S be a positive closed current of bidegree $(1,1)$ and unit mass on $\mathbb{P}^k$. If $\nu(S, p) = 0$ for all $p \in X_f$, then*

$$S_n := \frac{1}{\lambda^n}(f^n)^* S \longrightarrow T_f.$$

When $\dim X_f = 0$, T_f is the only f^*-invariant current of unit mass in $\mathbb{C}^k$.

Proof. — We assume I_f is f^{-1}-attracting in the following sense: there exists an open neighborhood V of I_f in $\mathbb{C}^k$ such that $\bigcap_{j \geq 0} f^{-j}(V) = \varnothing$. This implies that f is a normal mapping. Note also that f is proper in $\mathbb{C}^k$: if an algebraic curve $\mathcal{C} \subset \mathbb{C}^k$ were to be contracted to a point by f, $\mathcal{C}$ would intersect $(t = 0)$ within I_f, contradicting that I_f is f^{-1}-attracting. Therefore $\sup_{p \in \mathbb{C}^k} \nu(S_n, p) \to 0$ as observed above. If $q \in (t = 0) \setminus I_{f^\ell}$ then

$$\nu\big((f^\ell)^* S, q\big) \leq C_{f,q} \nu(S, f^\ell(q)) = 0, \text{ since } f^\ell(q) \in X_f = X_\ell.$$

The latter inequality is due to Favre [4] and Kiselman [16]. Therefore $\sup_{p\in\mathbb{P}^k\setminus I_{f^\ell}} \nu(S_n,p)\to 0$ so $S_n \to T_f$ by Theorem 1.4.

When $\dim X_f = 0$, X_f is reduced to a point which does not belong to I_f. In this case $\nu(S,X_f)=0$ is also a necessary condition to get $S_n\to T_f$. Indeed if $\nu(S,X_f)=\gamma>0$ then for all $q\in(t=0)\setminus I_{f^\ell}$,

$$\nu\big((f^\ell)^*S,q\big)\geq \nu\big(S,f^\ell(q)\big)=\gamma>0 \quad \text{so} \quad (f^\ell)^*S\geq\gamma[t=0]$$

hence S_n does not converge to T_f. In particular if S is f^*-invariant and does not charge the hyperplane at infinity then $\nu(S,X_f)=0$ hence $S=S_n\to T_f$, so $S=T_f$. Therefore every f^*-invariant current is a linear combination of T_f and $[t=0]$. □

Remark 2.6. — The previous result applies *e.g.* to mappings

$$f=f_1\circ\cdots\circ f_s,\quad f_j:(z,w)\in\mathbb{C}^2\mapsto\big(P_j(w),Q_j(z)+R_j(w)\big)\in\mathbb{C}^2,$$

where P_j,Q_j,R_j are polynomials of degree p_j,q_j,λ_j with $p_jq_j<\lambda_j$. One gets here

$$d_t(f)=\prod_{j=1}^s p_jq_j \quad\text{and}\quad \lambda:=\delta_1(f)=\prod_{j=1}^s\lambda_j>d_t(f).$$

The set $I_f=[1:0:0]$ is f^{-1}-attracting (see Example 4.1 in [12]) and $X_f=[0:1:0]$. An invariant measure μ such that $\log|J_{FS}(f)|\in L^1(\mu)$ is constructed in [12] (Theorem 5.3). Note that when $p_j=q_j=1$ then f is a complex Hénon mapping. A different proof of Theorem 2.5 was given in this case in the paper [6], where f^*-invariant currents are characterized for every birational mapping of $\mathbb{C}^2$.

We now consider the case of polynomial automorphisms of $\mathbb{C}^k$. We can prove finer volume estimates on the set of points whose orbit accumulates I_f so that it is not necessary to assume f is normal.

Theorem 2.7. — *Let $f:\mathbb{C}^k\to\mathbb{C}^k$ be an algebraically stable polynomial automorphism of $\mathbb{C}^k$ such that X_f is an f-attracting set. Let S be a positive closed current of bidegree $(1,1)$ and unit mass on $\mathbb{P}^k$. If $\nu(S,p)=0$ for all $p\in X_f$, then*

$$S_n:=\frac{1}{\lambda^n}(f^n)^*S\longrightarrow T_f.$$

If $\dim X_f=0$ then T_f is the only f^*-invariant current of unit mass in $\mathbb{C}^k$.

Proof. — As observed in Remark 1.3, it is not necessary to assume the existence of an invariant measure μ such that $\log|J_{FS}(f)|\in L^1(\mu)$. We use instead the fact that the euclidean jacobian of f is constant in $\mathbb{C}^k$ to derive a lower bound on $\int_B\log|J_{FS}(f^n)|\omega^k$.

We let $B(X_f)$ denote the basin of attraction of X_f in $\mathbb{C}^k$ and set

$$\mathcal{K}_f := \mathbb{C}^k \setminus B(X_f).$$

Since f is proper in $\mathbb{C}^k$ with $d_t(f) = 1 < \lambda$, we get as before

$$\sup_{p\in\mathbb{P}^k\setminus I_{f^\ell}} \nu(S_n, p) \longrightarrow 0.$$

Therefore $S_n \to 0$ in $B(X_f)$. Now we must make sure that $S_n \to 0$ on any relatively compact open subset B of $\mathcal{K}_f$ (if any). Write

$$S = \omega + \mathrm{dd}^c u,$$

where $u \le 0$, $u \in L^1(\mathbb{P}^k)$. We need to prove that for all $\varepsilon > 0$,

$$\operatorname{vol}\big(B \cap \{u_n < -\varepsilon\}\big) \longrightarrow 0,$$

where $u_n = \lambda^{-n} u \circ f^n$. Now $f^n(B \cap \{u_n < -\varepsilon\}) \subset \{u < -\varepsilon\lambda^n\}$ has volume $\le C_1 \exp(-\varepsilon\lambda^n)$ because the Lelong numbers of S_n are bounded from above by 1. On the other hand

$$\begin{aligned}\operatorname{vol}\big(f^n(B \cap \{u_n < -\varepsilon\})\big) &= \int_{B\cap\{u_n<-\varepsilon\}} \big|J_{FS}(f^n)\big|^2 \omega^k \\ &= |a|^{2n} \int_{B\cap\{u_n<-\varepsilon\}} \Big(\frac{1+\|z\|^2}{1+\|f^n(z)\|^2}\Big)^{k+1} \omega^k \\ &\ge C_B^{(\lambda-1)^n} \operatorname{vol}\big(B \cap \{u_n < -\varepsilon\}\big),\end{aligned}$$

where $|a| := |J_{\mathrm{eucl}}(f)| > 0$. The latter inequality follows from the fact that f has slow growth on $\mathcal{K}_f$ (see below). This yields $\operatorname{vol}(B \cap \{u_n < -\varepsilon\}) \le \alpha_\varepsilon^{\lambda^n}$ where $0 < \alpha_\varepsilon < 1$ so that $u_n \to 0$ on $\mathcal{K}_f$.

We finally have to prove that f does not grow too fast on $\mathcal{K}_f$. We can write $f = P + Q$, where P is a homogeneous mapping of degree λ and Q is a polynomial mapping of degree $\le \lambda - 1$. Identifying $(t = 0)$ with $\mathbb{P}^{k-1}$, we can assume $P(t = 0) = X_f$. Fix $z_0 \in \mathcal{K}_f$ and $z_n = f^n(z_0)$. We can assume z_n accumulates only at $I_f \setminus V$, where V is a small neighborhood of X_f. Define

$$\zeta_n = P(z_{n-1}), \quad \zeta_n' = Q(z_{n-1}),$$

so $z_n = \zeta_n + \zeta_n'$. There is a constant $C > 0$ such that

$$|\zeta_n|_+ \le C|\zeta_n'|_+,$$

where $|p|_+ := \max(\|p\|, 1)$. If not then for a subsequence $|\zeta_n|/|\zeta_n'| \to \infty$, so

$$\frac{z_n}{\|z_n\|} = \frac{\zeta_n + \zeta_n'}{\|z_n\|} = \frac{\zeta_n}{\|\zeta_n\|} + o(1) = \frac{P(z_{n-1})}{\|P(z_{n-1})\|} + o(1).$$

Hence z_n converges to X_f, contradicting our assumption $z_0 \in \mathcal{K}_f$. We infer

$$|\zeta_{n+1}'|_+ = \big|Q(z_{n-1})\big|_+ \le C_1 |z_{n-1}|_+^{\lambda-1} \le C_2 |\zeta_{n-1}'|_+^{\lambda-1},$$

hence $|f^n(p)|_+ \le C_2' |p|_+^{(\lambda-1)^n}$ on $\mathcal{K}_f$. □

Remark 2.8. — When f is a "weakly regular" polynomial automorphism (*i.e.* when $X_f \cap I_f = \varnothing$, see [13]) then a necessary and sufficient condition to get convergence to T_f is that $\sup_{p \in X_f} \nu(S_n, p) \to 0$. This condition can be understood in terms of the local topological degrees of $f_0 := f_{|X_f}$ which is then an holomorphic endomorphism of X_f. Note that X_f may be singular but $X_f \simeq \mathbb{P}^r$ if it is smooth, so we are back to the situation described in Section 2.1!

For a "generic" f, f_0 will have no exceptional set so that $S_n \to T_f$ iff $\nu(S, X_f) = \inf_{p \in X_f} \nu(S, p) = 0$. In this generic situation, T_f is the only f^*-invariant current of unit mass in $\mathbb{C}^k$ and $\lambda^{-n}(f^n)^*[H] \to T_f$ for every hyperplane $H \in (\mathbb{P}^k)^* \setminus \mathcal{E}_f$, where

$$\mathcal{E}_f := \big\{ H \in (\mathbb{P}^k)^* \,;\, X_f \subset H \big\}$$

is an algebraic subset of $(\mathbb{P}^k)^*$.

3. Appendix

For the reader's convenience we recall here some compactness criteria for families of quasiplurisubharmonic functions which play a central role in this note. We refer the reader to [20] for a systematic discussion of similar results.

Lemma 3.1. — *Let (φ_n) be a sequence of plurisubharmonic functions in some connected open subset Ω of a complex manifold X. If (φ_n) is locally uniformly bounded from above in Ω, then either (φ_n) uniformly converges to $-\infty$, or it is relatively compact for the $L^1_{\mathrm{loc}}(\Omega)$-topology.*

We refer the reader to [14] (Theorem 3.2.12) for a proof. There is no plurisubharmonic function on $\mathbb{P}^k$ (except constants). However there are plenty of quasiplurisubharmonic functions: these are functions φ which are locally the sum of a psh and a smooth function, so that their curvature is allowed to be negative but with a smooth control: $dd^c\varphi \geq -\theta$, where θ is a smooth form. Of particular interest for us is the following class

$$\mathcal{L}_\omega := \big\{ \varphi \in L^1(\mathbb{P}^k) \,;\, \varphi \text{ is u.s.c. and } dd^c\varphi \geq -\omega \big\},$$

where ω denotes as usual the Fubini-Study Kähler form on $\mathbb{P}^k$. Lemma 3.1 easily extends to qpsh functions whose curvature is bounded below by a fixed form, for instance $-\omega$. We make constant use of the following criteria which tell that $\mathcal{L}_\omega$ is a compact family once normalized.

Proposition 3.2. — *Let B be an open subset of $\mathbb{P}^k$. Then*

$$\mathcal{F}_1 := \big\{ \varphi \in \mathcal{L}_\omega \,;\, \sup_B \varphi = 0 \big\} \text{ and } \mathcal{F}_2 := \Big\{ \varphi \in \mathcal{L}_\omega \,;\, \int_B \varphi \omega^k = 0 \Big\}$$

are compact families of qpsh functions.

Proof. — Fix $B' \subset B$ a small ball. We can assume $B' = B(r)$ is the ball of radius r centered at the origin in some affine chart $\mathbb{C}^k \subset \mathbb{P}^k$.

Let $\varphi \in \mathcal{F}_1$. Then

$$\psi := \varphi + \frac{1}{2}\log\bigl[1 + \|z\|^2\bigr]$$

is a psh function with logarithmic growth in $\mathbb{C}^k$ (because $\omega = dd^c(\frac{1}{2}\log[1 + \|z\|^2])$ in $\mathbb{C}^k$). Moreover $\sup_{B'}\psi \le \sup_{B'}\varphi + \frac{1}{2}\log[1 + r^2] \le \frac{1}{2}\log[1 + r^2]$. Therefore

$$\psi(z) \le u_r(z) := \max\Bigl(\log\frac{\|z\|}{r}, 0\Bigr) + \frac{1}{2}\log[1 + r^2] \ \text{ in } \mathbb{C}^k.$$

Indeed one can check this on any line $L \simeq \mathbb{C}$ passing through the origin: the function $u_{r|L}$ is harmonic outside the disk $L \cap B(r)$ and dominates $\psi_{|L}$ on $\partial(L \cap B(r))$ and near infinity. This yields

$$\varphi \le \max\Bigl(\log\frac{\|z\|}{r}, 0\Bigr) + \frac{1}{2}\log[1 + r^2] - \frac{1}{2}\log\bigl[1 + \|z\|^2\bigr] \le c_r \ \text{ in } \mathbb{C}^k,$$

hence in $\mathbb{P}^k$. Therefore functions of $\mathcal{F}_1$ are uniformly bounded from above on $\mathbb{P}^k$. So if we fix a sequence (φ_j) of functions in $\mathcal{F}_1$, we can extract a convergent subsequence by Lemma 3.1. The cluster point is not $-\infty$ because $\sup_B \varphi_j = 0$ hence it belongs to $\mathcal{F}_1$.

The argument is similar for $\mathcal{F}_2$. We simply need to derive a uniform upper bound on $\mathbb{P}^k$. So let $\varphi \in \mathcal{F}_2$. Then $\varphi - \sup_B \varphi \in \mathcal{F}_1$ which is compact. Therefore there exists C such that

$$\int_B |\varphi - \sup_B \varphi|\omega^k \le C, \quad \text{for all } \varphi \in \mathcal{F}_2.$$

This yields $\sup_B \varphi \le C\,\mathrm{vol}(B)$ since $\int_B \varphi\,\omega^k = 0$. Moreover there exists $C' > 0$ such that $\sup_{\mathbb{P}^k}\psi \le C'$ for every $\psi \in \mathcal{F}_1$, so

$$\varphi = (\varphi - \sup_B \varphi) + \sup_B \varphi \le C' + C\,\mathrm{vol}(B) \text{ on } \mathbb{P}^k,$$

for every $\varphi \in \mathcal{F}_2$. This implies the compactness of $\mathcal{F}_2$. □

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