EQUIDISTRIBUTION TOWARDS
THE GREEN CURRENT

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Abstract. — Let $f : \mathbb{P}^k \to \mathbb{P}^k$ be a dominating rational mapping of first algebraic degree $\lambda \geq 2$. If $S$ is a positive closed current of bidegree $(1,1)$ on $\mathbb{P}^k$ with zero Lelong numbers, we show – under a natural dynamical assumption – that the pullbacks $\lambda^{-n}(f^n)^* S$ converge to the Green current $T_f$. For some families of mappings, we get finer convergence results which allow us to characterize all $f^*$-invariant currents.

Résultat (Equidistribution vers le courant de Green). — Soit $f : \mathbb{P}^k \to \mathbb{P}^k$ une application rationnelle dominante de premier degré algébrique $\lambda \geq 2$. Lorsque $S$ est un courant positif fermé de bidegré $(1,1)$ sur $\mathbb{P}^k$ dont les nombres de Lelong sont tous nuls, nous montrons, sous une hypothèse dynamique naturelle, que les pull-backs $\lambda^{-n}(f^n)^* S$ convergent vers le courant de Green $T_f$. Pour certaines familles d’applications, des résultats de convergence raffinés nous permettent de caractériser tous les courants $f^*$-invariants.

Introduction

Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be a rational map of degree $\lambda \geq 2$. A celebrated result of Brolin, Lyubich, Freire-Lopez-Mañé asserts that the preimages $\lambda^{-n}(f^n)^* \sigma$ of any probability measure $\sigma$ on $\mathbb{P}^1$ converge to an invariant measure $\mu_f$ as soon as $\sigma(E_f) = 0$, where $E_f$ is a (possibly empty) finite exceptional set. The purpose of this note is to prove similar results in higher dimension.
Let $f : \mathbb{P}^k \to \mathbb{P}^k$ be a rational mapping. It can be written $f = [P_0 : \cdots : P_k]$ in homogeneous coordinates, where the $P_j$'s are homogeneous polynomials of the same degree $\lambda$ (the first algebraic degree of $f$) with no common factor $P_0 \wedge \cdots \wedge P_k = 1$. Note that when $k \geq 2$, $f$ is not necessarily holomorphic: it is not well defined on the indeterminacy set $I_f = \bigcap_j P_j^{-1}(0)$ which is an algebraic subset of $\mathbb{P}^k$ of codimension $\geq 2$. There are several ways one can try to generalize the one-dimensional result. Given $Z$ an algebraic subset of $\mathbb{P}^k$ of pure codimension $p$, one can ask whether $f^{-n}(Z)$ (properly normalized) converges to an invariant current of bidegree $(p,p)$. In this note we focus on the case $p = 1$.

Given $S$ a positive closed current of bidegree $(1,1)$ on $\mathbb{P}^k$, we consider

$$S_n := \lambda^{-n}(f^n)^* S.$$  

This is a bounded sequence of positive closed currents of bidegree $(1,1)$ on $\mathbb{P}^k$. When $S = \omega$ is the Fubini-Study Kähler form, it was proved by Sibony [19] that $(\omega_n)$ converges to an invariant Green current $T_f$. On the other hand Russakovskii and Shiffman have shown [18] that $[H]_n - \omega_n \to 0$ for almost every hyperplane $H$ of $\mathbb{P}^k$. Our main result interpolates between these two extreme cases.

**Theorem 0.1.** — Let $f : \mathbb{P}^k \to \mathbb{P}^k$ be a dominating rational mapping with $\lambda \geq 2$. Assume there exists an invariant probability measure $\mu$ such that $\log |J_{FS}(f)| \in L^1(\mu)$. Let $S$ be a positive closed current of bidegree $(1,1)$ and unit mass on $\mathbb{P}^k$. If $\nu(S,p) = 0$ for all $p \in \mathbb{P}^k$, then

$$\frac{1}{\lambda^n}(f^n)^* S \rightharpoonup T_f \quad \text{in the weak sense of currents.}$$

Here $\nu(S,p)$ denotes the Lelong number of $S$ at point $p$ and $J_{FS}(f)$ denotes the jacobian of $f$ with respect to the Fubini-Study volume form $\omega$. Similar (weaker) results were previously obtained for Hénon mappings [1], [8], birational mappings [6], and holomorphic endomorphisms of $\mathbb{P}^k$ [9], [19], [7].

Although Theorem 0.1 does not imply directly Russakovskii-Shiffman’s result, the proof shows one essentially has to assume $\sup_{p \in \mathbb{P}^k \setminus E} \nu(S_n,p) \to 0$, where $E$ is some (possibly empty) exceptional set (see Theorem 1.4). The key ingredients of the proof are: a pluripotential estimate of the volume of sublevel sets of a quasipshurisubharmonic function [16] and a dynamical estimate on the decreasing of volumes under iteration (Theorem 1.2). Note that all the volumes are computed with respect to the Fubini-Study volume form $\omega$.

We prove the volume estimates and our main result in Section 1. We give refinements of the latter in Section 2 in case $f$ is an holomorphic endomorphism of $\mathbb{P}^k$ (Section 2.1) or a special type of polynomial endomorphism of $\mathbb{C}^k$ (Section 2.2). This allows us to characterize every $f^*$-invariant current. Such equidistribution results should be understood as strong ergodic properties of
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the Green current $T_f$. In dimension 1 indeed this implies that $T_f$ is strongly mixing (see Theorem VIII.22 in [2]). For the reader’s convenience we recall in an Appendix compactness criteria for families of qpsh functions. They are the higher dimensional counterparts of Montel’s Theorem.

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1. Equidistribution of pullbacks of currents

Let $f : \mathbb{P}^k \to \mathbb{P}^k$ be a rational mapping with first algebraic degree $\lambda \geq 2$. We always assume $f$ is dominating, i.e. its jacobian does not vanish identically in any coordinate chart. It follows that a generic point has $d_t(f)$ well defined preimages by $f$. Note that $d_t(f) = \lambda$ when $k = 1$ but these two degrees differ in general when $k \geq 2$.

Let $\omega$ denote the Fubini-Study Kähler form on $\mathbb{P}^k$. The smooth form $f^*\omega$ is well defined in $\mathbb{P}^k \setminus I_f$ and extends trivially through $I_f$ as a positive closed current of bidegree $(1, 1)$ and mass $\|f^*\omega\| = \int_{\mathbb{P}^k} f^*\omega \wedge \omega^{k-1} = \lambda$. So $\lambda^{-1} f^*\omega$ is cohomologous to $\omega$. Since $\mathbb{P}^k$ is Kähler, this can be written

$$\lambda^{-1} f^*\omega = \omega + dd^c G,$$

where $G \in L^1(\mathbb{P}^k)$ (see [11, p. 149]). The function $G$ is “quasiplushisubharmonic” (qpsh): it is locally given as the sum of a psh function (a local potential of $\lambda^{-1} f^*\omega \geq 0$) and a smooth function (a local potential of $-\omega$). In particular it is bounded from above on $\mathbb{P}^k$: replacing $G$ by $G - C_0$, we can therefore assume $G \leq 0$. Sibony [19] has shown that the decreasing sequence of qpsh functions

$$G_n := \sum_{j=0}^{n-1} \frac{1}{\lambda^j} G \circ f^j$$

converges in $L^1(\mathbb{P}^k)$ to a qpsh function $G_\infty \in L^1(\mathbb{P}^k)$. This shows that $\lambda^{-n}(f^n)^*\omega$ converges in the weak sense of (positive) currents to the so called Green current $T_f \geq 0$ which satisfies $f^* T_f = \lambda T_f$.

A natural question is then to look at the convergence of $S_n := \lambda^{-n}(f^n)^* S$, where $S$ is now any positive closed current of bidegree $(1, 1)$ and unit mass on $\mathbb{P}^k$. When $S = [H]$ is the current of integration along an hyperplane of $\mathbb{P}^k$, it was shown by Russakovskii and Shiffman [18] that $[H]_n \to T_f$ for every $H$ outside some pluripolar set $E \subset (\mathbb{P}^k)^*$. In order to prove convergence of $S_n$ for more general currents $S$, we first need to get control on the decreasing of volumes under iteration.

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1.1. Volume estimates. — Let \( f : \mathbb{P}^k \to \mathbb{P}^k \) be a dominating rational mapping with \( \lambda \geq 2 \). Let \( J_{FS}(f) \) denote its jacobian with respect to the Fubini-Study Kähler volume form. It is defined by

\[
f^* \omega^k = |J_{FS}(f)|^2 \omega^k.
\]

**Proposition 1.1.** — Fix \( B \) an open subset of \( \mathbb{P}^k \) and \( \delta_0 > 0 \). There exists \( C_0 > 0 \) such that for every open subset \( \Omega \) of \( \mathbb{P}^k \) with \( \text{vol}(\Omega) \geq \delta_0 \),

\[
\text{vol}(f^n(\Omega)) \geq (C_0)^{\lambda^n} \exp \left( \frac{1}{\text{vol}(\Omega)} \int_B \log |J_{FS}(f^n)|^2 \omega^k \right),
\]

for all \( n \in \mathbb{N} \).

**Proof.** — Fix \( C^k \) an affine chart of \( \mathbb{P}^k \). We have

\[
f = (P_1/P_0, \ldots, P_k/P_0)
\]

in \( C^k \) where the \( P_j \)'s are polynomials of degree \( \leq \lambda = \delta_1(f) \). Since \( \omega = dd^c \log[1 + \|z\|^2] \) in \( C^k \), we get

\[
\omega^k(z) = (1 + \|z\|^2)^{-(k+1)} dV,
\]

where \( dV \) denotes the euclidean volume form in \( C^k \). Therefore

\[
|J_{FS}(f)|^2 = |J_{\text{eucl}}(f)|^2 \left( \frac{1 + \|z\|^2}{1 + \|f(z)\|^2} \right)^{k+1}.
\]

We infer

\[
\log |J_{FS}(f)| = u - v,
\]

where \( u, v \) are qpsh functions such that \( dd^c u, dd^c v \geq -2\lambda k \omega \). Let \( \Omega \) be an open subset of \( \mathbb{P}^k \). We have

\[
\text{vol}(f^n(\Omega)) = \int_{f^n(\Omega)} \omega^k \geq \frac{1}{d_t(f)^n} \int_\Omega |J_{FS}(f^n)|^2 \omega^k,
\]

where the inequality follows from the change of variable formula. The concavity of the logarithm yields

\[
\text{vol}(f^n(\Omega)) \geq \frac{\text{vol}(\Omega)}{d_t(f)^n} \exp \left[ \frac{2D_n}{\text{vol}(\Omega)} \int_\Omega \frac{1}{D_n} (u_n - v_n) \omega^k \right],
\]

where

\[
\log |J_{FS}(f^n)| = u_n - v_n
\]

with \( dd^c u_n, dd^c v_n \geq -D_n \omega, D_n \leq 2\lambda^n k \). Observe that we can decompose

\[
\frac{1}{D_n} \log |J_{FS}(f^n)| = \varphi_n - \psi_n + \frac{1}{\text{vol}(B)} \int_B \frac{1}{D_n} \log |J_{FS}(f^n)| \omega^k,
\]
where
\[ \varphi_n = D_n^{-1}u_n - \log \|z\| - \frac{1}{\text{vol}(B)} \int_B (D_n^{-1}u_n - \log \|z\|) \omega^k, \]
\[ \psi_n = D_n^{-1}v_n - \log \|z\| - \frac{1}{\text{vol}(B)} \int_B (D_n^{-1}v_n - \log \|z\|) \omega^k. \]

The functions \( \varphi_n, \psi_n \) are quasipsh functions on \( \mathbb{P}^k \) (\( \dd^c \varphi_n, \dd^c \psi_n \geq -\omega \)) with \( \int_B \varphi_n = \int_B \psi_n = 0 \). It follows therefore from Proposition 3.2 (Appendix) that they belong to a compact family of qphs functions, so there exists \( C_\Omega \in \mathbb{R} \) such that \( \int_\Omega (\varphi_n - \psi_n) \omega^k \geq C_\Omega \), for all \( n \in \mathbb{N} \). Since \( D_n \leq 2k \lambda^n \), this yields the desired inequality.

It remains to get a lower bound on \( \int_B \log |J_{FS}(f^n)| \omega^k \), where \( B \) is an open subset which we may fix as we like.

**Theorem 1.2.** — Let \( f : \mathbb{P}^k \to \mathbb{P}^k \) be a dominating rational mapping with \( \lambda \geq 2 \). Assume there exists an invariant probability measure \( \mu \) such that \( \log |J_{FS}(f)| \in L^1(\mu) \). Fix \( \delta_0 > 0 \). Then there exists \( C_0 > 0 \) such that for every open subset \( \Omega \) of \( \mathbb{P}^k \) with \( \text{vol}(\Omega) \geq \delta_0 \),
\[ \text{vol}(f^n(\Omega)) \geq C_0 \lambda^n, \quad \forall n \in \mathbb{N}. \]

**Proof.** — Using Proposition 1.1, it is sufficient to find \( M > 0 \) such that for all \( n \), \( \int_B \log |J_{FS}(f^n)| \omega^k \geq -M \lambda^n \). We take here \( B = \mathbb{P}^k \) (but other normalisations could be useful, see Remark 1.3 below).

We define \( \lambda^{-1} \log |J_{FS}(f)| = \nu - v + C \), where \( u, v \) are qphs functions (\( \dd^c u, \dd^c v \geq -\omega \)) such that \( \sup_{\mathbb{P}^k} u = \sup_{\mathbb{P}^k} v = 0 \) and \( C \in \mathbb{R} \). Thus we get
\[ \frac{1}{\lambda^n} \log |J_{FS}(f^n)| = \frac{1}{\lambda^n} \sum_{j=0}^{n-1} \log |J_{FS}(f) \circ f^j| \geq \sum_{j=0}^{n-1} \frac{1}{\lambda^{n-j}} u_j + \frac{n}{\lambda^n} C, \]
where \( u_j := \lambda^{-j} u \circ f^j \). It is therefore sufficient to get a uniform lower bound on \( \int_{\mathbb{P}^k} u_j \omega^k \). This is a consequence of the fact that \( (u_j) \) is relatively compact in \( L^1(\mathbb{P}^k) \). Indeed \( \dd^c u_j \geq -\lambda^{-j} (f^j)^* \omega \), so \( u_j + G_j \) is qphs. By Lemma 3.1 (Appendix), the sequence \( (u_j + G_j) \) is either relatively compact or uniformly converges to \( -\infty \). Since \( G_j \to G_\infty \in L^1(\mathbb{P}^k) \), the sequence \( (u_j) \) is either relatively compact or converges to \( -\infty \). But the latter can not happen since \( u \in L^1(\mu) \) and \( \int u_j \dd \mu = \lambda^{-j} \int u \dd \mu \to 0 \). The desired control on \( \int_{\mathbb{P}^k} \log |J_{FS}(f^n)| \omega^k \) follows.

**Remark 1.3.** — The assumption on the existence of \( \mu \) is natural in our dynamical context. Observe that it is satisfied if e.g. there exists a non critical periodic point.

Other assumptions could be made to obtain the final lower bound on \( \int_B \log |J_{FS}(f^n)| \omega^k \). If \( f \mid_{\mathbb{C}^k} \) is polynomial, it is enough to assume that...
sup_B |J_{eucl}(f^n)| \geq \alpha^{\lambda^n} for some relatively compact open subset B of \mathbb{C}^k. This holds in particular when f is a polynomial automorphism of \mathbb{C}^k.

1.2. Equidistribution of pullbacks. — A major difficulty in higher dimensional complex dynamics lies in the presence of points of indeterminacy and in the difficulty of analyzing the dynamics near

I_f^\infty := \bigcup_{n \geq 0} I_{f^n}.

Following Fornaess and Sibony [9], a rational mapping f : \mathbb{P}^k \to \mathbb{P}^k is said to be normal if for every p \in \mathbb{P}^k \setminus \overline{I_f}, there exists a neighborhood W of p and V of \overline{I_f} such that \phi^n(W) \cap V = \emptyset for all n \geq 0. Note that the condition is empty if \overline{I_f} = \mathbb{P}^k, so one usually assumes further that \overline{I_f} is “small”, e.g. vol(\overline{I_f}) = 0. Examples of such mappings include holomorphic endomorphisms of \mathbb{P}^k (for which I_f = I_f^\infty = \emptyset) or some polynomial endomorphisms of \mathbb{C}^k with small topological degree (e.g. Hénon mappings in \mathbb{C}^2, see other examples in [12], [13] and Section 2.2 below).

Theorem 1.4. — Let f : \mathbb{P}^k \to \mathbb{P}^k be a dominating rational mapping with \lambda \geq 2. Assume there exists an invariant probability measure \mu on \mathbb{P}^k such that log |J_{FS}(f)| \in L^1(\mu). Let S be a positive closed current of bidgree (1,1) and unit mass on \mathbb{P}^k. If \nu(S, p) = 0 for all p \in \mathbb{P}^k, then

S_n := \frac{1}{\lambda^n} (f^n)^* S \to T_f.

Moreover if f is normal with vol(\overline{I_f}) = 0 then S_n \to T_f if and only if \nu(S_n, p) \to 0 uniformly on compact subsets of \mathbb{P}^k \setminus \overline{I_f}.

Proof. — Write S = \omega + dd^c u, where u \leq 0 is quasiplushisubharmonic on \mathbb{P}^k. Then \lambda^{-n} (f^n)^* (S - \omega) = dd^c u_n, where u_n = \lambda^{-n} u \circ f^n \leq 0. So we need to show that u_n \to 0 in L^1(\mathbb{P}^k).

Observe that dd^c (u_n + G_n) \geq -\omega, so (u_n) is either relatively compact or uniformly converges to \omega (Lemma 3.1). It is therefore sufficient to prove that for all \varepsilon > 0, vol(\Omega^\varepsilon_n) \to 0, where

\Omega^\varepsilon_n := \{ p \in \mathbb{P}^k : \frac{1}{\lambda^n} u \circ f^n (p) < -\varepsilon \}.

Assume on the contrary that vol(\Omega^\varepsilon_n) \geq \delta_0 for some fixed \varepsilon, \delta_0 > 0 and n \to +\infty. Observe that

f^n(\Omega^\varepsilon_n) \subset \{ p \in \mathbb{P}^k : u(p) < -\varepsilon \lambda^{n_i} \}.

If \nu(S, p) = 0 for all p \in \mathbb{P}^k, it follows from Skoda’s integrability Theorem (see Theorem 3.1 in [16]) that for every A > 0, there exists C_A > 0 such that

vol(f^n(\Omega^\varepsilon_{n_i})) \leq C_A \exp(-A \varepsilon \lambda^{n_i}).
On the other hand, since \( \operatorname{vol}(\Omega^i_{\varepsilon_n}) \geq \delta_0 > 0 \), there follows from Theorem 1.2 that there exists \( C_0 > 0 \) such that
\[
\operatorname{vol}(f^n(\Omega^i_{\varepsilon_n})) \geq C_0 \lambda^n_i, \text{ for all } i \in \mathbb{N}.
\]
Taking \( A > -\log C_0 \) yields a contradiction.

Assume now \( f \) is normal. It was proved by Favre [5] that \( \nu(T_f, p) = 0 \) for all \( p \in \mathbb{P}^k \setminus \overline{T_f} \). Therefore it is necessary, for \( S_\alpha \) to converge to \( T_f \), that for every open neighborhood \( V \) of \( \overline{T_f} \), \( \sup_{p \in \mathbb{P}^k \setminus V} \nu(S_\alpha, p) \to 0 \). This is because \( (S, p) \mapsto \nu(S, p) \) is upper semi-continuous (u.s.c.). Assume it is the case. Fix \( W \) a relatively compact open subset of \( \mathbb{P}^k \setminus \overline{T_f} \). Since Vol(\( \overline{T_f} \)) = 0, it is sufficient to prove that \( u_\alpha \to 0 \) over such \( W \). Since \( f \) is normal, we can fix \( V \) an open neighborhood of \( \overline{T_f} \) such that \( V \cap f^n(W) = \emptyset \), for all \( n \geq 0 \). We need to prove that \( \operatorname{vol}(W \cap \Omega^i_{\varepsilon_n}) \to 0 \). Now
\[
f^n(W \cap \Omega^i_{\varepsilon_n}) \subset \{ p \in \mathbb{P}^k \setminus V ; u(p) < -\varepsilon \lambda^n \},
\]
so the previous proof applies if \( \sup_{p \in \mathbb{P}^k \setminus V} \nu(S, p) \) is small enough. When \( \sup_{p \in \mathbb{P}^k \setminus V} \nu(S, p) \) is not small enough, we replace \( S \) by \( S_{N_0} \), \( N_0 \gg 1 \).

2. Invariant currents

It is an interesting problem to characterize all positive closed currents \( S \) of bidegree \((1,1)\) on \( \mathbb{P}^k \) such that \( f^* S = \lambda S \). This can be done by using our equidistribution result (Theorem 1.4). We illustrate this on two families of mappings.

2.1. Holomorphic endomorphisms of \( \mathbb{P}^k \).— We assume here \( f : \mathbb{P}^k \to \mathbb{P}^k \) is holomorphic, i.e. \( f = \emptyset \). In this case the construction of the Green current \( T_f \) is due to Hubbard and Papadopol [15]: \( G \) is smooth on \( \mathbb{P}^k \), so \( (G_n) \) uniformly converges to \( G_\infty \) which is henceforth continuous (see \((*)\)).

Since the Green current \( T_f \) has continuous potential, all its Lelong numbers are 0. Moreover, it follows from the work of Bedford and Taylor that the measure \( \mu_f := T_f^k \) is well defined. The measure \( \mu_f \) is invariant and every qpsk function is \( \mu_f \) integrable (as follows from the Chern-Levine-Nirenberg inequalities, see the Appendix in [19]). Therefore \( f \) satisfies the assumptions of Theorem 1.4. Given \( S \) a positive closed current of bidegree \((1,1)\) and unit mass on \( \mathbb{P}^k \), Theorem 1.4 reads here
\[
\frac{1}{\lambda^n}(f^n)^* S \to T_f \iff \sup_{p \in \mathbb{P}^k} \nu(\lambda^{-n}(f^n)^* S, p) \to 0.
\]

It remains to understand the behavior of Lelong numbers under iteration. Since \( f \) is proper, one easily gets
\[
\nu((f^n)^* S, p) \leq d(f^n, p) \nu(S, f^n(p)),
\]

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where \( d(f^n, p) \) denotes the local topological degree of \( f^n \) at \( p \),
\[
d(f^n, p) = \prod_{j=0}^{n-1} d(f, f^j(p)).
\]
So we are done if \( d(f^n, p) = o(\lambda^n) \).

Analyzing the behavior of \( d(f^n, p) \) is quite easy in dimension 1 as shows the following elementary lemma whose proof is left to the reader.

**Lemma 2.1.** — Let \( f : \mathbb{P}^1 \to \mathbb{P}^1 \) be a rational map of degree \( \lambda \geq 2 \). Set
\[
\mathcal{E}_f := \{ p \in \mathbb{P}^1 : d(f, p) = d(f, f(p)) = d(f, f^2(p)) = \lambda \}.
\]
Then either
- \( \mathcal{E}_f \) is empty, or
- \( \mathcal{E}_f = 1 \) point, \( f \) is conjugate to a polynomial, or else
- \( \mathcal{E}_f = 2 \) points, \( f \) is conjugate to \( z^\lambda \) or \( z^{-\lambda} \).

Combining this with Theorem 1.4 yields the following celebrated result of Brolin [3], Lyubich [17] and Freire-Lopez-Mañé [10]. Note that positive closed currents of bidegree \((1, 1)\) and unit mass are simply probability measures on \( \mathbb{P}^1 \).

**Theorem 2.2.** — Let \( f : \mathbb{P}^1 \to \mathbb{P}^1 \) be a rational map of degree \( \lambda \geq 2 \). Let \( \sigma \) be a probability measure on \( \mathbb{P}^1 \). Then
\[
\frac{1}{\lambda^n}(f^n)^*\sigma \to T_f \iff \sigma(\mathcal{E}_f) = 0.
\]

When \( k \geq 2 \) the “crude” estimate \( d(f^n, p) \leq d_t(f^n) = \lambda^k \) becomes worse as the dimension grows. Nevertheless, one still has that \( d(f^n, p) = O((\lambda - 1)^n) \) for a “very generic” choice of \( f \) (i.e. for \( f \) outside a countable union of hypersurfaces), so we get the following result of Fornaess and Sibony [9].

**Corollary 2.3.** — Let \( f : \mathbb{P}^k \to \mathbb{P}^k \) be a “very generic” holomorphic mapping with \( \lambda = d_t(f) \geq 2 \). Then
\[
\lambda^{-n}(f^n)^*S \to T_f
\]
for every positive closed current \( S \) of bidegree \((1, 1)\) and unit mass on \( \mathbb{P}^k \).

In particular \( T_f \) is the only \( f^* \)-invariant current.

It turns out that looking at local topological degrees is not sufficient to settle the problem of convergence to \( T_f \) when \( k \geq 2 \). Our volume estimates (Theorem 1.2) nevertheless allow us to complete the recent work of Favre and Jonsson [7] in dimension 2.

**Theorem 2.4.** — Let \( f : \mathbb{P}^2 \to \mathbb{P}^2 \) be a holomorphic mapping with \( \lambda \geq 2 \). There exists a (possibly empty) totally invariant algebraic subset \( \mathcal{E}_f \) of \( \mathbb{P}^2 \) such that if \( S \) is a positive closed current of bidegree \((1, 1)\) and unit mass on \( \mathbb{P}^k \), then
\[
\nu(S, \mathcal{E}_f) = 0 \implies \frac{1}{\lambda^n}(f^n)^*S \to T_f.
\]
The set $\mathcal{E}_f$ can be decomposed as $\mathcal{E}_1 \cup \mathcal{E}_2$, where $\mathcal{E}_1$ is a union of at most three lines and $\mathcal{E}_2$ is a finite set. The condition $\nu(S, \mathcal{E}_f) = 0$ has to be understood as $\nu(S, p) = 0$ for every point $p \in \mathcal{E}_2$ and almost every point $p$ of $\mathcal{E}_1$. We refer the reader to [7] for a proof. The only new information we provide are sufficient volume estimates near points of $\mathcal{E}_2$, without the extra assumption made by Favre and Jonsson that $\mathcal{E}_2$ consists of “homogeneous points”.

2.2. Some polynomial endomorphisms of $\mathbb{C}^k$. — Let $f : \mathbb{C}^k \to \mathbb{C}^k$ be a proper polynomial mapping, $f = (P_1, \ldots, P_k)$, where the $P_i$’s are polynomials with $\lambda = \max(\deg P_i) \geq 2$. We let $d_i(f)$ denote the topological degree of $f$ and shall assume here that $d_i(f) < \lambda$. Given $S$ a positive closed current of bidegree $(1, 1)$ and unit mass on $\mathbb{P}^k$, we thus get

$$
\sup_{p \in \mathbb{C}^k} \nu(S_n, p) \leq \sup_{p \in \mathbb{C}^k} \frac{d(f^n, p)}{\lambda^n} \sup_{p \in \mathbb{C}^k} \nu(S, f^n(p)) \leq \left( \frac{d_i(f)}{\lambda} \right)^n \to 0,
$$

where $S_n := \lambda^{-n}(f^n)^* S$.

We still denote by $f$ the meromorphic extension of $f$ to $\mathbb{P}^k = \mathbb{C}^k \cup \{t = 0\}$, where $(t = 0)$ denotes the hyperplane at infinity. Since $f$ is polynomial (hence holomorphic) in $\mathbb{C}^k$, the indeterminacy set $I_f$ is located within $(t = 0)$. Define by induction

$$
X_1 := f((t = 0) \setminus I_f), \quad X_{j+1} := f(X_j \setminus I_f).
$$

This is a decreasing sequence of irreducible analytic subsets of $(t = 0)$. We denote by $X_f := X_\ell$ the limit set, which we assume is non empty (this is equivalent to saying that $f$ is algebraically stable, see [19]).

**Theorem 2.5.** — Let $f : \mathbb{C}^k \to \mathbb{C}^k$ be an algebraically stable polynomial endomorphism with $d_i(f) < \lambda = \delta_1(f)$. Assume $I_f$ is an $f^{-1}$-attracting set and there exists an invariant probability measure $\mu$ such that $\log |J_{FS}(f)| \in L^1(\mu)$. Let $S$ be a positive closed current of bidegree $(1, 1)$ and unit mass on $\mathbb{P}^k$. If $\nu(S, p) = 0$ for all $p \in X_f$, then

$$
S_n := \frac{1}{\lambda^n} (f^n)^* S \to T_f.
$$

When $\dim X_f = 0$, $T_f$ is the only $f^*$-invariant current of unit mass in $\mathbb{C}^k$.

**Proof.** — We assume $I_f$ is $f^{-1}$-attracting in the following sense: there exists an open neighborhood $V$ of $I_f$ in $\mathbb{C}^k$ such that $\bigcap_{j \geq 0} f^{-j}(V) = \emptyset$. This implies that $f$ is a normal mapping. Note also that $f$ is proper in $\mathbb{C}^k$: if an algebraic curve $C \subset \mathbb{C}^k$ were to be contracted to a point by $f$, $C$ would intersect $(t = 0)$ within $I_f$, contradicting that $I_f$ is $f^{-1}$-attracting. Therefore $\sup_{p \in C^k} \nu(S_n, p) \to 0$ as observed above. If $q \in (t = 0) \setminus I_{f^\ell}$ then

$$
\nu((f^\ell)^* S, q) \leq C_q \nu(S, f^\ell(q)) = 0, \quad \text{since } f^\ell(q) \in X_f = X_\ell.
$$
The latter inequality is due to Favre [4] and Kiselman [16]. Therefore
\[ \sup_{p \in \mathbb{P}^k \setminus I_f} \nu(S_n, p) \to 0 \] so \( S_n \to T_f \) by Theorem 1.4.

When \( \dim X_f = 0 \), \( X_f \) is reduced to a point which does not belong to \( I_f \). In this case \( \nu(S, X_f) = 0 \) is also a necessary condition to get \( S_n \to T_f \). Indeed if
\[ \nu(S, X_f) = \gamma > 0 \] then for all \( q \in \{ t = 0 \} \setminus I_f \),
\[ \nu((f^t)^* S, q) \geq \nu(S, f^t(q)) = \gamma > 0 \] so \( (f^t)^* S \geq \gamma [t = 0] \)
hence \( S_n \) does not converge to \( T_f \). In particular if \( S \) is \( f^* \)-invariant and does not charge the hyperplane at infinity then \( \nu(S, X_f) = 0 \) hence \( S = S_n \to T_f \), so \( S = T_f \). Therefore every \( f^* \)-invariant current is a linear combination of \( T_f \)
and \([t = 0]\).

**Remark 2.6.** — The previous result applies e.g. to mappings
\[ f = f_1 \circ \cdots \circ f_s, \quad f_j : (z, w) \in \mathbb{C}^2 \mapsto (P_j(w), Q_j(z) + R_j(w)) \in \mathbb{C}^2, \]
where \( P_j, Q_j, R_j \) are polynomials of degree \( p_j, q_j, \lambda_j \) with \( p_j q_j < \lambda_j \). One gets here
\[ d_t(f) = \prod_{j=1}^{s} p_j q_j \quad \text{and} \quad \lambda := \delta_1(f) = \prod_{j=1}^{s} \lambda_j > d_t(f). \]
The set \( I_f = [1 : 0 : 0] \) is \( f^{-1} \)-attracting (see Example 4.1 in [12]) and \( X_f = [0 : 1 : 0] \). An invariant measure \( \mu \) such that \( \log |J_{FS}(f)| \in L^1(\mu) \) is constructed in [12] (Theorem 5.3). Note that when \( p_j = q_j = 1 \) then \( f \) is a complex Hénon mapping. A different proof of Theorem 2.5 was given in this case in the paper [6], where \( f^* \)-invariant currents are characterized for every birational mapping of \( \mathbb{C}^2 \).

We now consider the case of polynomial automorphisms of \( \mathbb{C}^k \). We can prove finer volume estimates on the set of points whose orbit accumulates \( I_f \) so that it is not necessary to assume \( f \) is normal.

**Theorem 2.7.** — Let \( f : \mathbb{C}^k \to \mathbb{C}^k \) be an algebraically stable polynomial automorphism of \( \mathbb{C}^k \) such that \( X_f \) is an \( f \)-attracting set. Let \( S \) be a positive closed current of bidegree \((1, 1)\) and unit mass on \( \mathbb{P}^k \). If \( \nu(S, p) = 0 \) for all \( p \in X_f \), then
\[ S_n := \frac{1}{\lambda^n} (f^n)^* S \longrightarrow T_f. \]
If \( \dim X_f = 0 \) then \( T_f \) is the only \( f^* \)-invariant current of unit mass in \( \mathbb{C}^k \).

**Proof.** — As observed in Remark 1.3, it is not necessary to assume the existence of an invariant measure \( \mu \) such that \( \log |J_{FS}(f)| \in L^1(\mu) \). We use instead the fact that the euclidean jacobian of \( f \) is constant in \( \mathbb{C}^k \) to derive a lower bound on \( \int_B \log |J_{FS}(f^n)| \omega^k \).
We let $B(X_f)$ denote the basin of attraction of $X_f$ in $\mathbb{C}^k$ and set
\[ K_f := \mathbb{C}^k \setminus B(X_f). \]
Since $f$ is proper in $\mathbb{C}^k$ with $d(f) = 1 < \lambda$, we get as before
\[ \sup_{p \in \mathbb{P}^k \setminus K_f} \nu(S_n, p) \to 0. \]
Therefore $S_n \to 0$ in $B(X_f)$. Now we must make sure that $S_n \to 0$ on any relatively compact open subset $B$ of $K_f$ (if any). Write
\[ S = \omega + \dd^c u, \]
where $u \leq 0$, $u \in L^1(\mathbb{P}^k)$. We need to prove that for all $\varepsilon > 0$,
\[ \vol(B \cap \{u_n < -\varepsilon\}) \to 0, \]
where $u_n = \lambda^{-n} u \circ f^n$. Now $f^n(B \cap \{u_n < -\varepsilon\}) \subset \{u < -\varepsilon \lambda^n\}$ has volume $\leq C_1 \exp(-\varepsilon \lambda^n)$ because the Lelong numbers of $S_n$ are bounded from above by 1. On the other hand
\[ \vol(f^n(B \cap \{u_n < -\varepsilon\})) = \int_{B \cap \{u_n < -\varepsilon\}} |J_{FS}(f^n)|^2 \omega^k \]
\[ = |a|^{2n} \int_{B \cap \{u_n < -\varepsilon\}} \left(1 + \frac{\|z\|^2}{1 + \|f^n(z)\|^2}\right)^{k+1} \omega^k \]
\[ \geq C_B^{(\lambda^{-1})^n} \vol(B \cap \{u_n < -\varepsilon\}), \]
where $|a| := |J_{\text{eucl}}(f)| > 0$. The latter inequality follows from the fact that $f$ has slow growth on $K_f$ (see below). This yields $\vol(B \cap \{u_n < -\varepsilon\}) \leq \alpha \varepsilon_n$ where $0 < \alpha \varepsilon < 1$ so that $u_n \to 0$ on $K_f$.

We finally have to prove that $f$ does not grow too fast on $K_f$. We can write $f = P + Q$, where $P$ is a homogeneous mapping of degree $\lambda$ and $Q$ is a polynomial mapping of degree $\leq \lambda - 1$. Identifying $(t = 0)$ with $\mathbb{P}^{k-1}$, we can assume $P(t = 0) = X_f$. Fix $z_0 \in K_f$ and $z_n = f^n(z_0)$. We can assume $z_n$ accumulates only at $I_f \setminus V$, where $V$ is a small neighborhood of $X_f$. Define
\[ \zeta_n = P(z_n-1), \quad \zeta_n' = Q(z_n-1), \]
so $z_n = \zeta_n + \zeta_n'$. There is a constant $C > 0$ such that
\[ |\zeta_n|_+ \leq C |\zeta_n'|_+, \]
where $|p|_+ := \max(\|p\|, 1)$. If not then for a subsequence $|\zeta_n|/|\zeta_n'| \to \infty$, so
\[ \frac{z_n}{\|z_n\|} = \frac{\zeta_n + \zeta_n'}{\|z_n\|} = \frac{\zeta_n}{\|\zeta_n\|} + o(1) = \frac{P(z_n-1)}{\|P(z_n-1)\|} + o(1). \]
Hence $z_n$ converges to $X_f$, contradicting our assumption $z_0 \in K_f$. We infer
\[ |\zeta_n'|_+ = |Q(z_n-1)|_+ \leq C_1 |z_n-1|^\lambda - 1 \leq C_2 |\zeta_n'-1|^\lambda - 1, \]
hence $|f^n(p)|_+ \leq C_2 |p|^\lambda - 1$ on $K_f$. \(\square\)
Remark 2.8. — When $f$ is a “weakly regular” polynomial automorphism (i.e. when $X_f \cap I_f = \emptyset$, see [13]) then a necessary and sufficient condition to get convergence to $T_f$ is that $\sup_{p \in X_f} \nu(S_n, p) \to 0$. This condition can be understood in terms of the local topological degrees of $f_0 := f|_{X_f}$ which is then an holomorphic endomorphism of $X_f$. Note that $X_f$ may be singular but $X_f \simeq \mathbb{P}^r$ if it is smooth, so we are back to the situation described in Section 2.1!

For a “generic” $f$, $f_0$ will have no exceptional set so that $S_n \to T_f$ iff $\nu(S, X_f) = \inf_{p \in X_f} \nu(S, p) = 0$. In this generic situation, $T_f$ is the only $f^*$-invariant current of unit mass in $\mathcal{C}^k$ and $\lambda^{-n}(f^n)^*[H] \to T_f$ for every hyperplane $H \in (\mathbb{P}^k)^* \setminus \mathcal{E}_f$, where

$$\mathcal{E}_f := \{ H \in (\mathbb{P}^k)^* \setminus X_f \subset H \}$$

is an algebraic subset of $(\mathbb{P}^k)^*$.

3. Appendix

For the reader’s convenience we recall here some compactness criteria for families of quasipositive functions which play a central role in this note. We refer the reader to [20] for a systematic discussion of similar results.

Lemma 3.1. — Let $(\varphi_n)$ be a sequence of plurisubharmonic functions in some connected open subset $\Omega$ of a complex manifold $X$. If $(\varphi_n)$ is locally uniformly bounded from above in $\Omega$, then either $(\varphi_n)$ uniformly converges to $-\infty$, or it is relatively compact for the $L^1_{\text{loc}}(\Omega)$-topology.

We refer the reader to [14] (Theorem 3.2.12) for a proof. There is no plurisubharmonic function on $\mathbb{P}^k$ (except constants). However there are plenty of quasipositive functions: these are functions $\varphi$ which are locally the sum of a psh and a smooth function, so that their curvature is allowed to be negative but with a smooth control: $\dd c \varphi \geq -\theta$, where $\theta$ is a smooth form. Of particular interest for us is the following class

$$\mathcal{L}_\omega := \{ \varphi \in L^1(\mathbb{P}^k); \varphi \text{ is u.s.c. and } \dd c \varphi \geq -\omega \},$$

where $\omega$ denotes as usual the Fubini-Study Kähler form on $\mathbb{P}^k$. Lemma 3.1 easily extends to qpsh functions whose curvature is bounded below by a fixed form, for instance $-\omega$. We make constant use of the following criteria which tell that $\mathcal{L}_\omega$ is a compact family once normalized.

Proposition 3.2. — Let $B$ be an open subset of $\mathbb{P}^k$. Then

$$\mathcal{F}_1 := \{ \varphi \in \mathcal{L}_\omega; \sup_B \varphi = 0 \} \text{ and } \mathcal{F}_2 := \{ \varphi \in \mathcal{L}_\omega; \int_B \varphi \omega^k = 0 \}$$

are compact families of qpsh functions.
Proof. — Fix $B' \subset B$ a small ball. We can assume $B' = B(r)$ is the ball of radius $r$ centered at the origin in some affine chart $\mathbb{C}^k \subset \mathbb{P}^k$.

Let $\varphi \in \mathcal{F}_1$. Then

$$\psi := \varphi + \frac{1}{2} \log[1 + \|z\|^2]$$

is a psh function with logarithmic growth in $\mathbb{C}^k$ (because $\omega = \ddc\left(\frac{1}{2} \log[1 + \|z\|^2]\right)$ in $\mathbb{C}^k$). Moreover $\sup_{B'} \psi \leq \sup_{B'} \varphi + \frac{1}{2} \log[1 + r^2] \leq \frac{1}{2} \log[1 + r^2]$. Therefore

$$\psi(z) \leq u_r(z) := \max \left( \log \frac{\|z\|}{r}, 0 \right) + \frac{1}{2} \log[1 + r^2] \text{ in } \mathbb{C}^k.$$ 

Indeed one can check this on any line $L \cong \mathbb{C}$ passing through the origin: the function $u_r|_L$ is harmonic outside the disk $L \cap B(r)$ and dominates $\psi|_L$ on $\partial(L \cap B(r))$ and near infinity. This yields

$$\varphi \leq \max \left( \log \frac{\|z\|}{r}, 0 \right) + \frac{1}{2} \log[1 + r^2] - \frac{1}{2} \log[1 + \|z\|^2] \leq c_r \text{ in } \mathbb{C}^k,$$

hence in $\mathbb{P}^k$. Therefore functions of $\mathcal{F}_1$ are uniformly bounded from above on $\mathbb{P}^k$. So if we fix a sequence $(\varphi_j)$ of functions in $\mathcal{F}_1$, we can extract a convergent subsequence by Lemma 3.1. The cluster point is not $-\infty$ because $\sup_{B} \varphi_j = 0$ hence it belongs to $\mathcal{F}_1$.

The argument is similar for $\mathcal{F}_2$. We simply need to derive a uniform upper bound on $\mathbb{P}^k$. So let $\varphi \in \mathcal{F}_2$. Then $\varphi - \sup_{B} \varphi \in \mathcal{F}_1$ which is compact. Therefore there exists $C$ such that

$$\int_B |\varphi - \sup_{B} \varphi| \omega^k \leq C, \text{ for all } \varphi \in \mathcal{F}_2.$$ 

This yields $\sup_{B} \varphi \leq C \operatorname{vol}(B)$ since $\int_B \varphi \omega^k = 0$. Moreover there exists $C' > 0$ such that $\sup_{B} \psi \leq C'$ for every $\psi \in \mathcal{F}_1$, so

$$\varphi = (\varphi - \sup_{B} \varphi) + \sup_{B} \varphi \leq C' + C \operatorname{vol}(B) \text{ on } \mathbb{P}^k,$$

for every $\varphi \in \mathcal{F}_2$. This implies the compactness of $\mathcal{F}_2$. 

\[ \square \]

**BIBLIOGRAPHY**


**BULLETIN DE LA SOCIÉTÉ MATHEMATIQUE DE FRANCE**