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FREE DECAY OF SOLUTIONS TO WAVE EQUATIONS ON A CURVED BACKGROUND

BY SERGE ALINHAC

ABSTRACT. — We investigate for which metric g (close to the standard metric g_0) the solutions of the corresponding d'Alembertian behave like free solutions of the standard wave equation. We give rather weak (*i.e.*, non integrable) decay conditions on $g - g_0$; in particular, $g - g_0$ decays like $t^{-\frac{1}{2}-\varepsilon}$ along wave cones.

RÉSUMÉ (*Décroissance des solutions des équations d'ondes sur un arrière-plan courbe*)
Nous étudions pour quelles métriques g (proches de la métrique standard g_0) les solutions du d'Alembertien pour g se comportent comme des solutions libres de l'équation des ondes standard. Nous proposons des conditions de décroissance assez faibles (*i.e.*, non intégrables) sur $g - g_0$; en particulier, $g - g_0$ décroît comme $t^{-\frac{1}{2}-\varepsilon}$ le long des cônes d'onde.

Introduction

We consider the wave equation L_g associated with a given Lorentzian metric g on $\mathbb{R}_t \times \mathbb{R}^3_x$. Our aim is to answer the question: under which conditions on g do the solutions of $L_g u = 0$ behave like free solutions of the standard wave equation L_0 ? One can of course use the energy method of Klainerman, commuting the standard "Z"-fields with the equation, and putting on g strong enough decay assumptions (relative to the standard metric) to obtain finally a

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SERGE ALINHAC, Département de Mathématiques, Université Paris-Sud, 91405 Orsay (France) • *E-mail :* `serge.alinhac@math.u-psud.fr`

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control of $|\partial Z^k u|_{L^2}$, which implies in turn, thanks to Klainerman's inequality, the behavior

$$|\partial u| \leq C\big(1+\big|t-|x|\big|\big)^{-\frac{1}{2}}\big(1+t+|x|\big)^{-1}.$$

What we have in mind is to impose as little decay as possible on g, getting close to what seems to be a *critical level.* The framework we choose here is one where a "1D-situation" occurs, in the sense of [2]. This means that we can prove for L_g an energy inequality in which three special derivatives G (the "good" derivatives) are better controlled than in the standard $L^\infty_t L^2_x$-norm: only one "bad" derivative is left. This idea has been used already in [1], where we study the equation

$$\partial_t^2 u - c^2(u)\Delta u = 0.$$

This later work splits essentially into a linear part, where we study the operator $\partial_t^2 - c^2(u)\Delta$, and a nonlinear part which is a bootstrap on certain properties of u. Because of the very special form of the equation, it seemed to us that the treatment of the linear problem involved many miracles which were may be not likely to occur again in a more general case. Also, in this nonlinear problem, u was likely to decay roughly as t^{-1}, implying a similar decay for derivatives of $c(u)$. The general analysis below shows that one can relax this assumption down to an almost $t^{-\frac{1}{2}}$ decay of the metric (relative to the flat metric).

A more precise discussion of these issues will be offered in section 1.4 after our notations, assumptions and results have been stated. Let us just say here that the whole paper is strongly inspired by the geometric techniques of Christodoulou and Klainerman, developed in [4], [3] and also by related work of Klainerman and Sideris [10], Klainerman and Nicolò [8] and Klainerman and Rodnianski [9].

1. Framework and main result

1.1. The general framework. — We work in $\mathbb{R}_t \times \mathbb{R}^3_x$ where

$$x^0 = t, \quad x = (x^1, x^2, x^3), \quad \partial_\alpha = \frac{\partial}{\partial x^\alpha}, \quad r = |x|, \quad r\omega = x, \quad \sigma = \langle r-t\rangle,$$

where here and below we use

$$\langle s\rangle = (1+s^2)^{\frac{1}{2}}.$$

As usual, the greek indices will run from 0 to 3, while the latin one will run only from 1 to 3.

We consider a metric $g = g_0 + \gamma$ which is a (small) perturbation of the standard Minkowski metric g_0 defined by

$$(g_0)_{00} = -1, \quad (g_0)_{ii} = 1, \quad (g_0)_{0i} = 0.$$

The inverse matrix to $g_{\alpha\beta}$ is denoted by $g^{\alpha\beta}$. We will write

$$\langle X, Y\rangle = g(X, Y)$$

and denote by D the connexion associated to g. Recall that for a function a, the gradient of a and the Hessian of a are defined by

$$\nabla a = g^{\alpha\beta}(\partial_\alpha a)\partial_\beta, \quad \nabla^2 a(X,Y) = XYa - (D_X Y)a.$$

We denote by L_0 the d'Alembertian associated to g_0 (the standard wave equation), and by

$$L_g u = g^{\alpha\beta}\nabla^2 u_{\alpha\beta}$$

the d'Alembertian associated to g. We assume

$$g^{00} = -1, \quad g^{0i}(x,t)\omega_i = 0,$$

and define

$$T = -\nabla t = \partial_t - g^{0i}\partial_i, \quad N = \frac{\nabla r}{|\nabla r|}, \quad L = T + N, \quad L_1 = T - N.$$

Note that our assumption $g^{0i}\omega_i = 0$ allows us to express $T - \partial_t$ and $N - c\partial_r$ using the standard rotations, a fact which will be important later on. As shown in [2], we have the easy properties

$$\langle T,T\rangle = -1, \quad T(r) = 0 = \langle N,T\rangle, \quad D_T T = 0,$$
$$\langle L,L\rangle = 0 = \langle L_1,L_1\rangle, \quad \langle L,L_1\rangle = -2.$$

We use the frame

$$e_1, e_2, L_1, L,$$

where the e_i form an orthonormal basis on the standard spheres $t = t_0$, $r = r_0$.

Three quantities play an important role in the following:

- the radial sound speed c defined by

$$c = |\nabla r|, \quad c^2 = g^{ij}\omega_i\omega_j,$$

- the second fundamental form k of the hypersurfaces $t = \text{Constant}$,

$$k(X,Y) = \langle D_X T, Y\rangle, \quad k_{ij} = \frac{1}{2}g^{0\alpha}(\partial_i g_{\alpha j} + \partial_j g_{\alpha i} - \partial_\alpha g_{ij}),$$

- the second fundamental form of the standard spheres $t = t_0$, $r = r_0$ in $\{t = t_0\}$

$$\theta(e,e') = \langle D_e N, e'\rangle,$$

where e and e' are tangent to the sphere.

We denote by $\bar{k}$ and $\bar{\theta}$ the traces of theses forms

$$\bar{k} = k(N,N) + k(e_1,e_1) + k(e_2,e_2), \quad \bar{\theta} = \theta(e_1,e_1) + \theta(e_2,e_2).$$

In the frame (e_i, L_1, L), the d'Alembertian L_g is

$$L_g = -LL_1 + \Delta_S - \bar{k}T + (k_{NN} + \bar{\theta})N + \sum_{i=1,2}\Big(2k_{iN} - \frac{e_i(c)}{c}\Big)e_i,$$

where Δ_S is the Laplacian on the standard spheres corresponding to the restriction of g to these spheres. Finally, we recall the definitions of the standard fields

$$R_i = (x \wedge \partial)_i, \quad S = t\partial_t + r\partial_r.$$

1.2. Assumptions on the metric. — The behavior of the metric and of the solution will be discussed in terms of the two parameters

$$\sigma = \big(1 + (r-t)^2\big)^{\frac{1}{2}}, \quad 1 + t + r.$$

Because of this, we distinguish three zones I, II and III, respectively defined by

$$r \leq \tfrac{1}{2}(1+t), \quad \tfrac{1}{2}(1+t) \leq r \leq \tfrac{3}{2}(1+t), \quad r \geq \tfrac{3}{2}(1+t),$$

which we also call "interior", "middle zone" and "exterior". The reason for using these parameters is that in nonlinear applications, the coefficients γ will be functions of u or ∂u, and their behavior has to be discussed in the same terms as the behavior of u.

The time decay of certain quantities will be measured using a smooth increasing $\phi = \phi(t) > 0$ such that

$$(1.2)_{\rm a} \qquad \phi' > 0, \quad (1+t)\phi' \in S^0, \quad \frac{\phi''}{\phi'} \in S^{-1},$$

$$(1.2)_{\rm b} \qquad \forall \epsilon > 0, \quad \phi(t) \leq C_\epsilon + \epsilon \log(1+t).$$

Here, S^m denotes symbols of order m, that is, smooth functions $s(t)$ satisfying

$$\big|s^{(k)}(t)\big| \leq C_k \langle t\rangle^{m-k}, \quad k \in \mathbb{N}.$$

In [1], we take $\phi(t) = \epsilon \log(1+t)$. The "free case" corresponds to the choice ϕ' integrable. It seemed however relevant to us to incorporate in the present paper certain decay patterns which played in important role in [1].

There are three groups of assumptions on the metric:

- *General low decay.* — For some $\mu > \frac{1}{2}$, and all k,

$$|\Gamma^k \gamma^{\alpha\beta}| \leq \gamma_0\, \sigma^{\frac{1}{2}} (1+t+r)^{-\mu}, \quad |\Gamma^k\, \partial\gamma^{\alpha\beta}| \leq \gamma_0\, \sigma^{-\frac{1}{2}} (1+t+r)^{-\mu}.$$

Here, Γ^k means any product of k fields Γ among R_i, S or $\sigma^\mu \partial_\alpha$. In zones I or III, it is enough to take Γ among R_i, S or ∂_α.

- *Special high decay.* — For the quantities $\bar{k}$, $\bar{\theta}$ and c, we have *in the middle zone* the high decay

$$|\Gamma^k \bar{k}| \leq \gamma_0\, \sigma^{-\frac{1}{2}} (1+t)^{-1}, \quad |\Gamma^k \bar{\theta}| \leq \gamma_0 (1+t)^{-1},$$
$$|1-c| \leq \gamma_0\, \sigma^{\frac{1}{2}} \phi', \quad |\partial c| \leq \gamma_0\, \sigma^{-\frac{1}{2}} \phi',$$
$$|\Gamma^{k+1} c| \leq \gamma_0\, \sigma^{\frac{1}{2}} \phi' {\rm e}^{C\phi}, \quad |\Gamma^k\, \partial c| \leq \gamma_0\, \sigma^{-\frac{1}{2}} \phi' {\rm e}^{C\phi}.$$

- *Technical interior assumption.* — In the interior, we assume $r|\bar{\theta}| \leq C$.

REMARK. — One can observe in the assumptions above that whenever a quantity is bounded by $*\sigma^{\frac{1}{2}}$, its gradient is bounded by $*\sigma^{-\frac{1}{2}}$. This "homogeneity" is important and occurs naturally in the context of nonlinear equations, where energy methods and Klainerman's formula give no better than a $\sigma^{-\frac{1}{2}}$ control of ∂u (see Introduction); this does not allow in general anything better than u controlled by $\sigma^{\frac{1}{2}}$. We postpone to section 1.4 the discussion of these assumptions.

1.3. Main result. — Let u be the solution of the Cauchy problem

$$L_g u = 0, \quad u(x,0) = u_0(x), \quad (\partial_t u)(x,0) = u_1(x).$$

Assume the following decay on the smooth real functions u_0, u_1

$$\forall \alpha, \forall \beta,\ |\alpha| \le |\beta|, \quad x^\alpha \partial_x^\beta u_i \in L^2, \quad i = 1,2.$$

We have then the following "free" decay property.

THEOREM. — *For γ_0 small enough and $r \ge \frac{1}{2}(1+t)$, we have*

$$|\partial u| \le C\sigma^{-\frac{1}{2}}(1+t+r)^{-1}\mathrm{e}^{C\phi}$$

for some $C > 0$.

REMARK 1. — The "free decay" result annouced in the title is obtained by choosing ϕ' integrable, in which case ϕ is bounded and so is $\mathrm{e}^{C\phi}$.

REMARK 2. — There is little doubt that the same estimate holds also for $\partial Z^k u$, where $Z = R_i$, $Z = S$ or $Z = \partial_\alpha$. This can be proved using the "hat-calculus" of section 9; we dropped the proof of these additionnal estimates to make the paper a little lighter, if possible.

We did not attempt here to give a poor estimate in the interior zone; getting a good one there (without using the hyperbolic rotations) is a real difficulty, which has been completely skipped in [8] for instance, where the authors work only outside the interior zone. One can may be hope for some extension of the inequality proved in [6] for the wave equation, which displays an improved interior behavior of ∂u.

1.4. Discussion of the method of the proof, of the assumptions, and plan of the paper. — a) The method of proof uses energy inequalities for L_g. In the litterature, there are essentially two approaches:

i) One can use a conformal energy inequality (see [5]), which gives a control of $R_i u$, Su and $H_i u$, with $H_i = t\partial_i + x_i\partial_t$. This is the approach of [7], [8] and [9]. This is enough to get some decay on u, but not quite the precise t^{-1} decay we want (see [7]).

ii) One can use the standard energy inequality and commute the fields R_i, S and H_i to the operator. This is the classical approach for many cases, see for instance [5].

In this later case, the standard Klainerman's inequality

$$\sigma^{\frac{1}{2}}(1+t+r)|v| \leq C \sum_{|\alpha|\leq 3} |Zv|_{L^2_x}$$

yields the result. From the formula

$$\partial_i = \omega_i \partial_r - \frac{1}{r}(\omega \wedge R)_i,$$

we see that the control of $R_i u$ makes all derivatives of u look radial, if r is big enough (that is, outside the interior zone). In the interior zone, one uses instead

$$\frac{R_i}{r} = \frac{1}{t}(\omega \wedge H)_i.$$

Here, we are *not willing* to use the hyperbolic rotations H_i. They do not appear in the assumptions on the metric, and we *do not commute them* with L_g: it becomes then difficult to control what happens in the interior zone. We will come back to this in point e).

Here our goal is to obtain, in zones II and III, the estimates

$$|\partial Z^k u|_{L^2} + |\sigma \partial^2 Z^k u|_{L^2} \leq C\mathrm{e}^{C\phi},$$

for $Z = R_i$ or $Z = S$, which is enough to obtain the decay of the theorem (see [3]).

b) The main idea of the proof is that we are in a 1D-situation, where the good derivatives are R_i/r and L. This is a consequence of the inequality obtained in [2]. We choose as the bad derivative T rather than L_1, which has non smooth coefficients in the interior. As in [1], we try to commute with L_g modified fields Z_m instead of the standard R_i and S. Since we are in a 1D-situation, we think it enough to perturb the standard fields by a certain amount of T only, that is take $Z_m = R_i + a(R_i)T$ or $Z_m = S + a(S)T$ for appropriate functions a to be chosen in each case.

c) In [1], we analyze the commutator $[L_g, Z_m]$ by brute force, taking advantage of the simple structure of the operator. However, this is tedious and does not permit to fully understand why the terms in the commutator (especially the first order terms) behave properly. We use here a geometric approach giving a formula for $[L_g, Z_m]$ in terms of the deformation tensor of Z_m: see [7]. We can then compute the traces of the tensors in an appropriate frame (L_1, L, e_1, e_2), where (e_1, e_2) is an orthonormal basis on the standard spheres. Using again the fact that we are in a 1D-situation, we can essentially (though not completely) discard all terms involving rotations, and take advantage of the special structure of LL_1 (which is close to L_g itself) and of L^2. Thus, in $[L_g, Z_m]$, the only remaining bad terms are the ones in L_1^2 and in L_1. This explains why it is possible to cancel these bad terms by choosing only one function a. It turns out that we must ask

$$La + \frac{aTc}{c} + \frac{Zc}{c} = 0, \quad Z = R_i, \quad Z = S.$$

This is of course to be compared with the more geometric approach based on the construction of an optical function as in [7], [8].

d) To actually perform all the computations hinted at in c., and keep in mind the behavior of the coefficients in the formula, we must develop a symbolic calculus as in [1]. In fact, we develop three calculus: the standard one reflects the action of the fields Z_m, and is explained in section 5. Another one is necessary to establish the behavior of various derivatives of the perturbation coefficients a: we call it the "bar"-calculus, and we explain it in the Appendix to avoid confusion. A third calculus, the "hat"-calculus, is sketched in section 9 when we need it. We did not try to formalize the structure of such calculus, though it is rather easy to see how they are constructed. On the other hand, we do not see how the computations could be done without it.

e) It turns out that the coefficients a, along with their Z_m^k derivatives, behave essentially like $\sigma^{\frac{1}{2}}$ while ∂a behaves like $\sigma^{-\frac{1}{2}}$. This causes many problems. For instance,

$$\partial Ru = \partial(R + aT)u - (\partial a)Tu - a\partial Tu.$$

Since we want to write all formula in terms of the fields Z_m (which are the only fields we hope to control), we will have to use the above formula whenever ∂Ru occurs in the computations. But, since a is not bounded, we must know that ∂Tu behaves better than $\partial Z_m u$. This can be easily done if one uses all "Z"-fields, including the H_i, since we have the inequality (see [5])

$$\sigma|\partial v| \leq C \sum |Zv|.$$

If we do not use the H_i, we have to rely on the substitute formula found by Klainerman and Sideris [10]. These formula allow roughly to control $\sigma\partial^2 v$ by ∂Sv, ∂Rv, ∂v and $(1 + t + r)L_0 v$. Here, we have to adapt them to allow a control by L_g, and this is the reason why we require a special behavior of the lower order terms coefficients $\bar{\theta}$ and $\bar{k}$ in L_g, and why we make our technical interior assumption.

2. A convenient piece of notation

Definition 2.1. — We will denote by f_0 any C^∞ function of the following arguments:

- $\gamma^{\alpha\beta}$, ω, $r/(1 + t)$, $\sigma/(1 + t)$,
- any 0-order symbol of t or $r - t$.

In the sequence, Z always means either R_i or S.

To simplify the formula, we often write fR for $\sum fR_i$, γ for $\gamma^{\alpha\beta}$, etc. and sum signs are dropped when this is not likely to cause any misunderstanding.

The following lemma indicates how such f_0 behave, when differentiated.

LEMMA 2.1. — *In zone* II, *we have the formula*

$$\partial_\alpha f_0 = f_0 \partial_\alpha \gamma + \frac{f_0}{\sigma}, \quad Z f_0 = f_0 Z\gamma + f_0.$$

Proof. — Since $\frac{1}{2} \leq \frac{r}{1+t} \leq \frac{3}{2}$, we have

$$\partial \omega = \frac{f_0}{r} = \frac{f_0}{1+t}, \quad Z\omega = f_0.$$

Since ∂r, ∂t are f_0, and $R_i r = R_i t = 0$, $Sr = r$, $St = t$,

$$\partial\Big(\frac{r}{1+t}\Big) = \frac{f_0}{1+t}, \quad Z\Big(\frac{r}{1+t}\Big) = f_0.$$

Recalling that $s = (1+s^2)^{\frac{1}{2}}$ and $\langle s\rangle' = s/\langle s\rangle$,

$$\partial\Big(\frac{\sigma}{1+t}\Big) = \frac{f_0}{1+t}, \quad Z\Big(\frac{\sigma}{1+t}\Big) = f_0.$$

Finally, since $S(r-t) = r-t$, for any 0-order symbol a,

$$\partial\big(a(r-t)\big) = a'(r-t)f_0 = \frac{f_0}{\sigma}, \quad S\big(a(r-t)\big) = a'(r-t)(r-t) = f_0$$

and similarly for $a(t)$. □

In the sequence, we quantity $r-ct$ will appear often in the computations in zone II, and we need to compare it to our standard σ. To this aim, we introduce the following definition.

DEFINITION 2.2. — We define f just as f_0, but containing also the additional argument $(1-c)/(\sigma^{\frac{1}{2}}\phi')$.

Let $\chi(s)$ be a smooth real increasing function, $\chi(s) = 0$ for $s \leq -1$, $\chi(s) = 1$ for $s \geq 1$, and $\chi - \frac{1}{2}$ odd. For technical reasons which will become clear later on, define $\widetilde{\sigma}$ by

$$\widetilde{\sigma} = \big(1-\chi(r-t)\big)\big(2-(r-ct)\big) + \chi(r-t)\big(2+(r-ct)\big).$$

The following lemma summarizes the relations between $r-ct$, σ, $\widetilde{\sigma}$.

LEMMA 2.2. — *We have*

$$r - ct = f\sigma, \quad \widetilde{\sigma} = f\sigma, \quad \sigma = f\widetilde{\sigma}.$$

Proof. — First,

$$r - ct = r - t + (1-c)t = \sigma\Big(\frac{r-t}{\sigma} + f\sigma^{-\frac{1}{2}}t\,\phi'\Big) = f\sigma$$

implies the first claim. For $r - t \leq 1$,

$$2-(r-ct) = \big(2-(r-t)\big)\Big(1-\frac{(1-c)t}{2-(r-t)}\Big), \quad \frac{(1-c)t}{2-(r-t)} = \frac{f\sigma^{\frac{1}{2}}}{2-(r-t)}$$

being bounded by $\frac{1}{2}$ for γ_0 small enough. Hence

$$\widetilde{\sigma} \geq \frac{1}{2}(1-\chi)\big(2-(r-t)\big) + \tfrac{1}{2}\chi\big(2+(r-t)\big) \geq 1,$$

and clearly $\liminf \widetilde{\sigma}/\sigma \geq \frac{1}{2}$ as σ goes to infinity. Hence $\widetilde{\sigma}/\sigma$, being an f bounded away from zero, satisfies $\sigma/\widetilde{\sigma} = f$. □

3. Two useful formula

In the flat case, denoting by $H_i = x^i\partial_t + t\partial_i$ the hyperbolic rotations, we have the two formula

i) $(r-t)\partial = f_0\partial + f_0R_i + f_0S + f_0H_i$,

ii) $(r+t)(\partial_t + \partial_r) = S + \sum \omega_i H_i$.

These show that the control of all fields R_i, S, H_i gives a control of ordinary derivatives *improved by* σ, and a control of $L_0 = \partial_t + \partial_r$ *improved by* t. In the present case, we do not use the H_i, and we need a substitute for these two formula. Klainerman and Sideris [10] have established a substitute for i) in the flat case. In the following proposition, we establish similar but more geometric formula involving L_g.

3.1. A formula of Klainerman-Sideris type. — For technical reasons, we introduce the "tangential" part $\widetilde{\Delta}$ of L_g defined by

$$c\widetilde{\Delta} = N^2 + \Delta + \bar{\theta}N - \frac{1}{c}\sum e_a(c)e_a.$$

We have thus $L_g = -T^2 - \bar{k}T + c\widetilde{\Delta}$.

Proposition 3.1. — • *In zone* I, *we have the pointwise a priori bounds*

$$\begin{aligned}\sigma|\partial\partial_t v| &\leq C|\partial v| + C|\partial S v| + C|\partial R v| + Cr|L_g v|,\\ \sigma|\widetilde{\Delta} v| &\leq C|\partial v| + C|\partial S v| + C|\partial R v| + Ct|L_g v| + C\gamma_0\,\sigma\sum|\partial^2_{ij}v|.\end{aligned}$$

- *In zone* II, *we have the formula* (*recall that* $Z = R_i$ *or* $Z = S$)

$$\begin{aligned}\sigma\partial T &= f(1+t)L_g + (1+t)(f\bar\theta + f(\partial c) + f\bar k)\partial \\ &\qquad + f\partial Z + f\partial T + (f + f(R\gamma) + f(ct-r)(\partial\gamma))\partial + f(\partial\gamma)R,\\ \sigma N^2 &= f(1+t)L_g + (1+t)(f\bar\theta + f(\partial c) + f\bar k)\partial \\ &\qquad + f\partial Z + f\partial T + (f + f(R\gamma) + f(ct-r)(\partial\gamma))\partial + f(\partial\gamma)R.\end{aligned}$$

- *In zone* III, *we have, for* γ_0 *small enough, the pointwise a priori bounds*

$$\sigma|\partial^2_{\alpha\beta}v| \le C|\partial v| + C|\partial Zv| + Cr|L_g v|.$$

Proof. — a) We first prove a number of elementary formula. Recall that

$$T = \partial_t + \frac{\gamma^{0i}}{r}(\omega\wedge R)_i, \quad N = c\partial_r - \frac{\gamma^{ij}\omega_i}{cr}(\omega\wedge R)_j.$$

Now

$$\begin{aligned}&\langle D_T N, N\rangle = 0, \quad \langle D_T N, T\rangle = -\langle N, D_T T\rangle = 0,\\ &\langle D_T N, e_a\rangle = \frac{1}{c}\langle D_T\nabla r, e_a\rangle = \frac{1}{c}\langle D_{e_a}\nabla r, T\rangle = -\langle N, D_{e_a}T\rangle = -k_{aN},\\ &D_T N = -\sum k_{aN}e_a, \quad \langle D_N T, T\rangle = 0, \quad \langle D_N T, N\rangle = k_{NN},\\ &\langle D_N T, e_a\rangle = k_{aN}, \quad D_N T = k_{NN}N + \sum k_{aN}e_a,\\ &[T,N] = D_T N - D_N T = -2\sum k_{aN}e_a - k_{NN}N,\\ &[L, L_1] = -2[T,N], \quad \langle D_N N, N\rangle = 0, \quad \langle D_N N, T\rangle = -k_{NN},\\ &\langle D_N N, e_a\rangle = \frac{1}{c}\langle D_N\nabla r, e_a\rangle = \frac{1}{c}\langle D_{e_a}\nabla r, N\rangle = \frac{1}{c}e_a(c),\\ &D_N N = k_{NN}T + \frac{1}{c}\sum e_a(c)e_a.\end{aligned}$$

b) We start by recalling the pointwise formula from [10] in the flat case:

$$\sigma\big(|\Delta u| + |\partial_t^2 u| + |\partial_i\partial_t u|\big) \le C|\partial u| + C|\partial Su| + C|\partial Ru| + C(r+t)|L_0 u|.$$

In zones II or III, we have in fact the pointwise estimates

$$\sigma|\partial^2 u| \le C|\partial u| + C|\partial Su| + C|\partial Ru| + Cr|L_0 u|.$$

This follows from the formula

$$\Delta u = \partial_r^2 u + \frac{2}{r}\partial_r u + \frac{1}{r^2}\sum R_i^2 u, \quad \frac{1}{r}R_i^2 u = h(\omega)\partial R_i u,$$

which imply

$$\sigma|\partial_r^2 u| \le C\sigma|\Delta u| + C|\partial u| + C|\partial Ru|.$$

Now $\partial_i = \omega_i\partial_r + h(\omega)R/r$, hence

$$|\partial_i\partial_j u| \le C|\partial_r^2 u| + \frac{C}{r}|\partial u| + \frac{C}{r}|\partial R u|,$$

which gives the result for $\partial^2 = \partial_i\partial_j$, all other derivatives being already estimated above. Finally, in zone III, since $L_g = L_0 + \gamma\partial^2 + h(\gamma)\partial\gamma\partial$,

$$r|L_0 v| \le r|L_g v| + C\gamma_0\Big(\frac{r}{\sigma}\Big)\sigma|\partial^2 v| + C\gamma_0 r\sigma^{-\frac12}(1+t+r)^{-\mu}|\partial v|,$$

which gives the result for γ_0 small enough.

c) We follow now the proof of [10], trying to replace ∂_t and ∂_r by T and N whenever possible. We thus write

$$TS = \partial_t + tT\partial_t + rT\partial_r, \quad NS = tN\partial_t + c\partial_r + rN\partial_r,$$
$$ctTS - rNS - ct\partial_t + rc\partial_r = rt(cT\partial_r - N\partial_t) + E, \quad E = ct^2T\partial_t - r^2N\partial_r.$$

Introducing $\delta = N\partial_r - \widetilde{\Delta}$, we write E in two different ways:

$$\begin{aligned} E &= ct^2T\partial_t - r^2\delta - r^2\widetilde{\Delta} = ct^2T\partial_t - r^2\delta - \frac{r^2}{c}(L_g + T^2 + \bar{k}T)\\ &= -\frac{r^2}{c}(L_g + \bar{k}T) - \frac{r^2}{c}T(T-\partial_t) + \frac{c^2t^2 - r^2}{c}T\partial_t - r^2\delta.\\ E &= -r^2\delta - ct^2T(T-\partial_t) - ct^2L_g - ct^2\bar{k}T + (c^2t^2 - r^2)\widetilde{\Delta}. \end{aligned}$$

d) We compute now $\widetilde{\Delta}$ and δ. For more precision, we denote by $h(\omega,\gamma)$ any smooth function of ω and the coefficients γ. We have

$$L_g u = -T^2u - \bar{k}Tu + c\widetilde{\Delta}u = g^{\alpha\beta}\nabla^2 u_{\alpha\beta}.$$

But $T = \partial_t - \gamma^{0i}\partial_i$,

$$\begin{aligned} T^2 &= \partial_t^2 - 2\gamma^{0i}\partial_{ti}^2 + \gamma^{0i}\gamma^{0j}\partial_{ij}^2 + h\partial\gamma\partial,\\ L_g &= g^{\alpha\beta}\partial_{\alpha\beta}^2 + h\partial\gamma\partial = -\partial_t^2 + \Delta + \gamma^{ij}\partial_{ij}^2 + 2\gamma^{0i}\partial_{it}^2 + h\partial\gamma\partial. \end{aligned}$$

Comparing the formula, we obtain

$$c\widetilde{\Delta} = \Delta + (\gamma^{0i}\gamma^{0j} + \gamma^{ij})\partial_{ij}^2 + h\partial\gamma\partial.$$

To compute Δ_S, we denote $\tilde{g}$ the induced metric on a given sphere, with corresponding connection $\widetilde{D}$, etc., we have

$$\Delta_S u = e_a^2u + e_b^2u - (\widetilde{D}_a e_a + \widetilde{D}_b e_b)u.$$

We claim that we can pick locally an orthonormal basis (e_1, e_2) of the form $e_a = (h/r)R$. For instance,

$$\begin{aligned} e_1 &= \frac{R_3}{|R_3|} = \frac{1}{r}\frac{x_1\partial_2 - x_2\partial_1}{(\omega_1^2 g_{22} + \omega_2^2 g_{11} - 2\omega_1\omega_2 g_{12})^{\frac12}},\\ e_2 &= \frac{-\langle R_3, R_2\rangle R_1 + \langle R_3, R_1\rangle R_2}{|\cdots|} = \frac{1}{r}\big(h_1(\omega,\gamma)R_1 + h_2(\omega,\gamma)R_2\big). \end{aligned}$$

We then obtain

$$\Delta_S = \frac{h}{r^2}R^2 + \frac{h}{r^2}R + \frac{h}{r^2}(R\gamma)R.$$

Since $N = c\partial_r + \overline{N}$, $\overline{N} = (\gamma/c)h(\omega)R/r = \gamma hR/r$, we obtain easily

$$rN\overline{N} = h\gamma\partial R + \Big(h\partial\gamma + \frac{h\gamma}{r} + \frac{h\gamma R\gamma}{r}\Big)R,$$

and similarly with $T = \partial_t + \overline{T} = \partial_t + h\gamma R/r$,

$$rT\overline{T} = h\partial R + h\partial + h(\partial\gamma)R, \quad cT\partial_r - N\partial_t = \frac{h}{r}\partial R.$$

Since $c = h(\omega,\gamma)$, we finally obtain

$$\begin{aligned} r\delta &= -rc^{-2}(Nc)N - rN\overline{N} - \frac{r}{c}\bar{\theta}N - \frac{r}{c}\Delta_S + \frac{r}{c^2}\sum e_a(c)e_a \\ &= -\frac{r}{c^2}(Nc)N - \frac{r}{c}\bar{\theta}N + h\partial R + h\partial + h(R\gamma)\partial + h(\partial\gamma)R. \end{aligned}$$

e) We prove now the estimates in zone I. First, using here our technical interior assumption, we have $|r\delta v| \le C|\partial v| + C|\partial Rv|$. Hence, using the first expression for E, we obtain

$$\sigma|T\partial_t v| \le C|\partial v| + C|\partial Rv| + C|\partial Sv| + Cr|L_g v|.$$

Let us write for short $\widetilde{\partial}_i = (1/r)(\omega\wedge R)_i$, thus $\partial_i = \omega_i\partial_r - \widetilde{\partial}_i$. Following [10], we write

$$(t-r)\widetilde{\partial}_i\partial_t = \widetilde{\partial}_i S - (\partial_t + \partial_r)(r\widetilde{\partial}_i), \quad \sigma|\widetilde{\partial}_i\partial_t v| \le C|\partial Sv| + C|\partial Rv|.$$

Since $T = \partial_t + \gamma^{0i}\,\widetilde{\partial}_i$, this yields $\sigma|\partial_t^2 v| \le C|\partial v| + C|\partial Sv| + C|\partial Rv| + Cr|L_g v|$. We now proceed to control ∂_{rt}^2, adapting again the proof of [10]. Substracting the formula for TS and NS above, we get

$$\begin{aligned} tN\partial_t - rT\partial_r &= (N-T)S + \partial_t - c\partial_r - r\delta + tT\partial_t - r\widetilde{\Delta}, \\ tT\partial_t - r\widetilde{\Delta} &= \Big(t - \frac{r}{c}\Big)T\partial_t - \frac{r}{c}(L_g + \bar{k}T + T\overline{T}). \end{aligned}$$

Since $T\partial_t$ is already controlled, we obtain

$$\big|(tT\partial_t - r\widetilde{\Delta})v\big| \le C|\partial v| + C|\partial Sv| + C|\partial Rv| + Cr|L_g v|,$$

and the same bound for $|(tN\partial_t - rT\partial_r)v|$. Now,

$$rT\partial_r = r\partial_r\partial_t + r\overline{T}\partial_r = \frac{r}{c}N\partial_t + h\gamma\partial R,$$

$$tN\partial_t - rT\partial_r = (ct-r)\partial_{rt}^2 + h(ct-r)\widetilde{\partial}_i\partial_t + h\gamma\partial R.$$

We finally obtain $\sigma|\partial_{rt}^2 v| \le C|\partial v| + C|\partial Sv| + C|\partial Rv| + Cr|L_g v|$, and the same bound for $\sigma|\partial\partial_t v|$. To finish the estimates in zone I, we use the second expression of E, which gives

$$\sigma|\widetilde{\Delta}v| \le C|\partial v| + C|\partial Rv| + Ct|L_g v| + Ct|T\overline{T}v|.$$

Now $T\overline{T} = h\partial\gamma\partial - \gamma^{0i}\partial^2_{it} + h\gamma\partial^2_{ij}$. Using the previously established estimates, we get the result.

f) In zone II, we need equalities, which will be later cast into the framework of the symbolic calculus. With $[T,\partial_t] = [\overline{T},\partial_t] = h(\partial\gamma)R/r$, we get

$$(ct-r)\partial_t T = f(1+t)L_g + f(1+t)\bar{\theta}\partial + f(1+t)(\partial c)\partial + f(1+t)\bar{k}\partial$$

$$+f\partial S + f\partial R + f\partial + f(R\gamma)\partial + f(\partial\gamma)R + (ct-r)f(\partial\gamma)\partial.$$

Similarly, we obtain exactly the same formula for $(ct-r)\widetilde{\Delta}$.

At this stage, we proceed as follows: assume that we have an identity of the form $(r-ct)A = B$. We also have

$$\begin{aligned}\big(1-\chi(r-t)\big)\big(2-(r-ct)\big)A &= \big(1-\chi(r-t)\big)(2A-B),\\ \chi(r-t)\big(2+(r-ct)\big)A &= \chi(r-t)(2A+B),\end{aligned}$$

hence $\widetilde{\sigma}A = fA + fB$, and, using Lemma 2.2, $\sigma A = fA + fB$. Since

$$c\widetilde{\Delta} = L_g + f\partial T + f\bar{k}\partial,$$

applying this procedure to the above identities, we get the desired formula for $\sigma\partial_t T$ and $\sigma\widetilde{\Delta}$. We have

$$\sigma N^2 = \sigma c\widetilde{\Delta} - \sigma\Delta_S - \sigma\bar{\theta}N + \frac{\sigma}{c}\sum e_a(c)e_a = c\sigma\widetilde{\Delta} + f\partial R + f\partial + f(R\gamma)\partial,$$

hence we also have the desired formula for σN^2.

To control $T\partial_r$, we write

$$(ct-r)T\partial_r = \frac{ct-r}{r}(TS-\partial_t) + \frac{t}{r}(ct-r)T\partial_t.$$

Since $[T,\partial_r] = [\overline{T},\partial_r] = f_0(\partial\gamma)\partial + f_0(\gamma/r)\partial$, we obtain the formula for $(ct-r)\partial_r T$. Finally,

$$\partial_i = \omega_i\partial_r + \frac{f_0 R}{r}, \quad [R,T] = \frac{(R\gamma)R}{r} + \frac{f_0\gamma R}{r}$$

gives $(ct-r)(f_0/r)RT = f_0\partial R + f_0\partial + f_0(R\gamma)\partial$ which concludes the proof. □

3.2. A formula for L^2.

PROPOSITION 3.2. — *In zones* II, *we have the formulas* (*with* $Z = R_i$ *or* $Z = S$)

$$\begin{aligned} L &= \frac{f}{1+t}S + \frac{r-ct}{r+ct}L_1 + \frac{f\gamma}{1+t}R, \\ L^2 &= \frac{f}{1+t}LZ + \frac{f\sigma}{(1+t)^2}\frac{R^2}{r} + \frac{f\sigma}{1+t}L_g + \frac{f}{1+t}L \\ &\quad + \frac{f}{1+t}(Z\gamma)\partial + \frac{f\sigma}{1+t}(\partial\gamma)\partial + \frac{f}{1+t}\gamma\partial \\ &\quad\quad + \frac{f\sigma}{1+t}\bar{\theta}\partial + \frac{f\sigma}{(1+t)^2}\partial + \frac{f\sigma}{(1+t)^2}(R\gamma)\partial. \end{aligned}$$

Proof. — a) With the notations of the proof of Proposition 3.1, we have

$$T = \partial_t + \frac{h\gamma}{r}R, \quad N = c\partial_r + \frac{h\gamma}{r}R,$$

hence

$$L = T + N = \partial_t + c\partial_r + \frac{h\gamma}{r}R, \quad (r+ct)L = cS + c^2t\partial_r + r\partial_t + \frac{f\gamma}{r}R.$$

Now $c^2t\partial_r + r\partial_t - cS = (r-ct)(\partial_t - c\partial_r)$, and finally

$$L = \frac{2c}{r+ct}S + \frac{r-ct}{r+ct}L_1 + \frac{f\gamma}{1+t}R,$$

which is the desired formula.

b) We have

$$[S,L] = -(\partial_t + c\partial_r) + f\gamma\partial + fS\gamma\partial, \quad [R,L] = f\gamma\partial + fR\gamma\partial,$$

$$[L, L_1] = -2[T,N] = f\partial\gamma\partial.$$

Hence

$$\begin{aligned} L^2 = \frac{f}{1+t}LZ + \frac{r-ct}{r+ct}LL_1 + \frac{f}{1+t}L \\ + \frac{f}{1+t}\gamma\partial + \frac{f}{1+t}(Z\gamma)\partial + f\frac{r-ct}{r+ct}(\partial\gamma)\partial. \end{aligned}$$

Replacing LL_1 in terms of L_g and using the formulas

$$\Delta_S = \frac{f}{r^2}R^2 + \frac{f}{r}\partial + \frac{f}{r}(R\gamma)\partial, \quad k_{ij} = f\partial\gamma,$$

and Lemma 2.2, we obtain the desired formula. □

4. Commutation formula

Since we will be working with the special frame (e_1, e_2, L_1, L), we need to use tools from differential geometry to express $[L_g, X]$. We recall first the definition of the deformation tensor π of a given field X

$$^{(X)}\pi_{\alpha\beta} = D_\alpha X_\beta + D_\beta X_\alpha.$$

We remark that, for any field X and $\pi = {}^{(X)}\pi$,

$$\pi^{\alpha\beta} = \partial^\alpha(X^\beta) + \partial^\beta(X^\alpha) - X(g^{\alpha\beta}). \tag{4.1}$$

In fact, $D_\alpha X_\beta = \langle D_\alpha X, \partial_\beta\rangle = g_{\beta\gamma}\partial_\alpha(X^\gamma) + X^\gamma\Gamma_{\beta\alpha\gamma}$. From the explicit expressions of the Γ's,

$$\Gamma_{\alpha\beta\gamma} = \frac{1}{2}(\partial_\beta g_{\alpha\gamma} + \partial_\gamma g_{\alpha\beta} - \partial_\alpha g_{\beta\gamma}),$$

we get

$$\Gamma_{\alpha\beta\gamma} + \Gamma_{\beta\alpha\gamma} = \partial_\gamma(g_{\alpha\beta}).$$

Since $Xg_{\alpha\delta} = -g_{\alpha\beta}Xg^{\beta\gamma}g_{\gamma\delta}$, we obtain the result.

We use in this paper the explicit commutation formula (see for instance [7])

$$[L_g, X]\phi = \pi^{\alpha\beta}\nabla^2\phi_{\alpha\beta} + D_\alpha\pi^{\alpha\beta}\partial_\beta\phi - \frac{1}{2}\partial^\beta(\operatorname{tr}\pi)\partial_\beta\phi,$$

where $\pi = {}^{(X)}\pi$ and $\operatorname{tr}\pi = \pi^\alpha_\alpha$. In what follows, we are concerned with the cases $X = T$, $X = R_i$ and $X = S$.

Proposition 4.1. — *With $\pi = {}^{(X)}\pi$ in each case, we have the commutation formula*

$$[L_g, X]v = \pi^{\alpha\beta}\nabla^2 v_{\alpha\beta} + AGv + B\partial v + \frac{1}{4}L_1(\pi_{LL})L_1 v.$$

Here, $G = e_1$, $G = e_2$ or $G = L$ stands for a good derivative, $Z = R_i$ or $Z = S$ and the coefficients A and B have the following form:

i) *For $X = T$,*

$$\begin{aligned} A &\equiv \overline{A} = f\partial^2\gamma,\\ B &\equiv \overline{B} = \frac{f}{1+t}(\partial\gamma) + \frac{f}{1+t}(Z\gamma)\partial\gamma\\ &\qquad + f(\partial\gamma)^2 + \frac{f}{1+t}Z\partial\gamma + \frac{f\sigma}{1+t}\partial^2\gamma + f\gamma\partial^2\gamma. \end{aligned}$$

ii) *For $X = R_i$ or $X = S$,*

$$\begin{aligned} A &= f\partial\gamma + f(Z\gamma)\partial\gamma,\\ B &= \frac{f\gamma}{1+t} + \frac{f}{1+t}Z\gamma + f\gamma\partial\gamma + f(Z\gamma)\partial\gamma + \frac{f\sigma}{1+t}\partial\gamma + \frac{f\sigma}{1+t}(Z\gamma)\partial\gamma\\ &\qquad + \frac{f}{1+t}(Z\gamma)^2 + f\gamma Z\partial\gamma + \frac{f\sigma}{1+t}(Z\partial\gamma) + \frac{f}{1+t}Z^2\gamma. \end{aligned}$$

Proof. — 1a) To use the commutation formula above, we compute the various components of ${}^{(T)}\pi$, ${}^{(R_i)}\pi$, ${}^{(S)}\pi$. Since $T = \partial_0 - \gamma^{0i}\partial_i$, the derivatives of the coefficients of T are $h\partial\gamma$, while $Tg^{\alpha\beta}$ is $h\partial\gamma$. Hence ${}^{(T)}\pi_{\alpha\beta} = h\partial\gamma$.

1b) If $X = R_i$, indicating by "bar" the lifting relative to the standard metric,

$$\partial^\alpha X^\beta + \partial^\beta X^\alpha = \bar{\partial}^\alpha X^\beta + \bar{\partial}^\beta X^\alpha + \gamma^{\alpha\mu}\partial_\mu X^\alpha + \gamma^{\beta\mu}\partial_\mu X^\alpha = \mathrm{I} + \mathrm{II}.$$

The term I corresponds to the commutator of the standard d'Alembertian with R_i, hence is zero. Hence we obtain $\pi_{\alpha\beta} = h\gamma + hR\gamma$.

1c) If $X = S$, we proceed as in 1b), with the difference that, since $X^\alpha = x^\alpha$, $\mathrm{I} = 2g^{\alpha\beta}$, and $\pi^{\alpha\beta} = 2g^{\alpha\beta} + h\gamma + hS\gamma$. Since the term $2g^{\alpha\beta}$ yields $2L_g$ in the commutator $[L_g, S]$, we will ignore it in the sequence and discuss in parallel the commutators with R_i and S.

2) The commutation formula involves π in the higher order terms, and derivatives of the tensor $D_\alpha\pi$ in the lower order terms. By definition,

$$D_\mu\pi_{\alpha\beta} = \partial_\mu(\pi_{\alpha\beta}) - \pi(D_\mu\partial_\alpha, \partial_\beta) - \pi(\partial_\alpha, D_\mu\partial_\beta).$$

Since the Γ's are just $h\partial\gamma$, the last two terms in the right-hand side of the above formula are products of components of π by $h\partial\gamma$. Since we use the frame e_α whose coordinates with respect to the ∂_α are h, we must include terms which are products of components of π by $\partial h = h\partial\gamma + h/r$. Altogether,

$$D_\mu\pi(e_\alpha, e_\beta) = \partial_\mu\big(\pi(e_\alpha, e_\beta)\big) + h\pi\partial\gamma + \frac{h\pi}{r}\cdot$$

Exactly the same analysis applies to the lower order terms arising from the terms $\pi^{\alpha\beta}\nabla^2\phi_{\alpha\beta}$, since $D_{e_\alpha}e_\beta$ is a sum of $h(\partial\gamma)\partial$ and $h(\partial h)\partial$. These lower order terms will eventually enter the terms $B\partial$.

For $X = R_i$ or $X = S$, the derivatives of the components of π, are just

$$\partial\pi = h\partial\gamma + \frac{h\gamma}{r} + h(\partial\gamma)X\gamma + \frac{h}{r}X\gamma + hX\partial\gamma.$$

For $X = T$, we obtain $\partial\pi = h(\partial\gamma)^2 + (h/r)\partial\gamma + h\partial^2\gamma$.

3) We have the formulas

$$e_a = \frac{h}{r}R, \quad L = \frac{f\sigma}{r}\partial + f\gamma\partial + \frac{f}{r}S.$$

This allows us to prove that a G-derivative of a component of π behaves better than just any derivative. For $X = R_i$ or S, the components of π are $h\gamma + hX\gamma$. Applying G, we find $Gh\gamma + GhX\gamma + hG\gamma + hGX\gamma$, that is finally

$$G\pi = \frac{f\gamma}{r} + \frac{f}{r}X\gamma + f\gamma\partial\gamma + f(X\gamma)\partial\gamma + \frac{f\sigma}{r}\partial\gamma + \frac{f\sigma}{r}(X\gamma)\partial\gamma + \frac{f}{r}(X\gamma)^2 + f\gamma X\partial\gamma + \frac{f\sigma}{r}X\partial\gamma + \frac{f}{r}X^2\gamma.$$

For T, we find

$$G\pi = \frac{f}{r}\partial\gamma + \frac{f}{r}(X\gamma)\partial\gamma + f(\partial\gamma)^2 + \frac{f}{r}X\partial\gamma + \frac{f\sigma}{r}\partial^2\gamma + f\gamma\partial^2\gamma.$$

Using the above commutation formula, written with respect to the frame (e_1, e_2, L_1, L), we obtain the proposition. □

5. Perturbation coefficients and symbolic calculus

Let us fix once for all a nonnegative cutoff function

$$\overline{\chi}(s) = \begin{cases} 1 & \text{if } \frac{3}{4} \leq s \leq \frac{5}{4}, \\ 0 & \text{if } s \leq \frac{1}{2} \text{ or } s \geq \frac{3}{2}. \end{cases}$$

Abusively, we will also denote by $\overline{\chi}$ the function $\overline{\chi}(r/(1+t))$.

Definition 5.1. — For $Z = R_i$ or $Z = S$, we define the corresponding perturbation coefficient $a = a(Z)$ by

$$La + \frac{aTc}{c} = -\overline{\chi}\frac{Zc}{c}, \tag{5.1}$$

$a(0, t) = 0$ and $a = 0$ close to $t = 0$.

Since, for γ_0 small enough, the middle zone is an influence domain of its boundary for L, this definition makes sense, and $\operatorname{supp} a$ is contained is the middle zone (the only place where we need perturb the standard fields R_i and S).

Definition 5.2. — We define the fields $\widetilde{Z}$ as

$$\widetilde{R}_i = R_i + a(R_i)T, \quad \widetilde{S} = S + a(S)T.$$

In the sequence, we will just write $\widetilde{Z} = Z + aT$. Finally, we define the full collection of the modified fields Z_m to be

$$\widetilde{R}_i,\ \widetilde{S},\ T.$$

We have already defined f in section 2.

Definition 5.3. — We define N_0 to be any of the terms

$$\sigma^{-\frac{1}{2}}(1+t)^\mu\gamma, \quad \sigma^{\frac{1}{2}}(1+t)^\mu\partial\gamma, \quad (\sigma^{\frac{1}{2}}/\phi')\partial c, \quad (1+t)\bar{\theta}, \quad \sigma^{\frac{1}{2}}(1+t)\bar{k}. \tag{5.2}$$

We define N_k $(k \geq 1)$ to be any of the terms

$$\sigma^{-\frac{1}{2}}(1+t)^\mu Z_m^k\gamma, \quad \sigma^{\frac{1}{2}}(1+t)^\mu Z_m^k\partial\gamma, \quad \sigma(1+t)^\mu Z_m^{k-1}\partial^2\gamma,$$

$$\sigma^{-\frac{1}{2}}Z_m^{k-1}a, \quad \sigma^{\frac{1}{2}}Z_m^{k-1}\partial a, \quad (\sigma^{\frac{1}{2}}\phi')^{-1}Z_m^k c, (\sigma^{\frac{1}{2}}/\phi')Z_m^k\partial c,$$

$$(1+t)Z_m^k\bar{\theta}, \quad \sigma^{\frac{1}{2}}(1+t)Z_m^k\bar{k}.$$

All these quantities will be used only in the middle zone. We have, between the quantities f, N_k and the fields Z_m, what we call a symbolic calculus, which means that we have the following lemma.

SYMBOLIC CALCULUS LEMMA. — *We have the relations:*

$$\begin{aligned}
&\text{i)} & Z_m^k f &= \sum f N_{k_1}\cdots N_{k_j}, & k_1+\cdots+k_j &\le k,\\
&\text{ii)} & Z_m^k N_p &= \sum f N_{k_1}\cdots N_{k_j}, & k_1+\cdots+k_j &\le k+p,\\
&\text{iii)} & Z_m^k(1+t) &= (1+t)\sum f N_{k_1}\cdots N_{k_j}, & k_1+\cdots+k_j &\le k,\\
&\text{iv)} & Z_m^k \sigma &= \sigma\sum f N_{k_1}\cdots N_{k_j}, & k_1+\cdots+k_j &\le k,\\
&\text{v)} & Z_m^k \phi' &= \phi'\sum f N_{k_1}\cdots N_{k_j}, & k_1+\cdots+k_j &\le k.
\end{aligned}$$

Proof. — In view of the very structure of the relations, it is enough to prove them for $k=1$ and any p. To prove i), we have to check first the effect of aT applied to all the arguments of f_0 but γ. Since $T=f_0\partial$, by applying aT we get $f_0a/\sigma = fN_1$, hence $Z_mf_0 = fN_1$. Now $Z(r-t) = f_0\,\sigma$, $aT(r-t) = f_0(a/\sigma)\sigma$, hence iv) is proved for $k=1$, and similarly for iii). Also,

$$Z\phi' = f_0t\phi'' = \phi' f_0 t\Big(\frac{\phi''}{\phi'}\Big) = \phi' f_0,$$

$$aT\phi' = a\phi'' = \phi'\Big(\frac{a}{1+t}\Big)\frac{(1+t)\phi''}{\phi'} = \phi' f_0 N_1,$$

thus v) is proved for $k=1$. To finish the proof of i), we have to take into account the argument $(1-c)/(\sigma^{\frac12}\phi')$. But then the result follows from iv) and v).

To prove ii) for $k=1$, it is enough to observe that this follows from the very definitions of the N_k, using iii), iv) and v). □

We finally define, for $k\ge 1$,

$$M_k = \sum f N_0^pN_1^qN_{\ell_1}\cdots N_{\ell_i},\quad p\ge 0,\ q\ge 0,\ \ell_i\ge 2,\ \sum(\ell_i-1)\le k-1.$$

Each M_k involves only finitely many terms; in the course of the proof, since only a few commutators will be computed, p and q will take only finitely many values that we need not make explicit here. Remark that, as a consequence of points i) and ii) of the lemma,

$$M_1 = \sum f N_0^pN_1^q,\quad M_kM_\ell = M_{k+\ell-1},$$

$$\sum_{k_1+\cdots+k_j\le k} fN_{k_1}\cdots N_{k_j} = M_k,\quad Z_mM_k = M_{k+1}, Z_m^pM_k = M_{k+p}.$$

As a consequence of the assumptions of g and of the definition of the a, we have

PROPOSITION 5.1. — *In zone* II, *for all k, we have for some constant $C = C_k$,*

$$|N_k| \le C\mathrm{e}^{C\phi}, \quad |M_k| \le C\mathrm{e}^{C\phi}.$$

Since the proof of Proposition 5.1 involves defining a symbolic calculus slightly different from the one above, we postpone it to the Appendix to avoid confusion.

REMARK. — It is only in the proof of this proposition that we need to use the fields $\sigma^\mu\partial_\alpha$ in the middle zone in the formulation of our general low decay assumption. The resulting assumption is still much weaker than would be the corresponding assumption using hyperbolic rotations.

6. Commutations with the modified fields

Recall that the modified fields Z_m are the fields

$$R_i + aT,\ S + aT,\ T.$$

PROPOSITION 6. — *We have in zone* II *the formula*

$$\begin{aligned}(6.1)\quad [L_g, Z_m] = M_1 L_g + \frac{M_1}{(1+t)^{\frac12+\mu}}\partial Z_m + fD\phi'\partial Z_m + \frac{f\sigma^{-\frac12}}{1+t}GZ_m \\ + \frac{M_3}{(1+t)^{\frac12+\mu}}\partial + M_1\phi'\partial + \frac{M_1\sigma^{-\frac12}}{(1+t)^{\mu}}G\cdot\end{aligned}$$

Here G means a good derivative L or R_i/r as before, and $D = (\sigma^{\frac12}/\phi')\partial c = N_0$.

Proof. — 1) We write for simplicity ${}^{(T)}\pi = \overline{\pi}$ and ${}^{(Z)}\pi = \pi$. Recall from [1, Lemma III], that $k_{NN} = -Tc/c$. Hence $\overline{\pi}_{LL} = 2\langle D_L T, L\rangle = 2k_{NN} = -2Tc/c$. Since $[R_i, L] = (R_i c/c)N + \cdots + R$ and $[S, L] = (Sc/c)N + \cdots + R$, we also have

$$\frac12\pi_{LL} = \langle D_L Z, L\rangle = \langle D_L Z - D_Z L, L\rangle = \big\langle [L,Z], L\big\rangle = -\frac{Zc}{c}\cdot$$

2) We use the formula $[L_g, aT] = a[L_g, T] + 2\nabla aT + (L_g a)T$, along with the formula given in Proposition 4.1. We obtain

$$\begin{aligned}\nabla a &= -\frac12(L_1 a)L - \frac12(La)L_1 + (e_1 a)e_1 + (e_2 a)e_2,\\ [L_g, Z + aT]\phi &= \big\{(\pi^{\alpha\beta} + a\overline{\pi}^{\alpha\beta})\nabla^2\phi_{\alpha\beta} + (A + a\overline{A})G + (B + a\overline{B})\partial\big\}\\ &\quad + \big\{(-(L_1 a)L - (La)L_1 + 2(e_1 a)e_1 + 2(e_2 a)e_2)T\\ &\qquad + \frac14[L_1(\pi_{LL}) + aL_1(\overline{\pi}_{LL})]L_1 + (L_g a)T\big\}\\ &= \{\mathrm{I}\} + \{\mathrm{II}\}.\end{aligned}$$

We express the higher order terms of the form $q^{\alpha\beta}\nabla^2\phi_{\alpha\beta}$ in the frame (e_1, e_2, L_1, L), which gives

$$\frac{1}{4}q_{L_1L_1}\nabla^2\phi_{LL} + \frac{1}{4}q_{LL}\nabla^2\phi_{L_1L_1} + \frac{1}{2}q_{LL_1}\nabla^2\phi_{LL_1}$$

$$-\sum_{i=1,2}[q_{Le_i}\nabla^2\phi_{L_1e_i} + q_{L_1e_i}\nabla^2\phi_{Le_i}] + \sum_{i,j=1,2} q_{e_ie_j}\nabla^2\phi_{e_ie_j}.$$

We pay a special attention to the terms involving L_1^2, which have as coefficient

$$\frac{1}{4}(\pi_{LL} + a\overline{\pi}_{LL}) - \frac{1}{2}La.$$

Since we have seen in 1) that $\pi_{LL} = -2Zc/c$ and $\overline{\pi}_{LL} = -2Tc/c$, this coefficient is just, taking into account the definition of a,

$$-\frac{1}{2}La - \frac{1}{2}\frac{Zc}{c} - \frac{1}{2}\frac{Tc}{c} = -\frac{1}{2}(1-\overline{\chi})\frac{Zc}{c}\cdot$$

We have

$$\begin{aligned}2(\nabla a)T = & -\frac{1}{2}(L_1a)L^2 - \frac{1}{2}(La)L_1^2 - \frac{1}{2}(La)(-[L, L_1] + LL_1)\\ & -\frac{1}{2}(L_1a)LL_1 + 2\sum(e_ia)e_iT\\ = & -\frac{1}{2}(L_1a)L^2 - \frac{1}{2}(La)L_1^2\\ & -(Ta)\Big\{-L_g - \bar{k}T + (k_{NN} + \bar{\theta})N + \Delta_S + \sum\Big(2k_{aN} - \frac{e_a(c)}{c}\Big)e_a\Big\}\\ & +(La)[N, T] + 2\sum(e_ia)e_iT.\end{aligned}$$

Since $[N, T] = k_{NN}N + 2\sum k_{e_iN}e_i$, the first order terms of the part II of $[L_g, \widetilde{Z}]$ are

$$(Ta)\Big\{\bar{k}T - (k_{NN} + \bar{\theta})N - \sum\Big(2k_{aN} - \frac{e_a(c)}{c}\Big)e_a\Big\}$$

$$+ (La)k_{NN}N + 2(La)\sum k_{e_iN}e_i - \frac{1}{2}\Big(L_1\frac{Zc}{c} + aL_1\frac{Tc}{c}\Big)L_1 + (L_ga)T.$$

The coefficient ϵ_1 of the L_1-component of these terms is

$$\epsilon_1 = \frac{1}{2}(Ta)(\bar{k} + k_{NN} + \bar{\theta}) + \frac{1}{2}L_ga - \frac{1}{2}(La)k_{NN} - \frac{1}{2}\Big(L_1\frac{Zc}{c} + aL_1\frac{Tc}{c}\Big).$$

Now

$$
\begin{aligned}
&L_g a + (Ta)(\bar{k} + k_{NN} + \bar{\theta}) \\
&\qquad = -LL_1 a + \Delta_S a + (k_{NN} + \bar{\theta})(La) + \sum \Big(2k_{iN} - \frac{e_i(c)}{c} \Big) e_i(a), \\
&L_1 La + aL_1 \frac{Tc}{c} + L_1 \frac{Zc}{c} = (1 - \overline{\chi}) L_1 \frac{Zc}{c} - (L_1 a) \frac{Tc}{c} - (L_1 \overline{\chi}) \frac{Zc}{c}, \\
&\epsilon_1 \;=\; -\frac{1}{2}[L, L_1] a - \frac{1}{2} \Big(L_1 La + L_1 \frac{Zc}{c} + aL_1 \frac{Tc}{c} \Big) + \frac{1}{2} \Delta_S a \\
&\qquad\qquad + \frac{1}{2}(La)(k_{NN} + \bar{\theta}) - \frac{1}{2}(La) k_{NN} + \frac{1}{2} \sum \Big(2k_{iN} - \frac{e_i(c)}{c} \Big) e_i(a) \\
&\quad = \frac{1}{2} \Delta_S a + \frac{1}{2}(La)\bar{\theta} - 2 \sum k_{e_i N} e_i a + \frac{1}{2} \frac{Tc}{c} (L_1 a + 2Na) \\
&\qquad\qquad + \frac{1}{2}(L_1 \overline{\chi}) \frac{Tc}{c} - \frac{1}{2}(1 - \overline{\chi}) L_1 \frac{Zc}{c} + \frac{1}{2} \sum \Big(2k_{iN} - \frac{e_i(c)}{c} \Big) e_i(a).
\end{aligned}
$$

The fact that ϵ_1 is smaller than it should be, due to the choice of the a, is crucial for our argument. The rest of the first order terms of II is

$$
\begin{aligned}
&2(La) \sum k_{e_i N} e_i + \epsilon_0 L - (Ta) \sum \Big(2k_{aN} - \frac{e_a(c)}{c} \Big) e_a, \\
&\epsilon_0 \;=\; \frac{1}{2} \Delta_S a - 2 \sum k_{e_i N}(e_i a) + \frac{1}{2}(L_1 \overline{\chi}) \frac{Zc}{c} \\
&\qquad\qquad + \frac{1}{2} \overline{\chi} L_1 \frac{Zc}{c} + \frac{1}{2} a L_1 \frac{Tc}{c} + \frac{1}{2} \Big(\frac{Tc}{c} - \bar{\theta} \Big) L_1 a.
\end{aligned}
$$

To summarize,

$$
\begin{aligned}
&[L_g, \widetilde{Z}] = \{I\} - \frac{1}{2}(L_1 a) L^2 - \frac{1}{2}(La) L_1^2 - (Ta)\Delta_S + (Ta) L_g \\
&+2 \sum (e_i a) e_i T + 2(Na) \sum k_{e_i N} e_i + \frac{Ta}{c} \sum e_a(c) e_a + \epsilon_0 L + \epsilon_1 L_1.
\end{aligned}
$$

3) We will now express all coefficients in terms of $\sigma, 1+t$ and the N_k. We will also need improved formula for L_1^2 and L^2. We have first

$$
\sigma \partial T = f(1+t) L_g + f \partial Z + f \partial T + M_1 \partial + \frac{f \sigma^{-\frac{1}{2}}}{(1+t)^{\mu}} N_0 R. \tag{6.1}
$$

In fact, the coefficient of the gradient term in the expression given in Proposition 3.1 is

$$
\begin{aligned}
&fr\bar{\theta} + fr\partial c + fr\bar{k} + f + fR\gamma + f\sigma\partial\gamma \\
&\qquad = fN_0 + f\sigma^{-\frac{1}{2}} N_0 + fN_0 + f + \frac{f\sigma^{\frac{1}{2}}}{(1+t)^{\mu}} N_1 + \frac{f\sigma^{\frac{1}{2}}}{(1+t)^{\mu}} N_0 = M_1.
\end{aligned}
$$

Next, we have

$$\partial Z = M_1 \partial Z_m + M_1 \sigma^{-\frac{1}{2}} \partial + M_1 \sigma^{-\frac{1}{2}} (1+t) L_g + \frac{M_1}{\sigma(1+t)^\mu} R. \tag{6.2}$$

To prove this, we write, using (6.1),

$$\begin{aligned}
\partial Z &= \partial \widetilde{Z} - (\partial a) T - \frac{a}{\sigma}(\sigma \partial T) \\
&= \partial \widetilde{Z} - (\partial a) T - \frac{a}{\sigma}\big(f(1+t)L_g + f\partial T + M_1 \partial + f(\partial\gamma)R\big) \\
&\qquad\qquad - \frac{fa}{\sigma}\big(\partial \widetilde{Z} - (\partial a)T - a\partial T\big) \\
&= \Big(1 + \frac{fa}{\sigma}\Big)\partial \widetilde{Z} + \Big(f + \frac{fa}{\sigma}\Big)(\partial a)T + \Big(\frac{fa}{\sigma} + fa\frac{a}{\sigma}\Big)\partial T \\
&\qquad\qquad + \frac{f(1+t)a}{\sigma} L_g + \frac{fa}{\sigma} M_1 \partial + \frac{fa}{\sigma}(\partial\gamma)R,
\end{aligned}$$

which gives the formula. We deduce from this the formula

$$\sigma \partial T = M_1 (1+t) L_g + M_1 \partial + M_1 \partial Z_m + \frac{M_1 \sigma^{-\frac{1}{2}}}{(1+t)^\mu} \widetilde{R}.$$

We have now

$$\begin{aligned}
\frac{R^2}{r} &= \frac{R}{r}(\widetilde{R} - aT) = \frac{R}{r}\widetilde{R} - \frac{Ra}{r}T - \frac{aR}{r}T, \\
RT &= [R,T] + TR = \big[f\gamma + f(R\gamma)\big]\frac{R}{r} + TR, \\
Ra &= \widetilde{R}a - aTa = M_2 \sigma^{\frac{1}{2}}, \\
\frac{R^2}{r} &= \frac{R}{r}\widetilde{R} + \frac{M_2 \sigma^{\frac{1}{2}}}{1+t}\partial + \frac{M_1 \sigma}{1+t}\partial Z_m.
\end{aligned}$$

This gives

$$\Delta_S = \frac{fR}{(1+t)r} Z_m + \frac{M_1 \sigma}{(1+t)^2}\partial Z_m + \frac{M_2}{(1+t)^{\frac{3}{2}}}\partial + \frac{M_1}{1+t}\frac{R}{r}, \tag{6.3}$$

and similar formula for products $e_i e_j$. From the expression of L_g we get now

$$LL_1 = fL_g + \frac{M_1}{1+t}\partial Z_m + \frac{M_2}{1+t}\partial + \frac{M_1 \sigma^{-\frac{1}{2}}}{(1+t)^\mu}\frac{R}{r}\cdot \tag{6.4}$$

Also, using (6.2),

$$\begin{aligned}
\partial e_i = \partial\Big(\frac{hR}{r}\Big) &= \frac{f}{1+t}\partial Z + M_1 \sigma^{-1}\frac{R}{r} \\
&= \frac{M_1}{1+t}\partial Z_m + M_1 \sigma^{-\frac{1}{2}} L_g + M_1 \sigma^{-1}\frac{R}{r} + \frac{M_1 \sigma^{-\frac{1}{2}}}{1+t}\partial.
\end{aligned} \tag{6.5}$$

We write $L_1^2 = L_1(2T-L) = 2L_1T+[L,L_1]-LL_1$ and $L_1T = f\partial T = (f/\sigma)\sigma\partial T$,

$$L_1^2 = \frac{M_1(1+t)}{\sigma}L_g + \frac{M_1}{\sigma}\partial Z_m + \frac{M_2}{\sigma}\partial + \frac{M_1\sigma^{-\frac{3}{2}}}{(1+t)^\mu}R. \tag{6.6}$$

To obtain a good formula for L^2, we use Proposition 3.2 and compute carefully the term LZ:

$$\begin{aligned}
LZ &= L(\widetilde{Z}-aT) = L\widetilde{Z} - (La)T - aLT,\\
LT &= \frac{f}{1+t}ZT + \frac{f\sigma}{1+t}\partial T\\
&= \frac{f}{1+t}\partial Z + \frac{f}{1+t}\partial + \frac{f}{1+t}(Z\gamma)\partial + \frac{f}{1+t}(\sigma\partial T)\\
&= M_1L_g + \frac{M_1}{1+t}\partial Z_m + \frac{M_1}{1+t}\partial + \frac{M_1\sigma^{-\frac{1}{2}}}{(1+t)^{1+\mu}}R,\\
LZ &= L\widetilde{Z} + M_1\sigma^{\frac{1}{2}}L_g + \frac{M_1\sigma^{\frac{1}{2}}}{1+t}\partial Z_m + \frac{M_1\sigma^{\frac{1}{2}}}{1+t}\partial + \frac{M_1}{(1+t)^{1+\mu}}R.
\end{aligned}$$

Replacing LZ by this expression into the formula for L^2, and using the above expression for R^2/r, we obtain finally

$$L^2 = \frac{M_1\sigma}{1+t}L_g + \frac{f}{1+t}LZ_m + \frac{M_1\sigma}{(1+t)^2}\partial Z_m + \frac{M_1\sigma^{\frac{1}{2}}}{(1+t)^{\frac{3}{2}}}\partial + \frac{f}{1+t}L. \tag{6.7}$$

4) We are now in a position to express the terms I and II in the expression of $[L_g,\widetilde{Z}]$. First, we rewrite the coefficients A, B of Proposition 4.1 using the notations of the symbolic calculus. We have

$$Z^2 = \widetilde{Z}^2 + fa\widetilde{Z}\partial + fa^2\partial^2 + f(\widetilde{Z}a)\partial + M_1a\partial + fa\partial a\partial + fa^2\partial\gamma\partial,$$

and $Z^2\gamma = M_2\sigma^{\frac{1}{2}}(1+t)^{-\mu}$. From this follows easily

$$\begin{aligned}
\overline{A} &= \frac{M_1\sigma^{-1}}{(1+t)^\mu}, \quad \overline{B} = \frac{M_1\sigma^{-\frac{1}{2}}}{(1+t)^{2\mu}},\\
A &= \frac{M_1\sigma^{-\frac{1}{2}}}{(1+t)^\mu}, \quad B = \frac{M_2\sigma^{\frac{1}{2}}}{(1+t)^{1+\mu}} + \frac{M_1}{(1+t)^{2\mu}}.
\end{aligned}$$

From the proof of Proposition 4.1, we get

$$\pi^{\alpha\beta} + a\overline{\pi}^{\alpha\beta} = \alpha_0 g^{\alpha\beta} + h\gamma + hR\gamma + ah\partial\gamma = \frac{M_1\sigma^{\frac{1}{2}}}{(1+t)^\mu},$$

with $\alpha_0 = 0$ for $X = R_i$ and $\alpha_0 = 2$ for $X = S$. On the other hand, the lower order terms arising from $q^{\alpha\beta}\nabla^2\phi_{\alpha\beta}$ are of the form

$$\frac{M_1\sigma^{\frac{1}{2}}}{(1+t)^\mu}(h\partial\gamma + h\partial h)\partial = \Big(\frac{M_1\sigma^{\frac{1}{2}}}{(1+t)^{1+\mu}} + \frac{M_1}{(1+t)^{2\mu}}\Big)\partial.$$

Hence

$$I = \frac{1}{4}q_{LL}(L_1^2 - D_{L_1}L_1) + \frac{\sigma^{\frac{1}{2}}}{(1+t)^\mu}[M_1L^2 + M_1LL_1 + M_1\partial e_i + M_1e_ie_j] + \frac{M_1\sigma^{-\frac{1}{2}}}{(1+t)^\mu}G + \Big(\frac{M_2\sigma^{\frac{1}{2}}}{(1+t)^{1+\mu}} + \frac{M_1}{(1+t)^{2\mu}}\Big)\partial.$$

Using the simplified formula

$$\begin{aligned}
L^2 &= M_1L_g + \frac{M_1}{1+t}\partial Z_m + \frac{M_1}{1+t}\partial,\\
LL_1 &= fL_g + \frac{M_1}{1+t}\partial Z_m + \frac{M_2}{1+t}\partial + \frac{M_1\sigma^{-\frac{1}{2}}}{(1+t)^\mu}\partial,\\
\partial e_i &= M_1L_g + \frac{M_1}{1+t}\partial Z_m + \frac{M_1}{1+t}\partial + \frac{M_1}{\sigma}\frac{R}{r},\\
e_ie_j &= \frac{M_1}{1+t}\partial Z_m + \frac{M_2}{1+t}\partial,
\end{aligned}$$

we obtain finally

$$I = \frac{1}{4}q_{LL}L_1^2 + \frac{M_1}{(1+t)^{\frac{1}{2}+\mu}}\partial Z_m + M_1L_g + \frac{M_1\sigma^{-\frac{1}{2}}}{(1+t)^\mu}G + \frac{M_2}{(1+t)^{\frac{1}{2}+\mu}}\partial.$$

5) To express II, we compute first ϵ_0 and ϵ_1. We have

$$\begin{aligned}
R^2 &= \widetilde{R}^2 + fa\widetilde{R}\partial + fa^2Z_m\partial + fa\partial + f(\widetilde{R}a)\partial\\
&\quad + fa(\widetilde{R}\gamma)\partial + fa(\partial a)\partial + fa^2(\partial\gamma)\partial + fa^2(Z_m\gamma)\partial,\\
R^2a &= M_3\sigma^{\frac{1}{2}}, \quad \Delta_S a = M_3\sigma^{\frac{1}{2}}(1+t)^{-2},\\
e_ia &= \frac{hRa}{r} = \frac{M_2\sigma^{\frac{1}{2}}}{1+t}, \quad L_1\overline{\chi} = \frac{f}{1+t},\\
L_1\Big(\frac{Zc}{c}\Big) &= f(\partial c)Zc + f[\partial, Z]c + fZ\partial c\\
&= f(\partial c)(\widetilde{Z}c) + fa(\partial c)^2 + f(\partial c) + f\widetilde{Z}\partial c + faZ_m\partial c\\
&= \frac{M_1\sigma^{-\frac{1}{2}}}{1+t} + M_1\phi'.\\
L_1\Big(\frac{Tc}{c}\Big) &= f(\partial c)^2 + f(\partial\gamma)\partial c + fZ_m\partial c = \frac{M_1\sigma^{-1}}{(1+t)^{1+\mu}} + M_1\sigma^{-\frac{1}{2}}\phi',\\
\frac{Tc}{c} - \bar{\theta} &= \frac{fN_0}{1+t}\cdot
\end{aligned}$$

Collecting terms, we get

$$\epsilon_0 = \frac{M_1\sigma^{-\frac{1}{2}}}{1+t} + M_1\phi' + \frac{M_3}{(1+t)^{\frac{3}{2}}}, \quad \epsilon_1 = M_1\phi' + \frac{M_3}{(1+t)^{\frac{3}{2}}},$$

since $(1-\overline{\chi})(\sigma^{-\frac{1}{2}}/(1+t)) = f/(1+t)^{\frac{3}{2}}$. For the terms $(e_i a)e_i T$ appearing in II, we write

$$(e_i a)e_i T = \frac{f}{(1+t)^2}(Ra)\big([R,T]+TR\big) = \frac{M_2}{(1+t)^{1+\mu}}\partial + \frac{M_2}{(1+t)^{\frac{3}{2}}}\partial Z$$

and we use again (6.2). Finally, using (6.3), (6.6) and (6.7) to express the terms $(L_1 a)L^2$, $(La)L_1^2$ and $(Ta)\Delta_S$ of II, we obtain the result for $[L_g, \widetilde{Z}]$.

6) To check that $[L_g, T]$ has also the form (6.1) is easy, since it is given by Proposition 4.1. All terms $\overline{\pi}^{\alpha\beta}\nabla^2\phi_{\alpha\beta}$ are the same as before, if we ignore the improvement by a factor σ^{-1}. The only difference are the two terms $\frac{1}{4}\overline{\pi}_{LL}L_1^2$ and $\frac{1}{4}L_1(\overline{\pi}_{LL})L_1$, which are not partially cancelled by other terms as before. Now $L_1(Tc/c)$ has already been computed. Also

$$\overline{\pi}_{LL}L_1^2 = fD\sigma^{-\frac{1}{2}}\phi' L_1^2, \quad D = (\sigma^{\frac{1}{2}}/\phi')\partial c = N_0.$$

Writing

$$L_1^2 = 2L_1T + [L, L_1] - LL_1$$

and using the previous formula, we get finally

$$\overline{\pi}_{LL}L_1^2 = fD\phi' L_1 T + M_1 L_g + \frac{M_2}{(1+t)^{1+\mu}}\partial + \frac{M_1}{(1+t)^2}\partial Z_m,$$

which is the desired result. □

We will also need the following commutation lemma, similar to the ones in [1].

Lemma 6.1. — *In zone* II, *we have*

$$[Z_m^k, \partial] = \sum M_p \partial Z_m^q,$$

where $p \geq 1$, $p+q \leq k$ in the sum.

Proof. — For $Z_m = Z + aT$, we have

$$[Z_m, \partial] = [Z, \partial] - (\partial a)T + a[T, \partial].$$

Since $[T, \partial] = f(\partial\gamma)\partial = M_1\partial$, $[Z, \partial] = f\partial$, we obtain

$$[Z_m, \partial] = f\partial + fa(\partial\gamma)\partial + f(\partial a)\partial = M_1\partial.$$

Since $[Z_m^{k+1}, \partial] = Z_m[Z_m^k, \partial] + [Z_m, \partial]Z_m^k$, the lemma is easily proved by induction. □

Lemma 6.2. — *In zone* II, *we have, for $G = R_i/r$ or $G = L$,*

$$[Z_m^k, G] = \sum M_p G Z_m^q + \frac{\sigma^{\frac{1}{2}}}{1+t}\sum M_{p+1}\partial Z_m^q,$$

where in both sums $p \geq 1$ and $p+q \leq k$.

Proof. — We have

$$\Big[Z + aT, \frac{R}{r}\Big] = \Big[Z, \frac{R}{r}\Big] + \frac{a}{r}\big[T, R\big] - \frac{Ra}{r}T.$$

Since $[Z, R] = fR$, $[T, R] = (f\gamma + fR\gamma)R/r$, $Ra = \sigma^{\frac{1}{2}}M_2$, we obtain

$$\Big[Z + aT, \frac{R}{r}\Big] = M_1\frac{R}{r} + \frac{M_2\sigma^{\frac{1}{2}}}{1+}\cdot$$

To estimate $[\widetilde{Z}, L]$, we use the formula already proved in the proof of Lemma A.2:

$$[R+aT, L] = fZ_m cL + f(Rc)\partial + Q, \quad [S+aT, L] = (fZ_m c - 1)L + f(Sc)\partial + Q,$$

where

$$Q = \Big[h\gamma + hZ\gamma + ha\partial\gamma + \frac{R}{r}ha\gamma(Z\gamma)r + \frac{R}{r}ha\gamma^2 r\Big]\frac{R}{r} = M_1\frac{R}{r}\cdot$$

Hence

$$[\widetilde{Z}, L] = M_1 L + M_1\frac{R}{r} + \frac{M_1\sigma^{\frac{1}{2}}}{1+t}\partial.$$

We note also, from the same proof,

$$\begin{aligned}[T, L] &= \frac{fN_0\,\sigma^{-\frac{1}{2}}}{1+t}\partial + \Big[h\partial\gamma + \frac{h\gamma\gamma}{r} + \frac{h\gamma^2}{r}\Big]\frac{R}{r}\\ &= \frac{fN_0\,\sigma^{-\frac{1}{2}}}{1+t}\partial + \frac{f\sigma^{-\frac{1}{2}}}{(1+t)^{\mu}}\frac{R}{r},\end{aligned}$$

which gives the formula for all Z_m and $k = 1$.

By induction, we get, using the Symbolic Calculus Lemma,

$$\begin{aligned}[Z_m^{k+1}, G] &= Z_m[Z_m^k, G] + [Z_m, G]Z_m^k,\\ Z_m\Big(M_1G + M_2\frac{\sigma^{\frac{1}{2}}}{1+t}\partial\Big) &= M_2G + M_1[Z_m, G] + M_1GZ_m\\ &\quad + \frac{\sigma^{\frac{1}{2}}}{1+t}\big(M_1M_2\partial + M_3\partial + M_2[Z_m, \partial] + M_2\partial Z_m\big),\end{aligned}$$

which gives the result. □

7. Commutators in zones I and III

In zones I or III, σ is big, so we need not use a special frame to express $[L_g, Z_m^k]$, which is simply given by the following lemma.

LEMMA 7. — *We have, in zones* I *or* III,

$$\begin{aligned}[Z_m^k, L_g] &= C Z_m^{\ell'} L_g \\ &\quad + \textstyle\sum_1 h(\gamma)(Z_m^{p_1}\gamma)\cdots(Z_m^{p_i}\gamma)(Z_m^{q_1}\partial\gamma)\cdots(Z_m^{q_j}\partial\gamma)\partial^2 Z_m^\ell \\ &\quad + \textstyle\sum_2 h(\gamma)(Z_m^{p_1}\gamma)\cdots(Z_m^{p_i}\gamma)(Z_m^{q_1}\partial\gamma) \\ &\qquad\qquad \cdots(Z_m^{q_j}\partial\gamma)(Z_m^{r_1}\partial^2\gamma)\cdots(Z_m^{r_s}\partial^2\gamma)\partial Z_m^\ell.\end{aligned}$$

Here, C is a constant, h stands for a smooth function, and $\ell' \le k-1$,

$$\sum p + \sum q + \sum r + \ell \le k-1$$

in both sums. Moreover, $i+j \ge 1$ in $\sum_1$, while $j+s \ge 1$ in $\sum_2$.

Proof. — 1) We denote here by h any smooth function of γ. We have

$$[Z_m, \partial] = (h + h\partial\gamma)\partial,$$

and by an easy induction argument,

$$[Z_m^k, \partial] = \sum h(Z_m^{p_1}\gamma)\cdots(Z_m^{p_i}\gamma)(Z_m^{q_1}\partial\gamma)\cdots(Z_m^{q_j}\partial\gamma)\partial Z_m^l,$$

where $\sum p + \sum q + \ell \le k-1$. Similarly, we have

$$[Z_m, \partial^2] = (h + h\partial\gamma)\partial^2 + (h\partial\gamma + h(\partial\gamma)^2 + h\partial^2\gamma)\partial,$$

and an easy induction argument gives also

$$\begin{aligned}[Z_m^k, \partial^2] &= \textstyle\sum_1 h(Z_m^{p_1}\gamma)\cdots(Z_m^{p_i}\gamma)(Z_m^{q_1}\partial\gamma)\cdots(Z_m^{q_j}\partial\gamma)\partial^2 Z_m^\ell \\ &\quad + \textstyle\sum_2 h(Z_m^{p_1}\gamma)\cdots(Z_m^{p_i}\gamma)(Z_m^{q_1}\partial\gamma) \\ &\qquad\qquad \cdots(Z_m^{q_j}\partial\gamma)(Z_m^{r_1}\partial^2\gamma)\cdots(Z_m^{q_s}\partial^2\gamma)\partial Z_m^\ell,\end{aligned}$$

where in both sums $\sum p + \sum q + \sum r + \ell \le k-1$.

2) We have $L_g = L_0 + \gamma\partial^2 + h\partial\gamma\partial$. Hence

$$\begin{aligned}[Z, L_g] &= [Z, L_0] + [Z, \gamma\partial^2] + [Z, h\partial\gamma\partial] \\ &= CL_0 + (Z\gamma)\partial^2 + h\gamma\partial^2 + h(Z\gamma)(\partial\gamma)\partial + h(Z\partial\gamma)\partial + h(\partial\gamma)\partial \\ &= CL_g + (h\gamma + hZ\gamma)\partial^2 + \big(h(Z\gamma)(\partial\gamma) + h(Z\partial\gamma) + h(\partial\gamma)\big)\partial.\end{aligned}$$

Also,

$$[T, L_g] = [T, L_0] + [T, \gamma\partial^2] + [T, h\partial\gamma\partial] = h(\partial\gamma)\partial^2 + \big(h\partial^2\gamma + h(\partial\gamma)^2\big)\partial.$$

Mixing both formula,

$$\begin{aligned}[Z_m, L_g] &= CL_g + \big(h\gamma + h(Z\gamma) + h(\partial\gamma)\big)\partial^2 \\ &\quad + \big(h(\partial\gamma) + h(Z\partial\gamma) + h(\partial\gamma)^2 + h(\partial\gamma)(Z\gamma) + h\partial^2\gamma\big)\partial,\end{aligned}$$

which is the result for $k=1$. Writing as usual

$$[Z_m^{k+1}, L_g] = Z_m[Z_m^k, L_g] + [Z_m, L_g]Z_m^k,$$

we obtain the result by induction. □

8. Control of $|\partial Z_m^k u|_{L^2}$

LEMMA 8. — *We have* $|\partial Z_m^k u|_{L^2} \le C\mathrm{e}^{C\phi}$.

Proof. — We can write

$$L_g Z_m^k u = [L_g, Z_m^k]u = \sum Z_m^{\ell_1}[L_g, Z_m]Z_m^{\ell_2},$$

with $\ell_1 + \ell_2 \le k-1$. Using cutoffs, we will study the commutator separately in zone I, II and III.

1) We first state the special energy inequality that we use:

$$E(t) + \int_{0\le t'\le t} \sigma^{-1-\epsilon} \sum |Gu|^2 \,\mathrm{d}x\,\mathrm{d}t' \le CE(0) + C\int_{0\le t'\le t} |L_g u|\cdot|\partial u|\,\mathrm{d}x\,\mathrm{d}t' + C\int_0^t A(t')E(t')\,\mathrm{d}t'.$$

Here, $E(t) = |(\partial u)(.\,,t)|_{L^2}^2$ is the standard energy of u at time t, and

$$A(t) = |\partial c|_{L_x^\infty} + \big|\sigma^{-1}(1-c)\big|_{L_x^\infty}.$$

This is a consequence of [2], the hypothesis of Theorem 4 of [2] being satisfied (it is understood of course that ϵ is chosen small enough with respect to $\mu - \frac{1}{2}$).

2) In zone I or III, we use Lemma 7. For $k=1$, we see that the coefficients of terms involving ∂u are bounded by $C\mathrm{e}^{C\phi}\sigma^{-\frac{1}{2}}(1+t+r)^{-\mu} \le C(1+t)^{-\nu}$, where we can take $\nu = 1 + \frac{1}{2}(\mu - \frac{1}{2}) > 1$. We write a term $b\partial^2 u = (b/\sigma)\sigma\partial^2 u$, and again

$$\Big|\frac{b}{\sigma}\Big| \le \frac{C}{(1+t)^\nu}.$$

In zone III, using the pointwise estimates of Propostion 3.1, we thus obtain

$$\big|[L_g, Z_m]u\big|_{L^2} \le \frac{C}{(1+t)^\nu} \sum_{\ell\le 1} |\partial Z_m^\ell u|_{L^2},$$

and the terms will be easily handled using Gronwall lemma. In zone I, if the second order derivative is of the form $\partial\partial_t u$, we can use the first formula of Proposition 3.1 and proceed exactly as before. If the second order derivative is $\partial_i\partial_j$, we have to proceed again as in [10]. We write, with L^2 norms in the whole space,

$$\begin{aligned}\sum |\sigma\partial_{ij}^2 v|_{L^2} &\le C|\partial v|_{L^2} + C|\sigma\Delta v|_{L^2}\\ &\le C|\partial v|_{L^2} + C|\sigma\widetilde{\Delta} v|_{L^2} + C\gamma_0 \sum |\sigma\partial_{ij}^2 v|_{L^2},\end{aligned}$$

and obtain for γ_0 small enough

$$\sum |\sigma\partial_{ij}^2 v|_{L^2} \le C|\partial v|_{L^2} + C|\sigma\widetilde{\Delta} v|_{L^2}.$$

To handle the term involving $\widetilde{\Delta}v$, we split it into three terms corresponding to the three zones. In zones I and III, we use again the pointwise estimates of Proposition 3.1. In zone II, we use the pointwise estimate following from the proof of Proposition 3.1 there

$$\sigma|\widetilde{\Delta}v| \leq C|\partial v| + C|\partial Zv| + C|\partial Tv| + Ct|L_g v| + C|\partial\gamma|\cdot|Rv|.$$

Taking again γ_0 small enough, the terms $\sum|\sigma\partial^2_{ij}u|$ disappear from the right-hand side. Using formula (6.2) to transform terms ∂Zu into $\partial Z_m u$, we see that we are left with easily handled terms, except for the terms

$$\frac{1}{(1+t)^\nu}|\partial\gamma|\cdot|Ru|.$$

But these terms are bounded by $C\sigma^{-\frac{1}{2}}(1+t)^{-\mu}|Gv|$, which can be easily handled using the inequality stated in 1). For $k \geq 2$, we proceed inductively along completely similar lines.

3) To analyze the term $Z_m^{\ell_1}[L_g, Z_m]v$, we apply Leibniz formula and use the commutators' Lemma 6.1 and 6.2. We distinguish the critical terms, involving ∂Z_m^k or GZ_m^k, from the noncritical terms, which, by induction, we can assumed to be already estimated. We say that a term $b\partial Z_m^\ell$ is integrable if $b = (1+t)^{-\nu}$ for some $\nu > 1$. The critical terms arise only if $Z_m^{\ell_1}$ goes through $[L_g, Z_m]$: they are of the form

$$\frac{M_1}{(1+t)^{\frac{1}{2}+\mu}}\partial Z_m^k, \quad fD\phi'\partial Z_m^k, \quad \frac{f\sigma^{-\frac{1}{2}}}{1+t}GZ_m^k.$$

Using the Symbolic Calculus Lemma and the commutation lemmas, we see that the non critical terms in $[L_g, Z_m^k]u$ are of the form

$$\frac{M_j}{(1+t)^{\frac{1}{2}+\mu}}\partial Z_m^\ell, \quad M_j\phi'\partial Z_m^\ell, \quad \frac{M_j\sigma^{-\frac{1}{2}}}{(1+t)^\mu}GZ_m^\ell,$$

for various j and ℓ. In fact, commuting Z_m^ℓ with ∂ yields only terms $M_j\partial Z_m^i$ of the desired form, while commuting Z_m^ℓ with G yields either terms $M_jGZ_m^i$ of the desired form or integrable terms $b\partial Z_m^i$.

To use the above energy inequality for $v = Z_m^k u$, we have to make sure that the energy norm at time $t=0$ is bounded. Since the a are zero close to $t=0$, $Z_m = R, S$ or $T = \partial_t + \gamma\partial$. Moreover, the equation $L_g u = 0$ can be written $\partial_t^2 u = \gamma\partial^2 u + h(\gamma)\partial\gamma\partial$, with ∂^2 containing at most one t-derivative. Hence we have the formula

$$Z_m^k u_{|\,t=0} = \sum h(\gamma)(x^{\alpha_1}\partial_{x,t}^{\beta_1}\gamma)\cdots(x^{\alpha_i}\partial_{x,t}^{\beta_i}\gamma)x^\alpha\partial_x^\beta\partial_t^\ell u,$$

where $|\alpha_j| \leq |\beta_j|$, $|\alpha| \leq |\beta|$, $l \leq 1$. The same formula holds for $\partial_t Z_m^k u$ as well. From the assumptions on γ, all coefficients involving γ in the above formula are bounded. Hence the norms $|\partial Z_m^k u|_{L^2}$ are bounded as a consequence of the assumptions on u_0, u_1.

Adding all inequalities on ∂Z_m^ℓ for $\ell \leq k$, we see that all integrable terms and all terms involving G-dervatives are easily absorbed using Gronwall Lemma. For the critical terms $fD\phi'\partial Z_m^k u$, we note that fD is bounded, thus the use of Gronwall lemma yields a bound $C\mathrm{e}^{C\phi}$, which is what is claimed. For the noncritical terms $M_j\phi'\partial Z_m^\ell u$, they are bounded by $C\phi'\mathrm{e}^{C\phi}$, the integral of which is bounded by $C\mathrm{e}^{C\phi}$. This completes the proof. □

9. End of the proof

We have now to recover the standard fields Z from the Z_m. More precisely, our aim is to obtain L^2 estimates in zones II and III of $\partial Z^2 u$ and $\sigma\partial^2 Zu$.

9.1. The first step is to modify the pointwise estimates of Proposition 3.1 to adapt them to zone II.

LEMMA 9.1.1. — *For any v, we have in zones* II *and* III *the pointwise estimates*

$$\sigma|\partial_{ij}^2 v| \leq C|\partial v| + C|\partial Rv| + C\sigma|N^2 v|,$$

$$\sigma|\partial\partial_t v| \leq C|\partial v| + C|\partial Rv| + C\sigma|\partial Tv|.$$

Proof. — From the proof of Proposition 3.1, it is enough to control $\sigma\partial_r^2 v$ to prove the first formula. But

$$\frac{N}{c} = \partial_r + \frac{h\gamma R}{r}, \quad \partial_r^2 = -\frac{Nc}{c^3}N + \frac{1}{c^2}N^2 + \Big(f(\partial\gamma) + \frac{f\gamma}{r}\Big)\frac{R}{r} + f\Big(\frac{\gamma}{r}\Big)\partial R$$

give the estimate. Similarly,

$$\partial T = \partial\partial_t + \partial\Big(h\gamma\frac{R}{r}\Big) = \partial\partial_t + \Big(f\partial\gamma + \frac{f\gamma}{r}\Big)\frac{R}{r} + f\Big(\frac{\gamma}{r}\Big)\partial R,$$

which completes the proof. □

LEMMA 9.1.2. — *On the boundary $r = \frac{3}{2}(1+t)$ of zone* III*, we have*

$$|Z^k u| \leq C(1+t)^{-\frac{1}{2}}\mathrm{e}^{C\phi}.$$

Proof. — From Sections 7 and 8, we know that in zone III we have for all k

$$|\partial Z^k u|_{L^2} + |\sigma\partial^2 Z^k u|_{L^2} \leq C\mathrm{e}^{C\phi}.$$

Using Lemma 2.3 of [3], we deduce $|\partial Z^k u| \leq Cr^{-\frac{3}{2}}\mathrm{e}^{C\phi}$. Integrating from infinity, we get the estimate. □

We need now to control a function in zone II from its gradient and its boundary values.

LEMMA 9.1.3. — *Let $C_0 < 1 < C_1$, $\alpha > \frac{1}{2}$, and denote by D_t the domain*

$$C_0(1+t) \leq r \leq C_1(1+t).$$

For all v, we have

$$\Big|\frac{\sigma^{-\alpha}}{(1+t)^{\frac{1}{2}}}v\Big|^2_{L^2(D_t)} \leq C|v_r|^2_{L^2(D_t)} + C(1+t)\int v^2\big(C_1(1+t),\omega,t\big)\,\mathrm{d}\omega.$$

Proof. — With $M_i = M_i(t) = C_i(1+t)$, we write

$$v(r) = v(M_1) - \int_r^{M_1} v_r\,\mathrm{d}s,\quad v(r)^2 \leq 2v(M_1)^2 + 2(M_1 - r)\int_r^{M_1} v_r^2\,\mathrm{d}s.$$

Hence

$$\int_{M_0}^{M_1}\frac{\sigma(r)^{-2\alpha}}{(1+t)}v(r)^2\,\mathrm{d}r \leq \frac{Cv(M_1)^2}{1+t}\int_{M_0}^{M_1}\frac{\mathrm{d}r}{\sigma(r)^{2\alpha}} + C\int_{M_0}^{M_1} v_r(s)^2\,\mathrm{d}s\int_{M_0}^{s}\frac{\mathrm{d}r}{\sigma(r)^{2\alpha}}\cdot$$

Since

$$\int_{M_0}^{M_1}\frac{\mathrm{d}r}{\sigma(r)^{2\alpha}} \leq \int\frac{\mathrm{d}u}{(1+u^2)^{\alpha}} \leq C$$

we get the result by multiplying the inequality by $(1+t)^2$ and integrating in ω. □

9.2. Though we could manage without it, it is more transparent to introduce here a "hat-calculus", analogous to the symbolic calculus of Section 5 and to the "bar-calculus" of the Appendix. We keep f as before, and define the fields $\widehat{Z}_m$ to be $Z + aT$ or $\sigma^\mu T$ (thus they are the same as the fields $\overline{Z}_m$). We define $\widehat{N}_0$ (exactly as N_0) to be any of the terms

$$\sigma^{-\frac{1}{2}}(1+t)^\mu\gamma,\quad \sigma^{\frac{1}{2}}(1+t)^\mu\partial\gamma,\quad \frac{\sigma^{\frac{1}{2}}}{\phi'}\partial c,\quad (1+t)\bar{\theta},\quad \sigma^{\frac{1}{2}}(1+t)\bar{k}.$$

For $k \geq 1$, we define $\widehat{N}_k$ to be any of the terms

$$\sigma^{-\frac{1}{2}}(1+t)^\mu\widehat{Z}_m^k\gamma,\quad \sigma^{\frac{1}{2}}(1+t)^\mu\widehat{Z}_m^k\partial\gamma,\quad \sigma(1+t)^\mu\widehat{Z}_m^{k-1}\partial^2\gamma,$$

$$\sigma^{-\frac{1}{2}}\widehat{Z}_m^{k-1}a,\quad \sigma^{\frac{1}{2}}\widehat{Z}_m^{k-1}\partial a,\quad \frac{\sigma^{-\frac{1}{2}}}{\phi'}\widehat{Z}_m^k c,\quad \frac{\sigma^{\frac{1}{2}}}{\phi'}\widehat{Z}_m^k\partial c, (1+t)\widehat{Z}_m^k\bar{\theta},\quad \sigma^{\frac{1}{2}}(1+t)\widehat{Z}_m^k\bar{k}.$$

In other words, the quantity are the same as in the Standard Calculus, but the fields are different. Just as for the calculus in the Appendix, we have a Symbolic Calculus Lemma for the "hat-Calculus". We also have the following easy lemma.

LEMMA. — *Recalling that Z_0 means R, S or $\sigma^\mu\partial$, we have*

$$\widehat{Z}_m^k = \sum f\widehat{N}_{k_1}\cdots\widehat{N}_{k_i}\,\widehat{Z}_m^{r_1}(\sigma^{-\frac{1}{2}}a)\cdots\widehat{Z}_m^{r_j}(\sigma^{-\frac{1}{2}}a)Z_0^p,$$

where in the sum $p \geq 1$, $\sum k_\ell + \sum r_m + p \leq k$.

Since the quantities $\widehat{Z}_m^r(\sigma^{-\frac12}a) \equiv \overline{Z}_m^r(\sigma^{-\frac12}a)$ are already estimated, we obtain by induction from the above Lemma the estimates $|\widehat{M}_k| \le C\mathrm{e}^{C\phi}$. Note also that $N_0 = \widehat{N}_0$, any $N_1 = f\widehat{N}_1$, and more generally, $M_k = \sum \widehat{M}_k$. This allows to use the previouly obtained formula, established with M_k coefficients, replacing these coefficients by $\widehat{M}_k$.

9.3. We prove now formula for ∂Z, ∂Z^2, $\sigma\partial^2 Z$.

LEMMA 9.3. — *In zone* II, *we have the formulas* (*recalling that* $Z = R_i$ *or* $Z = S$)

$$\begin{aligned}
\partial Z &= \widehat{M}_1\partial + \widehat{M}_1\partial Z_m + \widehat{M}_1\sigma^{-\frac12}(1+t)L_g + \widehat{M}_1\frac{\sigma^{-1}}{(1+t)^\mu}\widetilde{Z},\\
\partial Z^2 &= \sum_{\ell+\ell'\le 2}\widehat{M}_{\ell+1}\partial Z_m^{\ell'} + \frac{\sigma^{-1}}{(1+t)^\mu}\sum_{\ell+\ell'\le 2}\widehat{M}_{\ell+1}Z_m^{\ell'}\\
&\qquad + \big(\sigma^{-\frac12}(1+t)\big)\sum_{\ell+\ell'\le 1}\widehat{M}_{\ell+1}\widehat{Z}_m^{\ell'}L_g,\\
\sigma\partial^2 Z &= \widehat{M}_2\partial + \widehat{M}_2\partial Z_m + \widehat{M}_1\sigma\partial^2 + \widehat{M}_1\sigma\partial^2 Z_m\\
&\qquad + (1+t)(\widehat{M}_2 + \widehat{M}_1\sigma^{\frac12}\partial)L_g + \widehat{M}_2\sigma^{-\frac12}(1+t)^{-\mu}Z_m.
\end{aligned}$$

Proof. — The first formula has already been established in (6.2), taking into account that $\sigma^{-1}(1+t)^{-\mu}aT = \widehat{M}_1\partial$. To get the second, we apply Z to the left, using also that $Z = \widetilde{Z} - aT = \widetilde{Z} - (a/\sigma^\mu)(\sigma^\mu T) = \widehat{M}_1\widehat{Z}_m$. We obtain, using the "hat"-symbolic calculus,

$$\begin{aligned}
Z\partial Z &= f\partial Z + \partial Z^2\\
&= \widehat{M}_2\partial + \widehat{M}_2\partial Z_m + \widehat{M}_2\sigma^{-\frac12}(1+t)L_g + \widehat{M}_2\frac{\sigma^{-1}}{(1+t)^\mu}\widetilde{R}\\
&\quad + \widehat{M}_1(\partial + \partial Z) + \widehat{M}_1(\partial Z_m + \partial Z Z_m) + \sigma^{-\frac12}(1+t)(\widehat{M}_1 + \widehat{M}_1\widehat{Z}_m)L_g\\
&\qquad + \frac{\sigma^{-1}}{(1+t)^\mu}(\widehat{M}_1\widetilde{Z} + \widehat{M}_1 Z_m^2) + \widehat{M}_1\partial Z_m.
\end{aligned}$$

We express now the terms ∂Z and $\partial Z Z_m$ using the first formula, and the formula already established for $[L_g, Z_m]$. This produces desired terms, except for the terms $\sigma^{-\frac12}(1+t)\sigma^{-\frac12}(1+t)^{-\mu}G$. If $G = R/r$, such a term is already in the expression for ∂Z^2. If

$$G = L = \frac{f}{1+t}Z + \frac{f\sigma}{1+t}\partial + \frac{f\sigma^{\frac12}}{(1+t)^\mu}R,$$

we also get the desired terms.

To prove the third formula, we write

$$\begin{aligned}\sigma\partial^2 Z &= \sigma\partial^2(\widetilde{Z}-aT)=\sigma\partial^2\widetilde{Z}-\sigma(\partial^2 a)T+f(\partial a)\sigma\partial T+fa\sigma\partial^2 T\\ &=\sigma\partial^2\widetilde{Z}+\widehat{M}_2\partial+(f\partial a+fa/\sigma)(\sigma\partial T)+fa\partial(\sigma\partial T),\end{aligned}$$

where we have used $[\sigma,\partial]=f$. Now, using the formula for $\sigma\partial T$, we have

$$\begin{aligned}\partial(\sigma\partial T) &= \partial\big(\widehat{M}_1\partial+\widehat{M}_1\partial Z_m+\widehat{M}_1(1+t)L_g+\widehat{M}_1\sigma^{-\frac12}(1+t)^{-\mu}\widetilde{Z}\big)\\ &=(\sigma^{-\mu}\widehat{M}_2)\Big(\partial+\partial Z_m+(1+t)L_g+\frac{\sigma^{-\frac12}}{(1+t)^{\mu}}\widetilde{Z}\Big)\\ &\qquad+\widehat{M}_1L_g+\widehat{M}_1(1+t)\partial L_g+\widehat{M}_1\partial^2+\widehat{M}_1\partial^2 Z_m\\ &\qquad+\widehat{M}_1\frac{\sigma^{-\frac32}}{(1+t)^{\mu}}\widetilde{Z}+\widehat{M}_1\frac{\sigma^{-\frac12}}{(1+t)^{\mu}}\partial\widetilde{Z}.\end{aligned}$$

Replacing this in the above formula for $\sigma\partial^2 Z$, we obtain the desired formula. □

9.4. In [3], Lemma 2.3 contains the following *a priori* inequality.

LEMMA. — *For some C, we have for $r\geq 1+\frac12 t$ and all v, the inequality*

$$\big|v(x,t)\big|\leq\frac{C\sigma^{-\frac12}}{r}\Big[\sum_{\ell\leq 2}|R^{\ell}v|_{L^2}+\sum_{\ell\leq 1}|\sigma R^{\ell}\partial_r v|_{L^2}\Big],$$

where the L^2 norms are taken over $r\geq 1+\frac12 t$.

This lemma, combined with the following lemma, finishes the proof.

LEMMA 9.4. — *We have, in zones* II *and* III*,*

$$\sum_{k\leq 2}\big|Z^k(\partial u)\big|_{L^2}+\sum_{k\leq 1}\big|\sigma\partial Z^k(\partial u)\big|_{L^2}\leq C\mathrm{e}^{C\phi}.$$

Proof. — We have already proved the estimate in zone III. In zone II, using Lemma 9.3, it is enough to control the L^2 norms of the terms

$$\sigma^{-\frac12}(1+t)^{-\mu}\sum_{k\leq 2}Z_m^k u, \tag{9.4.1}$$

$$\sigma\partial^2 u,\ \sigma\partial^2 Z_m u. \tag{9.4.2}$$

To estimate the terms (9.4.2), we use Lemma 9.1.1 with $v=u$ or $v=Z_m u$; taking into account the formula for ∂Z, $\sigma\partial T$ and σN^2, we get more terms (9.4.1) to estimate along with terms $\widehat{M}_1(1+t)L_g Z_m u$. Using Lemma 9.1.3 for $v=Z_m^k u$, $k\leq 2$, and Lemma 9.1.2, we see that the L^2 norm of terms (9.4.1) is bounded by

$$C|\partial Z_m^k u|_{L^2}+C\mathrm{e}^{C\phi}\leq C\mathrm{e}^{C\phi}.$$

To estimate the term involving $L_g Z_m u = [L_g, Z_m]u$, we inspect formula (6.1): we see that it is enough to control $\sigma^{-\frac{1}{2}}(1+t)^{1-\mu} G Z_m^\ell u$, $\ell \le 1$. Now, as above, if $G = R/r$, this term reduces to already estimated terms. If $G = L$, we proceed as before without difficulties. □

Appendix
Proof of Proposition 5.1

A.1. We define as in Section 5 the elements of a symbolic calculus. The symbol f has the same meaning as before, but the fields are different, for a reason which will become clear in the proof of Lemma A.2 below.

We denote by $\overline{Z}_m$ either

$$R_i + aT,\ \ S + aT\ \text{ or }\ \sigma^\mu T.$$

We define $\overline{N}_0$ to be any of the terms

$$\sigma^{-\frac{1}{2}}(1+t)^\mu \gamma, \quad \sigma^{\frac{1}{2}}(1+t)^\mu \partial\gamma,$$

and, for $k \ge 1$, we define $\overline{N}_k$ to be any of the terms

$$\sigma^{-\frac{1}{2}}(1+t)^\mu \overline{Z}_m^k \gamma, \quad \sigma^{\frac{1}{2}}(1+t)^\mu \overline{Z}_m^k \partial\gamma, \quad \sigma^{-\frac{1}{2}} \overline{Z}_m^{k-1} a, \quad \frac{\sigma^{-\frac{1}{2}}}{\phi'} \overline{Z}_m^k c.$$

We prove now a symbolic calculus lemma for the system $f, \overline{Z}_m, \overline{N}_k$, which is exactly the same as Lemma 5.1.

LEMMA A.1. — *We have the formulas, where $k_1 + \cdots + k_j \le k$,*

i) $\overline{Z}_m^k f = \sum f \overline{N}_{k_1} \cdots \overline{N}_{k_j}$,

ii) $\overline{Z}_m^k \overline{N}_p = \sum f \overline{N}_{k_1} \cdots \overline{N}_{k_j}$,

iii) $\overline{Z}_m^k t = (1+t) \sum f \overline{N}_{k_1} \cdots \overline{N}_{k_j}$,

iv) $\overline{Z}_m^k \sigma = \sigma \sum f \overline{N}_{k_1} \cdots \overline{N}_{k_j}$,

v) $\overline{Z}_m^k \phi' = \phi' \sum f \overline{N}_{k_1} \cdots \overline{N}_{k_j}$.

Proof. — To check i), iii), iv) and v) for $k = 1$, we just have to check the case $\overline{Z}_m = \sigma^\mu T$, since $\overline{Z}_m = Z + aT$ is very similar to what has been done before. But $\sigma^\mu T t = \sigma^\mu = f_0(1+t)$, $\sigma^\mu T \sigma = f_0\,\sigma$, $\sigma^\mu T \phi' = \sigma^\mu(\phi''/\phi')\phi' = f_0 \phi'$. It remains to check ii) for $k = 1$, but this is obvious from the definition, using i), iii), iv) and v). □

Using this lemma, we define $\overline{M}_k$ from the quantities $\overline{N}_k$ exactly as we have defined M_k from N_k in Section 5, with the same properties with respect to the action of the fields $\overline{Z}_m$.

A.2. We perform now a careful computation of $[\overline{Z}_m, L]$.

Lemma A.2. — *We have*

$$[\overline{Z}_m, L] = \overline{M}_1 L + \frac{\overline{M}_1}{(1+t)^{\frac{1}{2}+\mu}} \overline{Z}_m + f A_0 \overline{Z}_m,$$

where $A_0 = Tc + f_0(c-1)/\sigma = \overline{M}_1 \phi'$.

Proof. — We use as in Section 3.1 the notation h for any smooth function ω and γ. We have

$$T = \partial_t + \frac{h\gamma R}{r}, \quad N = c\partial_r + \frac{h\gamma R}{r}\cdot$$

Hence

$$\begin{aligned} [R_i, T] &= h\gamma \frac{R}{r} + h(R\gamma)\frac{R}{r}, \\ [R_i, N] &= \frac{R_i c}{c}\Big(N + h\gamma \frac{R}{r}\Big) + h\gamma \frac{R}{r} + h(R\gamma)\frac{R}{r}, \\ [R_i, L] &= \frac{R_i c}{c} N + h\gamma \frac{R}{r} + h(R\gamma)\frac{R}{r}\cdot \end{aligned}$$

Since $[S, \partial_t] = -\partial_t$ and $[S, \partial_r] = -\partial_r$, we obtain analogously

$$\begin{aligned} [S, L] &= \frac{Sc}{c}\Big(N + h\frac{\gamma R}{r}\Big) - (\partial_t + c\partial_r) + h\gamma\frac{R}{r} + h(R\gamma)\frac{R}{r} \\ &= \frac{Sc}{c} N - L + h\gamma \frac{R}{r} + h(R\gamma)\frac{R}{r}\cdot \end{aligned}$$

Since $[T, L] = [T, N] = (Tc/c)N + h(\partial\gamma)R/r + h\gamma(R\gamma)R/r^2 + h(\gamma/r)^2 R$, we obtain

$$\begin{aligned} [R_i + aT, L] &= \Big(\frac{R_i c}{c} + \frac{aTc}{c}\Big)L - \Big(La + \frac{aTc}{c} + \frac{R_i c}{c}\Big)T + Q, \\ [S + aT, L] &= \Big(\frac{Sc}{c} + \frac{aTc}{c} - 1\Big)L - \Big(La + \frac{aTc}{c} + \frac{Sc}{c}\Big)T + Q, \end{aligned}$$

where Q has the form (with $Z = R_i$ or $Z = S$)

$$Q = \Big[h\gamma + hZ\gamma + ha\partial\gamma + \frac{ha\gamma(Z\gamma}{r} + \frac{ha\gamma^2}{r}\Big]\frac{R}{r}\cdot$$

We see that the coefficient of R in Q is

$$\frac{f_0\,\sigma^{\frac{1}{2}}}{(1+t)^{\mu+1}}(\overline{N}_0 + \overline{N}_0\overline{N}_1 + \overline{N}_0\overline{N}_1^2 + \overline{N}_0\overline{N}_1^3 + \overline{N}_1 + \overline{N}_1\overline{N}_0^2 + \overline{N}_1^2).$$

Now $R_i = R_i + aT - (a/\sigma^\mu)\sigma^\mu T = \overline{Z}_m - (a/\sigma^\mu)\overline{Z}_m = f_0\overline{Z}_m + f_0\overline{N}_1\overline{Z}_m$. Thus, replacing R by this expression in Q, we get

$$Q = (1+t)^{-\mu-\frac{1}{2}}\overline{M}_1\overline{Z}_m.$$

Finally,

$$[\sigma^\mu T, L] = \sigma^\mu [T,L] - \frac{\mu(L\sigma)}{\sigma}(\sigma^\mu T), \quad \frac{L\sigma}{\sigma} = \frac{(r-t)}{\langle r-t\rangle}\frac{c-1}{\sigma},$$

$$[\sigma^\mu T, L] = \sigma^\mu\Big(\frac{Tc}{c}\Big)L - \Big(\frac{Tc}{c} + \frac{f_0(c-1)}{\sigma}\Big)\sigma^\mu T + \frac{1}{(1+t)^{\frac32}}\overline{M}_1\overline{Z}_m,$$

$$\frac{Tc}{c} + \frac{f_0(c-1)}{\sigma} = f\sigma^{-\mu}Z_m c + f\phi' = f\phi' + f\sigma^{-\mu+\frac12}\phi'\overline{N}_1 = \overline{M}_1\phi'.$$

Note that, due to the relations defining a, the coefficient d of $\sigma^\mu T$ in $[R_i+aT, L]$ or $[S+aT,L]$ is

$$d = f_0(1-\overline{\chi})\sigma^{-\mu}Zc = f_0(1-\overline{\chi})\sigma^{-\mu}\big(\overline{Z}_m c - (a\sigma^{-\mu})\overline{Z}_m c\big).$$

Since, on the support of $1-\overline{\chi}$ in zone II, $\sigma/(1+t)$ is bounded and bounded away from zero,

$$(1-\overline{\chi})\sigma^{-\mu+\frac12}\phi' = \frac{(1-\overline{\chi})\sigma(1+t)\phi'}{(1+t)^{\mu-\frac12}(1+t)^{\mu+\frac12}} = \frac{f_0}{(1+t)^{\mu+\frac12}},$$

$$d = \frac{f_0\overline{N}_1 + f_0\overline{N}_1^2}{(1+t)^{\mu+\frac12}} = \frac{\overline{M}_1}{(1+t)^{\mu+\frac12}}.$$

It is to obtain this decay of d that we have introduced the field $\sigma^\mu T$ in the collection $\overline{Z}_m$. The proof is complete. □

A.3. More generally, we have

LEMMA A.3. — *For $k \ge 1$, we have*

$$[\overline{Z}_m^k, L] = \sum_{p\le k-1}\big[\overline{M}_{k-p}\overline{Z}_m^p L + \overline{M}_{k-p}(1+t)^{-\mu-\frac12}\overline{Z}_m^{p+1}\big] + \sum_{p\le k-2}\overline{M}_{k-p}\phi'\overline{Z}_m^{p+1} + fA_0\overline{Z}_m^k.$$

Proof. — The result is proved for $k=1$. But

$$[\overline{Z}_m^{k+1}, L] = \overline{Z}_m[\overline{Z}_m^k, L] + [\overline{Z}_m, L]\overline{Z}_m^k$$

gives by induction

$$\begin{aligned}[\overline{Z}_m^{k+1}, L] = \sum_{p\le k-1}\Big[&\overline{M}_{k-p+1}\overline{Z}_m^p L + \overline{M}_{k-p}\overline{Z}_m^{p+1}L + \overline{M}_{k-p+1}(1+t)^{-\mu-\frac12}\overline{Z}_m^{p+1}\\ &+ \overline{M}_{k-p}(1+t)^{-\mu-\frac12}\overline{M}_1\overline{Z}_m^{p+1} + \overline{M}_{k-p}(1+t)^{-\mu-\frac12}\overline{Z}_m^{p+2}\Big]\\ + \sum_{p\le k-2}&[\overline{M}_{k-p+1}\phi'\overline{Z}_m^{p+1} + \overline{M}_{k-p}\overline{M}_1\phi'\overline{Z}_m^{p+1} + \overline{M}_{k-p}\phi'\overline{Z}_m^{p+2}]\\ &+ \overline{M}_2\phi'\overline{Z}_m^k + fA_0\overline{Z}_m^{k+1} + \overline{M}_1(1+t)^{-\mu-\frac12}\overline{Z}_m^{k+1}\\ &+ \overline{M}_1\phi'\overline{Z}_m^{k+1} + \overline{M}_1\overline{Z}_m^k L + \overline{M}_1[L, \overline{Z}_m^k],\end{aligned}$$

which is a sum of terms of the desired form. □

A.4. In order to control a with the appropriate weight, we compute

$$\begin{aligned} L(\sigma^{-\frac{1}{2}}a) &= f_0\,\sigma^{-\frac{3}{2}}(c-1)a + \sigma^{-\frac{1}{2}}La \\ &= f\phi'(\sigma^{-\frac{1}{2}}a) + f_0\,\sigma^{-\frac{1}{2}}a(\partial_t c) + \frac{f_0\overline{N}_0}{(1+t)^{\mu+\frac{1}{2}}}(Rc)\sigma^{-\frac{1}{2}}a + f_0\,\sigma^{-\frac{1}{2}}Rc \\ &\equiv F. \end{aligned}$$

Since $|F| \le C\phi' + C\phi'(\sigma^{-\frac{1}{2}}a)$, we find $|\sigma^{-\frac{1}{2}}a| \le C\mathrm{e}^{C\phi}$, by integration, using Gronwall Lemma.

Since we wish to apply Z_m^k to the left, we need to evaluate the right-hand side from the assumptions on γ, which will be done using the following lemma.

LEMMA A.4. — *Let Z_0 denote one of the fields R_i, S or $\sigma^\mu\partial_\alpha$. We have the formula*

$$\overline{Z}_m^k = \sum f\overline{N}_{k_1}\cdots\overline{N}_{k_j}\overline{Z}_m^{r_1}(\sigma^{-\frac{1}{2}}a)\cdots\overline{Z}_m^{r_q}(\sigma^{-\frac{1}{2}}a)Z_0^p,$$

where in the sum $p \ge 1$ and $\sum k_i + \sum r_\ell + p \le k$.

Proof. — For $k = 1$,

$$\overline{Z}_m = Z + aT = Z_0 + f(\sigma^{-\mu}a)Z_0 = Z_0 + f(\sigma^{-\frac{1}{2}}a)Z_0, \quad \sigma^\mu T = f_0\,\sigma^\mu\partial$$

proves the result. By induction, using the calculus lemma, the result is obvious, since $\overline{Z}_m Z_0^p = Z_0^{p+1} + f(\sigma^{-\frac{1}{2}}a)Z_0^{p+1}$. □

Using Lemma A.4 for $k = 1$ and the assumptions on g, we get

$$|\overline{N}_0| \le C\mathrm{e}^{C\phi}, \quad |\overline{N}_1| \le C\mathrm{e}^{C\phi}.$$

Using the proof of Lemma A.2 and a direct computation, we write

$$\begin{aligned} L\big(\widetilde{Z}(\sigma^{-\frac{1}{2}}a)\big) &= \overline{M}_1 L(\sigma^{-\frac{1}{2}}a) + \frac{\overline{M}_1}{(1+t)^{\mu+\frac{1}{2}}}\overline{Z}_m(\sigma^{-\frac{1}{2}}a) \\ &\qquad + fA_0\overline{Z}_m(\sigma^{-\frac{1}{2}}a) + \widetilde{Z}\big(L(\sigma^{-\frac{1}{2}}a)\big), \\ \widetilde{Z}\big(L(\sigma^{-\frac{1}{2}}a)\big) &= \Big(f\phi' + f\partial_t c + \frac{f\overline{N}_0(Rc)}{(1+t)^{\mu+\frac{1}{2}}}\Big)\widetilde{Z}(\sigma^{-\frac{1}{2}}a) \\ &\quad + \Big(\overline{M}_1\phi' + \frac{\overline{M}_1(Rc)}{(1+t)^{\mu+\frac{1}{2}}} + f\widetilde{Z}(\partial_t c) + \frac{f\overline{N}_0\widetilde{Z}(Rc)}{(1+t)^{\mu+\frac{1}{2}}}\Big)(\sigma^{-\frac{1}{2}}a) \\ &\quad + \overline{M}_1\sigma^{-\frac{1}{2}}Rc + f\sigma^{-\frac{1}{2}}\widetilde{Z}(Rc), \end{aligned}$$

which gives finally

$$L\big(\widetilde{Z}(\sigma^{-\frac12}a)\big) = \Big(A_0 + f\phi' + f\,\partial_t c + \frac{\overline{M}_1}{(1+t)^{\mu+\frac12}} + \frac{f\,\overline{N}_0(Rc)}{(1+t)^{-\mu-\frac12}}\Big)\overline{Z}_m(\sigma^{-\frac12}a)$$
$$+ \Big(\overline{M}_1\phi' + \overline{M}_1\partial_t c + \frac{\overline{M}_1(Rc)}{(1+t)^{\mu+\frac12}} + f\widetilde{Z}(\partial_t c) + \frac{f\,\overline{N}_0\widetilde{Z}(Rc)}{(1+t)^{\mu+\frac12}}\Big)(\sigma^{-\frac12}a)$$
$$+ \overline{M}_1\sigma^{-\frac12}Rc + \overline{M}_1\sigma^{-\frac12}\widetilde{Z}(Rc). \tag{A.1}$$

As above, we have

$$[\sigma^{\frac12}T, L] = f_0\,\sigma^{\frac12}(Tc)L + f_0A_0(\sigma^{\frac12}T) + \frac{\overline{M}_1}{(1+t)^{\mu+\frac12}}\sigma^{-\frac12}R.$$

We write

$$\sigma^{-\frac12}Ra = R(\sigma^{-\frac12}a) = \widetilde{R}(\sigma^{-\frac12}a) + f_0aT(\sigma^{-\frac12}a)$$
$$= \widetilde{R}(\sigma^{-\frac12}a) + \overline{M}_1 + \overline{M}_1(\sigma^{\frac12}Ta).$$

Now

$$La = f_0\overline{Z}_m c + f_0a\sigma^{-\mu}\overline{Z}_m c = \overline{M}_1\sigma^{\frac12}\phi',$$
$$\sigma^{\frac12}T(La) = f_0(\partial c)(\sigma^{\frac12}Ta) + f_0a\sigma^{\frac12}\partial(f_0\partial c) + f_0\,\sigma^{\frac12}(\partial f_0)Zc + f_0\,\sigma^{\frac12}\partial(Zc)$$
$$= f_0(\partial c)(\sigma^{\frac12}Ta) + \overline{M}_1\sigma^{\frac12}\partial c + \overline{M}_1\sigma^{\frac12}Z_0(\partial c) + f_0\,\sigma^{-\frac12}Z_0c,$$

hence finally

$$L(\sigma^{\frac12}Ta) = \overline{M}_1(1+t)^{-\mu-\frac12}\widetilde{R}(\sigma^{-\frac12}a)$$
$$+ (fA_0 + f_0\partial c + \overline{M}_1(1+t)^{-\mu-\frac12})(\sigma^{\frac12}Ta)$$
$$+ \overline{M}_1\phi' + \overline{M}_1(1+t)^{-\mu-\frac12} + \overline{M}_1\sigma^{\frac12}\partial c$$
$$+ f_0\,\sigma^{-\frac12}Z_0c + \overline{M}_1\sigma^{\frac12}Z_0(\partial c). \tag{A.2}$$

A.5. We proceed by induction as follows. We have already a control of $\overline{M}_1$ and $\sigma^{-\frac12}a$. We assume that

$$\sum|\overline{M}_\ell| + \big|\overline{Z}_m^{\ell-1}(\sigma^{-\frac12}a)\big| + \big|\overline{Z}_m^{\ell-2}(\sigma^{\frac12}Ta)\big| \le C\mathrm{e}^{C\phi}, \quad \ell\le k, \tag{H_k}$$

and we are going to prove H_{k+1}. To this aim, we apply Z_m^{k-1} to the left of the equations (A.1) and (A.2). All critical terms have coefficients bounded by $C\phi' + C(1+t)^{-\mu-\frac12}$. All other terms can be computed using Leibniz formula and Lemma A.1; they involve at most quantities already controlled by the induction hypothesis, or terms involving $Z_m^\ell(\partial c)$, $Z_m^\ell(Z_0c)$, $Z_m^\ell Z_0(\partial c)$ for $\ell\le k$. These terms are estimated using Lemma A.4. Finally, we obtain

$$\big|L\big(\overline{Z}_m^{k-1}(\sigma^{\frac12}Ta)\big)\big| + \big|L(\overline{Z}_m^{k-1}\widetilde{Z}(\sigma^{-\frac12}a))\big|$$
$$\le C(\phi' + (1+t)^{-\mu-\frac12})\big(\big|Z_m^k(\sigma^{-\frac12}a)\big| + \big|Z_m^{k-1}(\sigma^{\frac12}Ta)\big|\big) + C\mathrm{e}^{C\phi}.$$

What is controlled by the action of L in the left-hand side is not quite what we have in the right-hand side. But $\overline{Z}_m^k$ is either $\overline{Z}_m^{k-1}\widetilde{Z}$ or $\overline{Z}_m^{k-1}(\sigma^\mu T)$; in the later case,

$$\sigma^\mu T(\sigma^{-\frac12}a) = f_0\,\sigma^{\frac12}Ta + f_0\,\sigma^{-\frac12}a,$$

so that

$$\big|\overline{Z}_m^{k-1}(\sigma^\mu T)(\sigma^{-\frac12}a)\big| \le C\big|\overline{Z}_m^{k-1}(\sigma^{\frac12}Ta)\big| + C\mathrm{e}^{C\phi}.$$

Integrating the equations, we get finally (omiting the sup norms in x to simplify)

$$E \equiv |\overline{Z}_m^{k-1}(\sigma^{\frac12}Ta)| + |\overline{Z}_m^{k}(\sigma^{-\frac12}a)| \le C\int_0^t \big(\phi' + (1+t)^{-\mu-\frac12}\big)E\,\mathrm{d}t' + C\mathrm{e}^{C\phi},$$

which gives, using Gronwall Lemma, part of the induction hypothesis (H_{k+1}). To control $\overline{M}_{k+1}$, we use Lemma A.4 and the assumptions on the metric to estimate the terms involving $\gamma, \partial\gamma, c$. Since the Symbolic Calculus Lemma implies

$$|\sigma^{-\frac12}\,\overline{Z}_m^k a| \le \big|\overline{Z}_m^k(\sigma^{-\frac12}a)\big| + C\mathrm{e}^{C\phi},$$

we obtain the full induction hypothesis (H_{k+1}).

A.6. To finish the proof of Proposition 5.1, we have to translate the results already obtained in terms of the fields Z_m. Since the Z_m are also $\overline{Z}_m$ except for $Z_m = T = \sigma^{-1}\overline{Z}_m$, we obtain easily the following lemmas.

Lemma A.6. — *We have for all k*

$$Z_m^k = \sum f\,\overline{N}_{k_1}\cdots\overline{N}_{k_j}\,\overline{Z}_m^p,$$

with $p \ge 1$ and $\sum k_i + p \le k$.

Lemma A.7. — *With the notation of Lemma A.4, we have*

$$Z_m^k = \sum f\,\overline{N}_{k_1}\cdots\overline{N}_{k_j}\,\overline{Z}_m^{r_1}(\sigma^{-\frac12}a)\cdots\overline{Z}_m^{r_q}(\sigma^{-\frac12}a)Z_0^p,$$

with $p \ge 1$ and $\sum k_i + \sum r_i + p \le k$.

Proof. — We use Lemma A.7 to evaluate all quantities in N_k which are expressed as normalized derivatives of $\gamma, \partial\gamma, \partial^2\gamma, c, \partial c, \bar\theta, \bar k$. To evaluate $Z_m^{k-1}a$, we use Lemma A.6. Finally,

$$\partial_t + c\partial_r = \frac{r-ct}{r}\partial_t + \frac{c}{r}S, \quad r - ct = f\sigma,$$

$$L = \partial_t + c\partial_r + \frac{f_0\gamma R}{r} = \frac{f_0}{r}Z + \frac{f\sigma}{r}T, \quad \sigma^{\frac12}L = \frac{\overline{M}_1}{(1+t)^{\frac12}}\overline{Z}_m + f(\sigma^{\frac12}T),$$

$$\sigma^{\frac12}N = \sigma^{\frac12}(L-T) = \frac{\overline{M}_1}{(1+t)^{\frac12}}\overline{Z}_m + f(\sigma^{\frac12}T), \quad \sigma^{\frac12}\partial_\alpha = \frac{\overline{M}_1}{(1+t)^{\frac12}}\overline{Z}_m + f(\sigma^{\frac12}T).$$

Using the symbolic calculus lemmas and the estimates (H_k), we obtain the result. □

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