RATIONAL BV-ALGEBRA IN STRING TOPOLOGY
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Abstract. — Let $M$ be a 1-connected closed manifold of dimension $m$ and $LM$ be
the space of free loops on $M$. M. Chas and D. Sullivan defined a structure of BV-
algebra on the singular homology of $LM$, $H_*(LM; k)$. When the ring of coefficients
is a field of characteristic zero, we prove that there exists a BV-algebra structure
on the Hochschild cohomology $HH^*(C^*(M); C^*(M))$ which extends the canonical
structure of Gerstenhaber algebra. We construct then an isomorphism of BV-algebras
between $HH^*(C^*(M); C^*(M))$ and the shifted homology $H_{*+m}(LM; k)$. We also
prove that the Chas-Sullivan product and the BV-operator behave well with a Hodge
decomposition of $H_*(LM)$.
Rezumé (BV-algèbres rationnelles en topologie des lacets libres)

Soit $M$ une variété simplement connexe compacte sans bord de dimension $m$. Définissons par $LM$ l’espace des lacets libres sur $M$. M. Chas et D. Sullivan ont défini une structure de BV-algèbre sur l’homologie singulière $H_*(LM;k)$. Lorsque l’anneau des coefficients $k$ est un corps de caractère nulle, nous établissons l’existence d’une structure de BV-algèbre sur la cohomologie de Hochschild $HH^*(C^*(M);C^*(M))$ qui étend la structure canonique d’algèbre de Gerstenhaber. De plus nous construisons un isomorphisme de BV-algèbres entre $H_{*+m}(LM;k)$ et $HH^*(C^*(M);C^*(M))$. Finalement nous démontrons que le produit de Chas-Sullivan ainsi que le BV-opérateur sont compatibles avec la décomposition de Hodge de $H_*(LM;k)$.

1. Introduction

Chas and Sullivan considered in [3] the free loop space $LM = map(S^1, M)$ for a smooth orientable closed manifold of dimension $m$. They use geometric methods to show that the shifted homology $H_*(LM)$ has the structure of a Batalin-Vilkovisky algebra (BV-algebra for short). Later on Cohen and Jones defined in [5] a ring spectrum structure on the Thom spectrum $LM^{-TM}$ which realizes the Chas-Sullivan product in homology. More recently, Gruher and Salvatore proved in [17] that the algebra structure (and thus the BV-algebra structure) on $H_*(LM)$ is natural with respect to smooth orientation preserving homotopy equivalences.

Assume that the coefficients ring is a field. By a result of Jones [10, Thm. 4.1] there exists a natural linear isomorphism

$HH_*(C^*(M);C^*(M)) \cong H^*(LM)$,

and by duality an isomorphism $H_*(LM) \cong HH^*(C^*(M);C_*(M))$. Here $HH_*(A;Q)$ (respectively $HH^*(A;Q)$) denotes the Hochschild homology (respectively cohomology) of a differential graded algebra $A$ with coefficients in the differential graded $A$-bimodule $Q$. $C^*(M)$ denotes the singular cochains algebra and $C_*(M)$ the complex of singular chains. The cap product induces an isomorphism of graded vector spaces (for instance see [11, Appendix]),

$HH^*(C^*(M);C_*(M)) \cong HH^{*-m}(C^*(M);C^*(M))$,

and therefore an isomorphism of graded vector spaces

$H_*(LM) \cong HH^*(C^*(M);C^*(M))$.

Since $HH^*(A;A)$ is canonically a Gerstenhaber algebra, for any differential graded algebra $A$, it is natural to ask:

**Question 1.** — *Does there exist an isomorphism of Gerstenhaber algebras between $H_*(LM)$ and $HH^*(C^*(M);C^*(M))$?*
Various isomorphisms of graded algebras have been constructed. The first one has been constructed by Merkulov for real coefficients \([24]\), \([13]\) using iterated integrals. An another isomorphism has been constructed for rational coefficients by M. Vigué and the two authors, \([12]\), using the chain coalgebra of the Quillen minimal model of \(M\).

Although \(\text{HH}^*(A;A)\) does not have, for any differential graded algebra \(A\), a natural structure of BV-algebra extending the canonical Gerstenhaber algebra, a second natural question is:

**Question 2.** — Does there exist on \(\text{HH}^*(C^*(M);C^*(M))\) a structure of BV-algebra extending the structure of Gerstenhaber algebra and an isomorphism of BV-algebras between \(\mathbb{H}_*(LM)\) and \(\text{HH}^*(C^*(M);C^*(M))\)?

The main result of this paper furnishes a positive answer to Question 2 and thus to Question 1 when the field of coefficients is assumed of characteristic zero.

**Theorem 1.** — If \(M\) is 1-connected and the field of coefficients has characteristic zero then

(i) Poincaré duality induces a BV-structure on \(\text{HH}^*(C^*(M);C^*(M))\) extending the structure of Gerstenhaber algebra;

(ii) there exists an isomorphism of BV-algebras

\[
\mathbb{H}_*(LM) \cong \text{HH}^*(C^*(M);C^*(M)).
\]

BV-algebra structures on the Hochschild cohomology \(\text{HH}^*(A;A)\) have been constructed by different authors under some conditions on \(A\). First of all, Tradler and Zeinalian \([29]\) did it when \(A\) is the dual of an \(A_\infty\)-coalgebra with co-duality (rational coefficients). This is in particular the case when \(A = C^*(M)\), see \([28]\). Menichi \([23]\) constructed also a BV-structure in the case when \(A\) is a symmetric algebra (any coefficients). Let us mention that Ginzburg \([16, \text{Thm. 3.4.3}]\) has proved that \(\text{HH}^*(A;A)\) is a BV-algebra for certain algebras \(A\). Using this result Vaintrob \([30]\) constructed an isomorphism of BV-algebras between \(\mathbb{H}_*(LM)\) and \(\text{HH}^*(A;A)\) when \(A\) is the group ring with rational coefficients of the fundamental group of an aspherical manifold \(M\). This is coherent with our Theorem 1 because in this case \(C_*(\Omega M)\) is quasi-isomorphic to \(A\) and using \([9, \text{Prop. 3.3}]\) we have isomorphisms of Gerstenhaber algebras

\[
\text{HH}^*(A;A) \cong \text{HH}^*(C_*(\Omega M);C_*(\Omega M)) \cong \text{HH}^*(C^*(M);C^*(M)).
\]

Extending Theorem 1 to finite fields of coefficients would be difficult. For instance Menichi \([22]\) proved that algebras \(\mathbb{H}_*(LS^2)\) and \(\text{HH}^*(H^*(S^2);H^*(S^2))\)
are isomorphic as Gerstenhaber algebras but not as BV-algebras for $\mathbb{Z}/2$-coefficients.

In this paper we work over a field of characteristic zero. We use rational homotopy theory for which we refer systematically to [7]. We only recall here that a morphism in some category of complexes is a quasi-isomorphism if it induces an isomorphism in homology. Two objects are quasi-isomorphic if they are related by a finite sequence of quasi-isomorphisms. We shall use the classical convention $V^i = V_{-i}$ for degrees and $V^\vee$ denotes the graded dual of the graded vector space $V$.

Let $C_\ast(A;A) := (A \otimes T(sA), \partial)$ be the Hochschild chain complex of a differential graded algebra $A$ with coefficients in $A$. Here $T(sA)$ denotes the free coalgebra generated by the graded vector space $sA$ with $A = \{A^i\}_{i \geq 1}$ and $(sA)^i = A^{i+1}$. We emphasize that $C_\ast(A;A) = A \otimes T(sA)$ is considered as a cochain complex for upper degrees.

Now by a recent result of Lambrechts and Stanley [20] there is a commutative differential graded algebra $A$ satisfying:

1) $A$ is quasi-isomorphic to the differential graded algebra $C^\ast(M)$.

2) $A$ is connected, finite dimensional and satisfies Poincaré duality in dimension $m$. This means there exists a $A$-linear isomorphism $\theta : A \to A^\vee$ of degree $-m$ which commutes with the differentials.

We call $A$ a Poincaré duality model for $M$.

The starting point of the proof is to replace $C^\ast(M)$ by $A$ because there is an isomorphism of Gerstenhaber algebras, [9, Prop. 3.3],

\begin{equation}
HH^\ast(A;A) \cong HH^\ast(C^\ast(M);C^\ast(M)).
\end{equation}

This will allows us to use Poincaré duality at the chain level.

Denote by $\mu$ the multiplication of $A$. This is a model of the diagonal map. We define then the linear map $\mu_A : A \to A \otimes A$ by the commutative diagram

\begin{equation}
A^\vee \xrightarrow{\mu^\vee} (A \otimes A)^\vee = A^\vee \otimes A^\vee
\end{equation}

\begin{equation}
\phi \begin{array}{c}
\cong \\
\cong
\end{array}
\begin{array}{c}
A \\
\mu_A
\end{array}
\end{equation}

By definition $\mu_A$ is a $A \otimes A$-linear map degree $m$ which commutes with the differentials (Here $A$ is a $A \otimes A$-module via $\mu$). This is a representative of the Gysin map associated to the diagonal embedding. With these notation we prove in §4:

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PROPOSITION 1. — 1) The cochain complex $C_*(A; A)$ is quasi-isomorphic to the complex $C^*(LM)$. In particular, there is an isomorphism of graded vector spaces

$$HH_*(A; A) \cong H^*(LM).$$

2) If $\mu$ denotes the multiplication of $A$ and $\phi$ denotes the coproduct of the coalgebra $T(sA)$ then the composite $\Phi$

$$\begin{align*}
A \otimes T(sA) & \xrightarrow{id \otimes \phi} A \otimes T(sA) \otimes T(sA) \cong A \otimes A^\otimes (A \otimes T(sA))^\otimes 2 \\
\Phi & \downarrow \quad \cong \\
(A \otimes T(sA))^\otimes 2 & \cong A^\otimes 2 \otimes A^\otimes 2 \quad \mu_A \otimes id
\end{align*}$$

is a linear map of degree $m$ which commutes with the differentials.

3) The isomorphism $HH_*(A; A) \cong H^*(LM)$, considered in 1), transfers the map induced by $\Phi$ on $HH_*(A; A)$ to the dual of the Chas-Sullivan product on $H^{*-m}(LM)$.

4) The duality isomorphism $HH_*(A; A)^\vee \cong HH^*(A; A^\vee)$ (6) transfers the map induced by $\Phi$ on $HH_*(A; A)^\vee$ to the Gerstenhaber product on $HH^*(A; A)$.

Denote by $\Delta : \mathbb{H}_*(LM) \to \mathbb{H}_{*+1}(LM)$ and $\Delta' : \mathbb{H}_*^*(LM) \to \mathbb{H}_{*+1}^*(LM)$ the morphisms induced by the canonical action of $S^1$ on $LM$. As proved by Chas and Sullivan this operator $\Delta$ defines on $\mathbb{H}_.(M)$ a structure of BV-algebra. In section 5 we prove:

PROPOSITION 2. — The isomorphism $HH_*(A; A) \cong H^*(LM)$, considered in Proposition 1, transfers Connes’ boundary $B : HH_*(A; A) \to HH_{*+1}(A; A)$ to the operator $\Delta'$.

L. Menichi [23] proved that the duality isomorphism

$$HH_*(A; A)^\vee \cong HH^*(A; A^\vee) \cong HH^*(A; A)$$

transfers $B^\vee : (HH_{*+1}(A; A)^\vee \to (HH_*(A; A))^\vee$ to a BV-operator on $HH^*(A; A)$ that defines a BV-structure extending the Gerstenhaber algebra structure. The isomorphisms of Gerstenhaber algebras (1) carries on the right hand term a structure of BV-algebra extending the Gerstenhaber algebra. This fact combined with Proposition 1 and 2 gives Theorem 1.

Since the field of coefficients is of characteristic zero, the homology of $LM$ admits a Hodge decomposition, $\mathbb{H}_*(LM) = \bigoplus_{r \geq 0} \mathbb{H}_r^r(LM)$ (see [33], [32], [15]
and [21, Thm. 4.5.10]). We prove that this decomposition behaves well with respect to the product \( \bullet \) and the BV-operator \( \Delta \) defined by Chas-Sullivan.

**Theorem 2.** — With the above notation, we have

1) \( \mathbb{H}_{[r]}(LM) \otimes \mathbb{H}_{[s]}(LM) \rightarrow \mathbb{H}_{[r+s]}(LM) \),
2) \( \Delta : \mathbb{H}_{[r]}(LM) \rightarrow \mathbb{H}_{[r+1]}(LM) \).

By definition \( \mathbb{H}_{[r]}(LM) \) is the image of \( H_{*+m}(M) \) by the homomorphism induced in homology by the canonical section \( M \rightarrow LM \). It has been proved in [10] that if \( \text{aut} M \) denotes the monoid of (unbased) self-equivalences of \( M \) then there exists a natural isomorphism of graded algebras

\[ \mathbb{H}_{[r]}(LM) \cong H_{*+m}(M) \otimes \pi_*(\text{aut} M). \]

For any \( r \geq 0 \), a description of \( \mathbb{H}_{[r]}(LM) \) can be obtained, using a Lie model \((L \hookrightarrow d)\) of \( M \), as proved in the last result.

**Proposition 3.** — The graded vector space \( \mathbb{H}_{[r]}(LM) \) is isomorphic to \( \text{Tor}^{UL}(\mathbb{k} \hookrightarrow \Gamma^r(L)) \) where \( \Gamma^r(L) \) is the sub-UL-module of UL for the adjoint representation that is the image of \( \wedge^r L \) by the classical Poincaré-Birkhoff-Witt isomorphism of coalgebras \( \wedge L \rightarrow UL \).

The text is organized as follows. Notation and definitions are made precise in sections 2 and 3. Proposition 1 is proved in Sections 4, Proposition 2 is proved in section 5. Theorem 2 and Proposition 3 are proved in the last section.

### 2. Hochschild homology and cohomology

#### 2.1. Bar construction. —
Let \( A \) be a differential graded augmented cochain algebra and let \( P \) (res. \( N \)) be a differential graded right (resp. left) \( A \)-module,

\[ A = \{ A^i \}_{i \geq 0}, \quad P = \{ P^j \}_{j \in \mathbb{Z}}, \quad N = \{ N^j \}_{j \in \mathbb{Z}} \quad \text{and} \quad \overline{A} = \ker(\varepsilon : A \rightarrow \mathbb{k}). \]

The two-sided (normalized) bar construction,

\[ B(P;A;N) = P \otimes T(s\overline{A}) \otimes N, \quad B_k(P;A;N)^f = (P \otimes T^k(s\overline{A}) \otimes N)^f, \]

is the cochain complex defined as follows. For \( k \geq 1 \), a generic element \( p[a_1|a_2| \cdots |a_k]|n \) in \( B_k(P;A;N) \) has (upper) degree \( |p| + |n| + \sum_{i=1}^{k}(|sa_i|) \). If \( k = 0 \), we write \( p[|n] = p \otimes 1 \otimes n \in P \otimes T^0(s\overline{A}) \otimes N \). The differential \( d = d_0 + d_1 \) is defined by

\[ B_k(P;A;N)^f \xrightarrow{d_0} B_k(P;A;N)^{f+1}, \]

where
\(d_0(p[a_1|a_2|\cdots|a_k]n) = d(p)[a_1|a_2|\cdots|a_k]n\)
\[- \sum_{i=1}^k (-1)^i p[a_1|a_2|\cdots|d(a_i)|\cdots|a_k]n\]
\[+ (-1)^{i+1} p[a_1|a_2|\cdots|a_k]d(n),\]
\[\mathbb{B}_k(P; A; N)^\ell \xrightarrow{d_1} \mathbb{B}_{k-1}(P; A; N)^{\ell+1},\]
\[d_1(p[a_1|a_2|\cdots|a_k]n) = (-1)^{|p|} p[a_1|a_2|\cdots|a_k]n\]
\[+ \sum_{i=2}^k (-1)^i p[a_1|a_2|\cdots|a_{i-1}a_i|\cdots|a_k]n\]
\[\quad - (-1)^{i'} p[a_1|a_2|\cdots|a_{k-1}a_k]n.\]

Here \(\epsilon_i = |p| + \sum_{j<i}(|a_j|).\)

In particular, considering \(k\) as a trivial \(A\)-bimodule we obtain the complex
\[\mathbb{B}A = \mathbb{B}(k; A; k)\]
which is a differential graded coalgebra whose comultiplication is defined by
\[\phi([a_1|\cdots|a_r]) = \sum_{i=0}^r [a_1|\cdots|a_i] \otimes [a_{i+1}|\cdots|a_r].\]

Recall that a differential \(A\)-module \(N\) is called semifree if \(N\) is the union of an increasing sequence of sub-modules \(N(i), i \geq 0,\) such that each \(N(i)/N(i-1)\) is an \(R\)-free module on a basis of cycles (see \([7]\)). Then,

**LEMMA 1** (see \([7, \text{Lemma 4.3}]) \text{—} The canonical map \(\varphi : \mathbb{B}(A; A; A) \to A\)
defined by \(\varphi[ ] = 1\) and \(\varphi([a_1|\cdots|a_k]) = 0\) if \(k > 0,\) is a semifree resolution of \(A\) as an \(A\)-bimodule.

### 2.2. Hochschild complexes

Let us denote by \(A^e = A \otimes A^{op}\) the envelopping algebra of \(A.\)

If \(P\) is a differential graded right \(A^e\)-module then the cochain complex
\[C_*(P; A) := (P \otimes T(sA), \partial) \overset{\text{def}}{=} P \otimes_{A^e} \mathbb{B}(A; A; A),\]
is called the **Hochschild chain complex of \(A\) with coefficients in \(P.\)** Its homology is called the **Hochschild homology of \(A\) with coefficients in \(P.\)** and is denoted by \(HH_*(A; P).\) When we consider \(C_*(A; A)\) as well as \(HH_*(A; A),\) \(A\) is supposed equipped with its canonical right \(A^e\)-module structure.

For sake of completeness, let us recall the definition of the Connes’ coboundary:

\[B : C_*(A; A) \to C_*(A; A).\]
One has $B(a_0 \otimes [a_1] \cdots [a_n]) = 0$ if $|a_0| = 0$ and
\[ B(a_0 \otimes [a_1] \cdots [a_n]) = \sum_{i=0}^{n} (-1)^{\tilde{e}_i} 1 \otimes [a_i] \cdots [a_n|a_0|a_1] \cdots [a_{i-1}] \]
if $|a_0| > 0$, where
\[ \tilde{e}_i = (|sa_0| + |sa_i| + \cdots + |sa_{i-1}|) \cdot (|sa_i| + \cdots + |sa_n|). \]
It is well known that $B^2 = 0$ and $B \circ \partial + \partial \circ B = 0$. We also denote by $B$ the induced operator in Hochschild homology $HH_\ast(A; A)$.

If $N$ is a (left) differential graded $A^\ast$-module then the $(\mathbb{Z}\text{-graded})$ complex
\[ C^\ast(A; N) := \left( \text{Hom}(T(sA), N), \delta \right) \overset{\text{def}}{=} \text{Hom}_{A^\ast}(\mathbb{B}(A; A; A), N), \]
is called the \textit{Hochschild cochain complex} of $A$ with coefficients in the differential graded $A$-bimodule $N$. Its homology is called the \textit{Hochschild cohomology of $A$ with coefficients in $N$} and is denoted by $HH^\ast(A; N)$. When we consider $C^\ast(A; A)$ as well as $HH^\ast(A; A)$, $A$ is supposed equipped with its canonical left $A^\ast$-bimodule structure.

Consider the graded dual, $V^\vee$, of the graded vector space $V = \{V^i\}_{i \in \mathbb{Z}}$, i.e. $V^\vee = \{V_i^\vee\}_{i \in \mathbb{Z}}$ with $V_i^\vee := \text{Hom}(V^i, k)$. The canonical isomorphism
\[ \text{Hom} \left( A \otimes_{A^\ast} \mathbb{B}(A; A; A), k \right) \to \text{Hom}_{A^\ast}(\mathbb{B}(A; A; A), A^\ast) \]
induces the isomorphism of complexes $C_\ast(A; A)^\vee \to C^\ast(A; A^\ast)$.

2.3. The Gerstenhaber algebra on $HH^\ast(A; A)$, — A Gerstenhaber algebra is a commutative graded algebra $H = \{H_i\}_{i \in \mathbb{Z}}$ with a bracket
\[ H_i \otimes H_j \to H_{i+j+1}, \quad x \otimes y \mapsto \{x, y\} \]
such that for $a, a', a'' \in H$:
\begin{enumerate}
\item[(a)] $\{a, a'\} = (-1)^{(|a| - 1)(|a'|-1)}\{a', a\}$;
\item[(b)] $\{a, \{a', a''\}\} = \{\{a, a'\}, a''\} + (-1)^{(|a| - 1)(|a''|-1)}\{a', \{a, a''\}\}$.
\end{enumerate}

For instance the Hochschild cohomology $HH^\ast(A; A)$ is a Gerstenhaber algebra [14]. The bracket can be defined by identifying $C^\ast(A; A)$ with a differential graded Lie algebra of coderivations (see [26] and [9, 2.4]).
2.4. BV-algebras and differential graded Poincaré duality algebras. — A Batalin-Vilkovisky algebra (BV-algebra for short) is a commutative graded algebra, $H$ together with a linear map (called a BV-operator)

$$\Delta : H^k \rightarrow H^{k-1}$$

such that:

1) $\Delta \circ \Delta = 0$;

2) $H$ is a Gerstenhaber algebra with the bracket defined by

$$\{a, a'\} := (-1)^{|a|} (\Delta(aa') - \Delta(a)a' - (-1)^{|a|} ab\Delta(a')).$$

3. The Chas-Sullivan algebra structure on $\mathbb{H}_*(LM)$ and its dual

We assume in this section and in the following ones that $k$ is a field of characteristic zero.

Denote by $p_0 : LM \rightarrow M$ the evaluation map at the base point of $S^1$, and recall that the space $LM$ can be replaced by a smooth manifold ([1], [25]) so that $p_0$ is a smooth locally trivial fibre bundle ([1], [25]).

The Chas-Sullivan product

$$\cdot : H_*(LM)^\otimes 2 \rightarrow H_{*-m}(LM), \quad x \otimes y \mapsto x \cdot y$$

was first defined in [3] by using “transversal geometric chains”. Then

$$\mathbb{H}_*(LM) := H_{*+m}(LM)$$

becomes a commutative graded algebra.

It is convenient for our purpose to introduce the dual of the loop product $H^*(LM) \rightarrow H^{*-m}(LM \times 2)$. Consider the commutative diagram

$$\begin{array}{cccc}
LM^{\times 2} & \xrightarrow{i} & LM \times_M LM & \xrightarrow{\text{Comp}} & LM \\
p_0^{\times 2} \downarrow & & p_0 \downarrow & & p_0 \\
M^{\times 2} & \xrightarrow{\Delta} & M & \xrightarrow{} & M
\end{array}$$

where

- Comp denotes composition of free loops,
- the left hand square is a pullback diagram of locally trivial fibrations,
- $i$ is the embedding of the manifold of composable loops into $LM \times LM$. 
The embeddings $\Delta$ and $i$ have both codimension $m$. Thus, using the Thom-Pontryagin construction we obtain the Gysin maps

$$
\Delta^! : H^k(M) \to H^{k+m}(M \times^2) \quad \text{and} \quad i^! : H^k(LM \times_M LM) \to H^{k+m}(LM \times^2).
$$

Thus diagram (1) yields the diagram

$$
\begin{array}{ccccccc}
H^{k+m}(LM \times^2) & \xleftarrow{i^!} & H^k(LM \times_M LM) & \xrightarrow{H^*(\text{Comp})} & H^k(LM) \\
\downarrow{H^*(p_0)^\otimes} & & \downarrow{H^*(p_0)} & & \downarrow{H^*(p_0)} \\
H^{k+m}(M \times^2) & \xleftarrow{\Delta^!} & H^k(M) & \to & H^k(M)
\end{array}
$$

Following [27], [6], the dual of the loop product is defined by composition of maps on the upper line:

$$
i^! \circ H^*(\text{Comp}) : H^*(LM) \to H^{*+m}(LM \times^2).
$$

4. Proof of Proposition 1 and the Cohen-Jones-Yan spectral sequence.

The composition of free loops $\text{Comp} : LM \times_M LM \to LM$ is obtained by pullback from the composition of paths $\text{Comp}^\prime : M^I \times_M M^I \to M^I$ in the following commutative diagram.

$$(\text{Comp})$$

Here $\Delta$ denotes the diagonal embedding, $j$ the obvious inclusions, $ev_1$ denotes the evaluation maps at $t$, and $\text{pr}_{13}$ the map defined by $\text{pr}_{13}(a, b, c) = (a, c)$.

Let $(A, d)$ be a commutative differential graded algebra quasi-isomorphic to the differential graded algebra $C^*(M)$. A cochain model of the right hand square in diagram (Comp) is given by the commutative diagram

$$
\begin{array}{ccccccc}
\mathbb{B}(A; A; A) & \xrightarrow{\Psi} & \mathbb{B}(A; A; A) \otimes A & \xrightarrow{\Psi} & A \otimes^3 \\
\uparrow & & \uparrow & & \uparrow \\
A \otimes^2 & \xrightarrow{\Psi} & A \otimes^3
\end{array}
$$

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where $\Psi$ and $\psi$ denote the homorphism of cochain complexes defined by
\[
\Psi(a \otimes [a_1] \cdots [a_k] \otimes a') = \sum_{i=0}^{k} a \otimes [a_1] \cdots [a_{i+1}] \otimes [a_i] \otimes [a_{i+2}] \otimes \cdots \otimes [a_k] \otimes a',
\]
\[
\psi(a \otimes a') = a \otimes 1 \otimes a'.
\]

We consider now the commutative diagram obtained by tensoring diagram (†) by $A$:
\[
A \otimes_{A^{\otimes 2}} \mathbb{B}(A; A; A) \overset{id \otimes \Psi}{\longrightarrow} A \otimes_{A^{\otimes 3}} (\mathbb{B}(A; A) \otimes_A \mathbb{B}(A; A))
\]
(‡)
\[
A \otimes_{A^{\otimes 2}} A^{\otimes 2} \overset{id \otimes \psi}{\longrightarrow} A \otimes_{A^{\otimes 3}} A^{\otimes 3}
\]

Since $\mathbb{B}(A; A; A)$ is a semifree model of $A$ as $A$-bimodule, we deduce from [8], p. 78, that diagram (‡) is a cochain model of the left hand square in diagram (Comp). Obviously, we have also the commutative diagram
\[
A \otimes_{A^{\otimes 3}} \mathbb{B}(A; A; A) \overset{id \otimes \Psi}{\longrightarrow} A \otimes_{A^{\otimes 3}} (\mathbb{B}(A; A) \otimes_A \mathbb{B}(A; A))
\]
\[
\overset{\cong}{\longrightarrow}
\]
\[
A \otimes T(s\overline{A}) \overset{id \otimes \phi}{\longrightarrow} A \otimes T(s\overline{A}) \otimes T(s\overline{A})
\]

where $\phi$ denotes the coproduct of the coalgebra $T(s\overline{A})$. Thus we have proved:

**Lemma 2.** — The cochain complex $C_*(A; A)$ is a cochain model of $LM$, (i.e. we have an isomorphism of graded vector spaces $HH_*(A; A) \cong H^*(LM)$.) Moreover, the composite
\[
C_*(A; A) \longrightarrow C_*(A; A) \otimes_A C_*(A; A)
\]
\[
\overset{\cong}{\longrightarrow}
\]
\[
A \otimes T(s\overline{A}) \overset{id \otimes \phi}{\longrightarrow} A \otimes T(s\overline{A}) \otimes T(s\overline{A})
\]
is model of the composition of free loops.

Recall now that the Gysin map $\Delta^!$ of the diagonal embedding $\Delta : M \to M \times M$ is the Poincaré dual of the homomorphism $H_*(\Delta)$. This means that the following diagram is commutative:
\[
H_*(M) \overset{H_*(\Delta)}{\longrightarrow} H_*(M \times M)
\]
\[
-\cap [M] \overset{\cong}{\longrightarrow} -\cap [M \times M]
\]
\[
H^*(M) \overset{\Delta^!}{\longrightarrow} H^*(M \times M)
\]
Let $A$ be a Poincaré duality model of $M$ and $\mu_A$ as defined by diagram (2) of the introduction. The linear map $\mu_A = A \to A \otimes A$ is a cochain model for $\Delta^1$. Next observe that, [26], we can choose the pullback of a tubular neighborhood of the diagonal embedding $\Delta$ as a tubular neighborhood of the embedding $i : LM \times_M LM \to LM \times LM$. Thus the Gysin map $i^!$ is obtained by pullback from $\Delta^1$. Therefore, since $A$ is graded commutative, then $C_*(A; A)$ is a $A$-semifree and we have proved:

**Lemma 3. —** The linear map of degree $m$

$$C_*(A; A) \otimes_A C_*(A; A) \xrightarrow{\cong} A \otimes_A A \otimes_A C_*(A; A) \xrightarrow{\mu_A \otimes \text{id}} C_*(A; A)^{\otimes 2}$$

commutes with the differential and induces $i^!$ in homology.

Then a combination of Lemmas 2, 3 and Lemma 4 below gives Proposition 1 of the introduction.

**Lemma 4. —** The duality isomorphism $(HH_{*+m}(A; A))^* \cong HH_{*+m}(A; A)^*$

$HH^*(A; A)$ transfers the map induced by $\Phi$ on $HH_*(A; A)$ to the Gerstenhaber product on $HH^*(A; A)$.

**Proof. —** Observe that the composite (dotted arrow in the next diagram) induces the Gerstenhaber product in $HH^*(A; A)$.

Then the remaining of the proof follows by considering an obvious commutative diagram.
Spectral sequence. — By putting $F_p := A \otimes (T(s\Lambda))^{\leq p}$, for $p \geq 0$, we define a filtration

$$A \otimes T(s\Lambda) \supset \cdots \supset F_p \supset F_{p-1} \supset \cdots \supset A = F_0$$

such that $\partial F_p \subset F_p$ and $\Phi(F_p) \subset \bigoplus F_k \otimes F_{\ell}$. The resulting spectral sequence

$$E_{p,q}^2 = H^q(M) \otimes H^p(\Omega M) \Longrightarrow H^{p+q}(LM)$$

is the comultiplicative “regraded” Serre spectral sequence for the fibration $p_0 : LM \to M$. It dualizes into a spectral sequence of algebras

$$H_{q+m}(M) \otimes H_p(\Omega M) \Longrightarrow \mathbb{H}_{p+q}(LM).$$

We recover in this way, for coefficients in a field of characteristic zero, the spectral sequence defined previously by Cohen, Jones and Yan [6].

5. Proof of Proposition 2.

Let $\rho : S^1 \times LM \to LM$ be the canonical action of the circle on the space $LM$. The action $\rho$ induces an operator $\Delta : \mathbb{H}_*(LM) \to \mathbb{H}_{*+1}(LM)$. The Chas-Sullivan product together with $\Delta$ gives to $\mathbb{H}_*(LM)$ a BV-structure [3].

Denote by $\mathcal{M}_M = (\bigwedge V, d)$ a (non necessary minimal) Sullivan model for $M$ [8, §12]. We put $sV = V$ and denote by $S$ the derivation of $\bigwedge V \otimes \bigwedge V$ defined by $S(v) = \bar{v}$ and $S(\bar{v}) = 0$ for $v \in V$ and $\bar{v} \in V$. Then a Sullivan model for $LM$ is given by the commutative differential graded algebra $(\bigwedge V \otimes \bigwedge V, \bar{d})$ where $\bar{d} = -S(dv)$ [34]. Moreover in [33] Burghelea and Vigué prove that a Sullivan model of the action $\rho : S^1 \times LM \to LM$ is given by

$$\mathcal{M}_\rho : (\bigwedge V \otimes \bigwedge V, \bar{d}) \to (\bigwedge u, 0) \otimes (\bigwedge V \otimes \bigwedge V, \bar{d}), \quad |u| = 1),$$

$$\mathcal{M}_\rho(\alpha) = 1 \otimes \alpha + u \otimes S(\alpha), \quad \alpha \in \bigwedge V \otimes \bigwedge V.$$
6. Hodge decomposition

With the notation of the previous sections, let \((\mathcal{M}_M \otimes \bigwedge V, \bar{d})\) be a Sullivan model for \(LM\). Denote by \(G^p = \bigwedge V \otimes \bigwedge^p V\) the subvector space generated by the words of length \(p\) in \(V\). The differential \(\bar{d}\) satisfies \(\bar{d}(G^p) \subset G^p\). Thus we put

\[ H^n_{[\bar{d}]}(LM) := H^n(G^p). \]

This decomposition splits \(H^*(LM; \mathbf{k})\) into summands given as eigenspaces of the maps \(LM \to LM\) induced from the \(n\)-power maps of the circle \(e^{it} \mapsto e^{int}\) [33]. It defines by duality a Hodge decomposition on \(H_*(LM)\). We are now ready to prove Theorem 2 of the introduction.

**Proof of Theorem 2.** — Recall that the differential \(\partial\) in \(C^*(\mathcal{M}_M; \mathcal{M}_M)\) decomposes into \(\partial = \partial_0 + \partial_1\) with \(\partial_0(\mathcal{M}_M \otimes T^p(s\Lambda V)) \subset \mathcal{M}_M \otimes T^p(s\Lambda V)\), and \(\partial_1(\mathcal{M}_M \otimes T^p(s\Lambda V)) \subset \mathcal{M}_M \otimes T^{p-1}(s\Lambda V)\).

We consider the quasi-isomorphism \(f : C^*(\mathcal{M}_M; \mathcal{M}_M) \to (\mathcal{M}_M \otimes \bigwedge \bar{V}, \bar{d})\) defined in Lemma 5. If we apply Lemma 5, when \(d = 0\) in \(\Lambda V\), we deduce that \(\text{Ker} f\) is \(\partial_1\)-acyclic.

**Lemma 6.** — Let us define \(K^{(p)} := \text{Ker} f \cap (\mathcal{M}_M \otimes T^p(s\Lambda V))\).

1) If \(\omega \in K^{(p)} \cap \text{Ker} \partial\) then there exists \(\omega' \in \bigoplus_{r \geq p+1} K^{(r)}\) such that \(\partial \omega' = \omega\).

2) \(f\) induces a surjective map

\[ (\mathcal{M}_M \otimes T^{\geq p}(s\Lambda V)) \cap \text{Ker} \partial \twoheadrightarrow (\mathcal{M}_M \otimes \Lambda^p sV) \cap \text{Ker} \bar{d}. \]

**Proof.** — If \(\omega \in K^{(p)} \cap \text{Ker} \partial\) then \(\omega = \partial(u + v)\) with \(u \in K^{(p)}\) and \(v \in K^{(\geq p+1)}\). Since \(\partial_1 u = 0\) we have \(u = \partial \beta_1\) some \(\beta \in K^{(p+1)}\) and thus \(\omega - d\beta_1 \in K^{(\geq p+1)}\). An induction on \(n \geq 1\) we prove that there exists \(\beta_n \in K^{(p+n)}\) such that \(\omega - d\beta_n \in K^{(p+n)}\). Since \(\Lambda V = 1\)-connected \((\mathcal{M}_M \otimes T^{p+n}(s\Lambda V))|\omega| = 0\) for some integer \(n_0\). We put \(\omega' = \beta_{n_0}\).

In order to prove the second statement, we consider a \(\bar{d}\)-cocycle \(\alpha \in \mathcal{M}_M \otimes \Lambda^p sV\) and we write \(\alpha = f(\omega)\) for some \(\omega \in \mathcal{M}_M \otimes T^p(s\Lambda V)\). It follows from the definition of \(f\) that \(\partial \omega \in K^{(p-1)}\). Thus, by the first statement, \(\partial \omega = \partial \omega'\) some \(\omega' \in K^{(\geq p)}\). Then \(\bar{w} = \omega - \omega'\) is \(\partial\)-cocycle of \(K^{\geq p}\) such that \(f(\bar{w}) = \alpha\).

To end the proof of Theorem 2, let us consider \(\alpha \in H^*_{[\bar{d}]}(LM)\). By Lemma 6, \(\alpha\) is the class of \(f(\beta)\) where \(\beta \in \mathcal{M}_M \otimes T^{\geq n}(s\Lambda V)\). Therefore \(\Phi(\beta)\) belongs to \(\bigoplus_{i+j \geq n} (\mathcal{M}_M \otimes T^i(s\Lambda V)) \otimes (\mathcal{M}_M \otimes T^j(s\Lambda V))\) (see Lemma 2). Now since \(f(\mathcal{M}_M \otimes T^p(s\Lambda V)) \subset \mathcal{M}_M \otimes \Lambda^p sV\),

\[ [\Phi(\alpha)] \in \bigoplus_{i+j \geq n} H^*_{[\bar{d}]}(LM) \otimes H^*_{[\bar{d}]}(LM). \]
Now, as announced in the introduction (Proposition 3) there is another interpretation of $H^n[p](LM)$ in terms of the cohomology of a differential graded Lie algebra.

Let $L$ be a differential graded algebra $L$ such that the cochain algebra $C^*(L)$ is a Sullivan model of $M$, [8, p. 322]. In particular, the homology of the enveloping universal algebra of $L$, denoted $UL$, is a Hopf algebra isomorphic to $H_*(\Omega M)$. We consider the cochain complex $C^*(L; UL_\wedge)$ of $L$ with coefficients in $UL^\wedge$ considered as an $L$-module for the adjoint representation. We have shown (see [12, Lemma 4]) that the natural inclusion $C^*(L) \hookrightarrow C^*(L; UL_\wedge)$ is a relative Sullivan model of the fibration $p_0 : LM \to M$. Write $C^*(L) = (\wedge V, d)$, then $V = (sL)^\wedge$ and $\nabla = L^\wedge$. There is also (Poincaré-Birkhoff-Witt Theorem) an isomorphism of graded coalgebras, [8, Prop. 21.2]:

$$\gamma : \wedge L \longrightarrow UL, \quad x_1 \wedge \cdots \wedge x_k \longmapsto \sum_{\sigma \in S_k} \varepsilon_{\sigma} x_{\sigma(1)} \cdots x_{\sigma(k)}.$$ 

If we put $\Gamma^p = \gamma(\wedge^p V)$ we obtain the following isomorphisms of cochain complexes

$$(\wedge V \otimes \wedge \nabla, d) \cong C^*(L; UL_\wedge), \quad \Gamma^p \cong C^*(L; (\Gamma^p)^\wedge)$$

which in turn induce the isomorphisms

$$\mathbb{H}^*(LM) \cong \text{Ext}_{UL}(k, UL_\wedge), \quad \mathbb{H}^*[p](LM) \cong \text{Ext}_{UL}(k, \Gamma^p(L)^\wedge)$$

and by duality,

$$H_*(LM) \cong \text{Tor}^{UL}(k, UL_\wedge), \quad H^*[p](LM) \cong \text{Tor}^{UL}(k, \Gamma^p).$$

BIBLIOGRAPHY


