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Kaïs Ammari & Mouez Dimassi

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WEYL FORMULA WITH OPTIMAL REMAINDER ESTIMATE OF SOME ELASTIC NETWORKS AND APPLICATIONS

BY KAÏS AMMARI & MOUEZ DIMASSI

ABSTRACT. — We consider a network of vibrating elastic strings and Euler-Bernoulli beams. Using a generalized Poisson formula and some Tauberian theorem, we give a Weyl formula with optimal remainder estimate. As a consequence we prove some observability and stabilization results.

RÉSUMÉ (*Formule de Weyl avec reste optimal de quelques réseaux élastiques et applications*)

Nous considérons un réseau de cordes et de poutres d'Euler-Bernoulli. En utilisant une formule de Poisson généralisée et un théorème taubérien nous prouvons une formule de Weyl avec reste optimal. Comme conséquence nous prouvons des résultats d'observabilités et de stabilisations.

1. Introduction

In the last years various models of multiple-link flexible structures have been given and developed. The structures which we have in mind consist of finitely many interconnected flexible elements like strings, beams, plates representative

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KAÏS AMMARI, Département de Mathématiques, Faculté des Sciences de Monastir, 5019 Monastir, Tunisie • *E-mail :* kais.ammari@fsm.rnu.tn

MOUEZ DIMASSI, Département de Mathématiques, Institut Galilée, Université Paris 13, 99, avenue Jean-Baptiste Clément 93430 Villetaneuse, France • *E-mail :* dimassi@math.univ-paris13.fr

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of trusses, frames, solar panels, antennae deformable mirrors, for more details concerning the models see [12]. "The spectral analysis of such models displays, in addition to its own mathematical interest, control and stabilization problems, see [9, 10, 11, 12], [15, 16, 18] and [1, 3, 4, 5].

First of all, we introduce some notations, which are simply those of [7], [14], we refer to [7] for more details, that is needed to formulate the problem under consideration.

Let Γ be a connected topological graph embedded in $\mathbb{R}^m$, $m \in \mathbb{N}^*$, with n verticies $\mathcal{E} = \{E_i,\, 1 \leq i \leq n\}$ and N edges $\mathcal{K} = \{k_i,\, 1 \leq i \leq N\}$. Each edge k_j is a Jordan curve in $\mathbb{R}^m$ and is assumed to be parametrized by its arc length parameter x_j, such that the parametrizations

$$\pi_j : [0, l_j] \rightarrow k_j : x_j \longmapsto \pi_j(x_j)$$

is $C^\nu([0,l_j], \mathbb{R}^m)$ for all $1 \leq j \leq N$.

We now define the C^ν-network G associated with Γ as the union

$$G = \cup_{j=1}^N k_j.$$

The incidence matrix $D = (d_{ij})_{n\times N}$ of Γ is defined by

$$d_{ij} = \begin{cases} 1 \text{ if } \pi_j(l_i) = E_i, \\ -1 \text{ if } \pi_j(0) = E_i, \\ 0 \,\text{otherwise.} \end{cases}$$

The adjacency matrix $\mathcal{E} = (e_{ih})_{n\times n}$ of Γ is given by

$$e_{ih} = \begin{cases} 1 \text{ if there exists an edge } k_{s(i,h)} \text{ between } E_i \text{ and } E_h \\ 0 \text{ otherwise.} \end{cases}$$

The valence [(*)] of the node E_i will be noted $\gamma(E_i)$. There are two types of nodes: the interior nodes $int\,\mathcal{E} = \{E_i \in \mathcal{E}; \gamma(E_i) > 1\}$ and the boundary nodes $\partial\mathcal{E} = \{E_i \in \mathcal{E} : \gamma(E_i) = 1\}$. In the following we will denote $I_{ext} = \{i \in \{1,\dots,n\} : \gamma(E_i) = 1\}$ and $I_{int} = \{1,\dots,n\} \setminus I_{ext}$. We denote by $N_i = \{j \in \{1,\dots,n\}, E_i \in k_j\}$ the set of edges adjacent to E_i. We remark that if $E_i \in \partial\mathcal{E}$, then N_i is a singleton which is denoted by $\{j_i\}$.

For a function $u : G \rightarrow \mathbb{R}$, we set $u_j = u \circ \pi_j : [0,l_j] \rightarrow \mathbb{R}$, its restriction to the edge k_j. We further use the abbreviations:

$$u_j(E_i) = u_j(\pi_j^{-1}(E_i)), u_{jx_j^{(n)}}(E_i) = \frac{d^n u_j}{dx_j^n}(\pi_j^{-1}(E_i)),\, n \in \mathbb{N}^*.$$

Finally, differentiations are carried out on each edge k_j with respect to the arc length parameter x_j.

(*) The valence of the node E_i is the cardinal of the set of edges adjacent to E_i.

We consider the following operator Δ_G on the Hilbert space $H = \prod_{j=1}^{N} L^2(0,l_j)$, endowed with the usual product norm.

$$D(\Delta_G) = \{u \in H,\, u_j \in H^2(0,l_j) \text{ satisfying } (1.1) - (1.3)\}$$

$$\Delta_G u = \Big(-u_{jx_j^{(2)}}\Big)_{j=1}^{N},\ \forall u \in D(\Delta_G).$$

If $O = (O_{ih})_{n\times n}$ is the orientation matrix defined by

$$O_{ih} = \begin{cases} 1 \text{ if } k_{s(i,h)} \text{ is directed from } E_i \text{ to } E_h \\ -1 \text{ if } k_{s(i,h)} \text{ is directed from } E_h \text{ to } E_i \\ 0 \text{ else} \end{cases}$$

$$u \text{ is continuous on } G, \tag{1.1}$$

$$\sum_{j=s(i,h)\in N_i} O_{ih}\, u_{jx_j}(E_i) = 0,\ \forall i = 1,\ldots,n, \tag{1.2}$$

$$u_{j_i}(E_i) = 0,\ \forall i \in I_{ext}. \tag{1.3}$$

We study a model of networks of strings and of Euler-Bernoulli beams.

More precisely we consider the following initial problems:

on a finite network, of length L, made of edges k_j, identified to a real interval of length l_j, $j = 1,\ldots,N$, (i.e. $L = \sum_{i=1}^{N} l_i$) we consider the eigenvalue problem

$$-\frac{d^2u_j}{dx_j^2} = \lambda u_j, \quad k_j,\, j = 1,\ldots,N, \tag{1.4}$$

$$u \text{ satisfies } (1.1) - (1.3) \tag{1.5}$$

and

$$\frac{d^4u_j}{dx_j^4} = \lambda u_j, \quad k_j,\, j = 1,\ldots,N, \tag{1.6}$$

$$O_{ih} u_{jx_j^{(2)}}(E_i) = O_{ik} u_{lx_l^{(2)}}(E_i), \text{ if } j = s(i,h),\ l = s(i,k), \tag{1.7}$$

$$\sum_{j=s(i,h)\in N_i} O_{ih} u_{jx_j^{(3)}}(E_i) = 0,\ \forall i = 1,\ldots,n, \tag{1.8}$$

$$u_{j_i}(E_i) = 0,\ u_{j_i,x_{j_i}^{(2)}}(E_i) = 0,\ \forall i \in I_{ext}, \tag{1.9}$$

$$u \text{ satisfies } (1.1) - (1.2). \tag{1.10}$$

In the present paper we give some asymptotic Weyl formula of some networks of strings and of Euler-Bernoulli beams.

The plan of the paper is as follows. In the following section we give precise statements of the main results. The two last sections are devoted to some applications and related question.

2. Asymptotics with optimal remainder estimates

Let $\lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots$ be the eigenvalues, repeated according to their multiplicity, of the self-adjoint operator Δ_G on a C^2-network G which is defined in Section 1.

We introduce the counting function of eigenvalues

$$N_{\Delta_G}(\lambda) := \#\sigma(\Delta_G) \cap]-\infty, \lambda],$$

where in general $\#A$ denotes the number of elements of A.

Our main result can now be stated as follows.

THEOREM 2.1. — *There exists* $\lambda_0 \gg 1$ *such that*

$$N_{\Delta_G}(\lambda) = \frac{L}{\pi}\sqrt{\lambda} + \mathcal{O}(1),$$

uniformly on $\lambda \in]\lambda_0, +\infty[$.

Proof. — Let $\chi \in C_0^\infty(]-\epsilon, \epsilon[)$, with $\chi(0) = 1$. We choose $\epsilon > 0$ small enough such that $l_j > \epsilon$ for all $j = 1, \ldots, N$. We may chose χ with the additional property $\widehat{\chi}(t) \geq 0$ and $\widehat{\chi}(0) > 0$. In fact, it suffices to choose $\chi = \psi \star \tilde{\psi}$ for a suitable $\psi \in C_0^\infty$. Here $\tilde{\psi}(t) = \psi(-t)$. We define

$$\mu(\lambda) = \#\{j; \mu_j \leq \lambda\} = \sum_{\mu_j \leq \lambda} (1)$$

where $\mu_j = \sqrt{\lambda_j}$. According to Lemma 4.1 in the Appendix, we have

$$\sum_{j=0}^{\infty} \widehat{\chi}(\lambda + \mu_j) + \sum_{j=0}^{\infty} \widehat{\chi}(\lambda - \mu_j) = 2L\chi(0) + (n - N)\widehat{\chi}(\lambda). \tag{2.1}$$

Since $\widehat{\chi}(0) > 0$, there exists $\delta > 0$ such that $\widehat{\chi}(t) > \frac{1}{2}\widehat{\chi}(0)$ for all $t \in [-\delta, \delta]$. Combining this with the fact that $\widehat{\chi}(t) \geq 0$ for all $t \in \mathbb{R}$, and using (2.1), we obtain

$$\begin{aligned} \#\{j; \mu_j \in [\lambda - \delta, \lambda + \delta]\} &\leq \frac{2}{\widehat{\chi}(0)} \sum_j \Big(\widehat{\chi}(\lambda + \mu_j) + \widehat{\chi}(\lambda - \mu_j)\Big) \\ &\leq \frac{2}{\widehat{\chi}(0)} \Big(2L\chi(0) + (n - N)\widehat{\chi}(\lambda)\Big) = \mathcal{O}(1), \end{aligned} \tag{2.2}$$

uniformly on $\lambda \in \mathbb{R}$. Without any loss of generality, we may assume that $\delta = 1$. Writing

$$\mu(\lambda) \leq \sum_{0}^{[\lambda]} \mu(j+1) - \mu(j),$$

with $[\lambda]$=largest integer $\leq \lambda$, we obtain

$$\mu(\lambda) = \mathcal{O}(\lambda), \tag{2.3}$$

which yields

$$\mu_j = \mathcal{O}(1)j.$$

An immediate consequence of the above equality is that

$$\sum_{j=0}^{\infty} \widehat{\chi}(\lambda - \mu_j) = \mathcal{O}(\langle\lambda\rangle^{-\infty}), \quad \text{for } \lambda < 0, \tag{2.4}$$

$$\sum_{j=0}^{\infty} \widehat{\chi}(\lambda + \mu_j) = \mathcal{O}(\langle\lambda\rangle^{-\infty}), \quad \text{for } \lambda > 0.$$

Here $\langle\lambda\rangle = (1 + |\lambda|^2)^{\frac{1}{2}}$. Combining this with (2.1), we obtain

$$\sum_{j=0}^{\infty} \widehat{\chi}(\lambda - \mu_j) = 2L\chi(0) + \mathcal{O}(\langle\lambda\rangle^{-\infty}), \quad \text{for } \lambda > 0. \tag{2.5}$$

Put $K(\lambda) = \int_{-\infty}^{\lambda} \widehat{\chi}(\tau) d\tau$. We have

$$\int K(\lambda - x) d\mu(x) = \sum K(\lambda - \mu_n) \tag{2.6}$$

$$= \sum \int_{-\infty}^{\lambda - \mu_n} \widehat{\chi}(\tau) d\tau = \sum \int_{-\infty}^{\lambda} \widehat{\chi}(x - \mu_n) dx.$$

We recall that $\mu(\lambda) = \sum_{\mu_n \leq \lambda} (1)$. The estimates (2.4) and (2.5) yield

$$\int K(\lambda - x) d\mu(x) = 2L\chi(0) \int_{0}^{\lambda} dx + \mathcal{O}(1) \tag{2.7}$$

$$= 2L\chi(0)\lambda + \mathcal{O}(1), \lambda \to +\infty.$$

We rewrite the left hand side of (2.6) as

$$\int K(\lambda - x) d\mu(x) = \int \mu(\lambda - \tau) \widehat{\chi}(\tau) d\tau \tag{2.8}$$

$$= \int (\mu(\lambda - \tau) - \mu(\lambda)) \widehat{\chi}(\tau) d\tau + 2\pi\mu(\lambda),$$

since $\int \widehat{\chi}(\tau)d\tau = 2\pi\chi(0) = 2\pi$. For $\lambda \gg 1$, we get from (2.2)

$$|\mu(\lambda-\tau)-\mu(\lambda)| \leq C(1+|\tau|).$$

Consequently,

$$\int (\mu(\lambda-\tau)-\mu(\lambda))\widehat{\chi}(\tau)d\tau = \mathcal{O}(1), (\lambda \to +\infty). \tag{2.9}$$

Putting together (2.7), (2.8) and (2.9), we get

$$\mu(\lambda) = \frac{L}{\pi}\lambda + \mathcal{O}(1).$$

Combining this with the fact that $\mu(\sqrt{\lambda}) = N_{\Delta_G}(\lambda)$, we get Theorem 2.1. □

As a consequence we have the following result concerning the beams networks:

Corollary 2.2. — *There exists $\lambda_0 \gg 1$ such that*

$$N_{\Delta_G^2}(\lambda) := \#\sigma(\Delta_G^2) \cap]-\infty,\lambda] = \frac{L}{\pi}\lambda + \mathcal{O}(1),$$

uniformly on $\lambda \in]\lambda_0, +\infty[$ and where $D(\Delta_G^2) = \left\{u \in H,\, u_j \in H^4(0,l_j) \text{ satisfying } (1.1)-(1.2) \text{ and } (1.7)-(1.8)\right\}$.

Remark 2.3. — *We remark that our method is valid for all elliptic operator in a graph.*

As consequence of Theorem 2.1 we have the following result:

Corollary 2.4. — *There exist $M \in \mathbb{N}^*$ and $\eta > 0$ such that*

$$\mu_{n+M} - \mu_n \geq \eta\, M, \forall\, n \geq 0,$$

where $\mu_k = \sqrt{\lambda_k}$, $\forall\, k \geq 0$.

Proof. — From Theorem 2.1, there exists $n_0 \in \mathbb{N}^*$ and $C > 0$ such that

$$-C \leq \mu_n - \frac{\pi}{L}n \leq C,$$

for all $n \geq n_0$. Changing C by $\tilde{C} > C$, we may assume that $n_0 = 0$. Using the above inequality, we obtain

$$\frac{\pi}{L}M - 2\,\tilde{C} \leq \mu_{n+M} - \mu_n,$$

for all $n \geq 0, M \in \mathbb{N}^*$. Choosing M large enough so that $\frac{\pi}{L}M - 2\,\tilde{C} > \eta\, M$ we obtain the result. □

3. Applications

Let H be a Hilbert space equipped with the norm $\|.\|_H$, and let $A_1 : \mathscr{D}(A_1) \to H$ be a self-adjoint, positive and boundedly invertible operator with compact resolvent. We introduce the scale of Hilbert spaces H_α, $\alpha \in \mathbb{R}$, as follows: for every $\alpha \geq 0$, $H_\alpha = \mathscr{D}(A_1^\alpha)$, with the norm $\|z\|_\alpha = \|A_1^\alpha z\|_H$. The space $H_{-\alpha}$ is defined by duality with respect to the pivot space H a follows: $H_{-\alpha} = H_\alpha^*$ for $\alpha > 0$. The operator A_1 can be extended (or restricted) to each H_α, such that it becomes a bounded operator

$$A_1 : H_\alpha \to H_{\alpha-1} \qquad \forall\, \alpha \in \mathbb{R}. \tag{3.10}$$

The second ingredient needed for our construction is a bounded linear operator $B_1 : U \to H_{-\frac{1}{2}}$, where U is another Hilbert space which will be identified with its dual. The systems we consider are described by

$$\ddot{w}(t) + A_1 w(t) + B_1 y(t) = 0\,, w(0) = w_0,\, \dot{w}(0) = w_1, \tag{3.11}$$

$$y(t) = B_1^* \dot{w}(t), \tag{3.12}$$

where $t \in [0,\infty)$ is the time. The equation (3.11) is understood as an equation in $H_{-\frac{1}{2}}$, i.e., all the terms are in $H_{-\frac{1}{2}}$. Most of the linear equations modelling the damped vibrations of elastic structures can be written in the form (3.11), where w stands for the displacement field and the term $B_1 B_1^* \dot{w}(t)$, represents a viscous feedback damping. The system (3.11)-(3.12) is well-posed:

For $(w_0, w_1) \in H_{\frac{1}{2}} \times H$, the problem (3.11)-(3.12) admits a unique solution

$$w \in C([0,\infty); H_{\frac{1}{2}}) \cap C^1([0,\infty); H)$$

such that $B_1^* w(\cdot) \in H^1(0,T;U)$. Moreover w satisfies, for all $t \geq 0$, the energy estimate

$$\|(w_0,w_1)\|^2_{H_{\frac{1}{2}}\times H} - \|(w(t),\dot{w}(t))\|^2_{H_{\frac{1}{2}}\times H} = 2\int_0^t \left\| \frac{\mathrm{d}}{\mathrm{d}t} B_1^* w(s) \right\|_U^2 \mathrm{d}s\,. \tag{3.13}$$

From (3.13) it follows that the mapping $t \mapsto \|(w(t),\dot{w}(t))\|^2_{H_{\frac{1}{2}}\times H}$ is non increasing.

We consider the initial value problem

$$\ddot{\varphi}(t) + A_1 \varphi(t) = 0, \tag{3.14}$$

$$\varphi(0) = w_0,\, \dot{\varphi}(0) = w_1. \tag{3.15}$$

It is well known that (3.14)-(3.15) is well-posed in $H_1 \times H_{\frac{1}{2}}$ and in $H_{\frac{1}{2}} \times H$.

Consider now the unbounded linear operator

$$\mathscr{A}_d : \mathscr{D}(\mathscr{A}_d) \to H_{\frac{1}{2}} \times H,\ \mathscr{A}_d = \begin{pmatrix} 0 & I \\ -A_1 & -B_1 B_1^* \end{pmatrix}, \tag{3.16}$$

where

$$\mathscr{D}(\mathscr{A}_d) = \left\{ (u,v) \in H_{\frac{1}{2}} \times H,\ A_1 u + B_1 B_1^* v \in H,\ v \in H_{\frac{1}{2}} \right\}.$$

The result below, see [6], shows that, under a certain regularity assumption, the polynomial stability of (3.11)-(3.12) is a consequence of a weak observability inequality. More precisely, we have:

THEOREM 3.1. — *Assume that for any $\gamma > 0$ we have*

$$\sup_{\mathrm{Re}\lambda=\gamma} \left\| \lambda B_1^* (\lambda^2 I + A_1)^{-1} B_1 \right\|_{\mathscr{L}(U)} < \infty. \tag{3.17}$$

Then the following assertion holds true:
If there exist $T, C > 0, \alpha > -\frac{1}{2}$ such that: $\forall\, (w_0, w_1) \in H_1 \times H_{\frac{1}{2}}$ we have

$$||B_1^* \varphi'(t)||_{L^2(0,T;U)} \geq C\ ||(w_0,w_1)||_{H_{-\alpha} \times H_{-\alpha-\frac{1}{2}}}, \tag{3.18}$$

where $\varphi(t)$ is a solution of (3.14)-(3.15).

Then there exists a constant $C_1 > 0$ such that for all $t > 0$ and for all $(w^0, w^1) \in \mathscr{D}(\mathscr{A}_d)$ we have

$$\|(w(t), \dot{w}(t))\|_{H_{\frac{1}{2}} \times H} \leq \frac{C_1}{(1+t)^{\frac{1}{4\,\alpha+2}}} \|(w^0, w^1)\|_{\mathscr{D}(\mathscr{A}_d)}. \tag{3.19}$$

3.1. Application to a polynomial stabilization of a star-shaped network of strings. — We consider the following initial and boundary value problems:

$$\frac{\partial^2 u_i}{\partial t^2}(x,t) - \frac{\partial^2 u_i}{\partial x^2}(x,t) = 0, \quad 0 < x < l_i,\ t > 0, \tag{3.20}$$

$$u_i(l_i, t) = 0, \quad t > 0, \tag{3.21}$$

$$u_i(0,t) = u_j(0,t), \quad t > 0, \tag{3.22}$$

$$\sum_{i=1}^{N} \frac{\partial u_i}{\partial x}(0,t) = \frac{\partial u_1}{\partial t}(0,t), \quad t > 0, \tag{3.23}$$

$$u_i(x,0) = u_i^0(x), \frac{\partial u_i}{\partial t}(x,0) = u_i^1(x), \quad 0 < x < l_i, \tag{3.24}$$

for $i, j = 1, \ldots, N$ and where $u_i : [0, l_i] \times (0, +\infty) \to \mathbb{R}$, $i = 1, \ldots, N$, $N \geq 2$ be the displacement of the string of length l_i. Denote by $L = \sum_{i=1}^{N} l_i$.

Let $H = \prod_{i=1}^{N} L^2(0,l_i)$, $A_1 = -\dfrac{d^2}{dx^2}$,

$$\mathcal{D}(A_1) = \Big\{((u_i)_{i=1,\dots,N}) \in \prod_{i=1}^{N} H^2(0,l_i),\ u_i(0) = u_j(0),\ \forall\, i,j = 1,\dots,N,$$
$$u_i(l_i) = 0,\ \sum_{i=1}^{N} \frac{du_i}{dx}(0) = 0\Big\}.$$

$\mathcal{D}(A_1^{\frac{1}{2}})$ is the completed of $\mathcal{D}(A_1)$ with respect to the norm

$$\|z\|_{\mathcal{D}(A_1^{\frac{1}{2}})} = \sqrt{(A_1 z, z)_H} = \sqrt{\sum_{i=1}^{N} \int_0^{l_i} -\frac{d^2 z_i}{dx^2}\, z_i\, dx}.$$

Let $B_1 \in \mathcal{L}(\mathbb{R}, \mathcal{D}(A_1^{\frac{1}{2}})')$, $B_1 v = (A_1)_{-1}\mathcal{N} v$, $\forall\, v \in \mathbb{R}$, where $(A_1)_{-1} : \mathcal{D}(A_1^{\frac{1}{2}}) \to \mathcal{D}(A_1^{\frac{1}{2}})'$ is an extension of A_1 to $\mathcal{D}(A_1^{\frac{1}{2}})$, $\mathcal{N} \in \mathcal{L}(\mathbb{R}, \mathcal{D}(A_1^{\frac{1}{2}}))$ and $\mathcal{N} v$ is a solution of:

$$\frac{d^2(\mathcal{N} v)_i}{dx^2} = 0, \quad 0 < x < l_i, \tag{3.25}$$

$$(\mathcal{N} v)_i(l_i) = 0, \tag{3.26}$$

$$(\mathcal{N} v)_i(0) = (\mathcal{N} v)_j(0), \tag{3.27}$$

$$\sum_{i=1}^{N} \frac{d(\mathcal{N} v)_i}{dx}(0) = v, \tag{3.28}$$

for all $i,j = 1,\dots,N$, and

$$B_1^* \psi = \psi_1(0),\ \forall\, \psi \in \mathcal{D}(A_1^{\frac{1}{2}}).$$

We denote by $\lambda_k = \mu_k^2$ the eigenvalues of A_1. In the case: $\dfrac{l_i}{l_j} \notin \mathbb{Q}$, $\forall\, 1 \le i \ne j \le N$, the eigenvalues λ_k are simple (see [3]) and the corresponding eigenfunctions are given by:

$$\phi_k^i(x) = \frac{\sin(\mu_k(x - l_i))}{\sin(\mu_k l_i)\left(\displaystyle\sum_{i=1}^{N} \frac{l_i}{\sin^2(\mu_k l_i)}\right)^{\frac{1}{2}}},\ i = 1,\dots,N.$$

We define the energy of u_i, $i = 1,\dots,N$ of (3.20)-(3.24) at instant t by

$$E(t) = \sum_{i=1}^{N} \frac{1}{2} \int_0^{l_i} \left(\left|\frac{\partial u_i}{\partial t}(x,t)\right|^2 + \left|\frac{\partial u_i}{\partial x}(x,t)\right|^2 \right) dx. \tag{3.29}$$

The wellposedness space for (3.20)-(3.24) is $E = \mathcal{D}(A_1^{\frac{1}{2}}) \times \prod_{i=1}^{N} L^2(0,l_i)$. Denote

$$\mathcal{D}(\mathcal{A}_d) = \Bigg\{ ((u_i)_{i=1,\dots,N}, (v_i)_{i=1,\dots,N}) \in \Big[\mathcal{D}(A_1^{\frac{1}{2}}) \cap \prod_{i=1}^{N} H^2(0,l_i) \Big] \times \mathcal{D}(A_1^{\frac{1}{2}}),$$

$$\sum_{i=1}^{N} \frac{du_i}{dx}(0) = v_1(0) \Bigg\}. \tag{3.30}$$

The corresponding operator $\mathcal{A}_d$ is defined by

$$\mathcal{A}_d \begin{pmatrix} u_1 \\ \dots \\ u_N \\ v_1 \\ \dots \\ v_N \end{pmatrix} = \begin{pmatrix} v_1 \\ \dots \\ v_N \\ \frac{d^2 u_1}{dx^2} \\ \dots \\ \frac{d^2 u_N}{dx^2} \end{pmatrix}, \ \forall \, (u,v) \in \mathcal{D}(\mathcal{A}_d).$$

If $(u^0, u^1) \in E$, then the problem (3.20)-(3.24) admits a unique solution

$$u \in C(0,+\infty; \mathcal{D}(A_1^{\frac{1}{2}})) \cap C^1(0,+\infty; \prod_{i=1}^{N} L^2(0,l_i))$$

and we have: $\lim_{t \to +\infty} E(t) = 0$ holds true for any finite energy solution of (3.20)-(3.24) if and only if

$$\frac{l_i}{l_j} \notin \mathbb{Q}, \ \forall \, 1 \leq i \neq j \leq N, \tag{3.31}$$

where $\mathbb{Q}$ is the set of all rational numbers.

Denote by $\mathcal{S}$ the set of all numbers ρ such that $\rho \notin \mathbb{Q}$ and if $[0, a_1, \dots, a_n, \dots]$ is the expansion of ρ as a continued fraction, then (a_n) is bounded. Let us notice that $\mathcal{S}$ is obviously uncountable and, by classical results on diophantine approximation, its Lebesgue measure is equal to zero. Roughly speaking the set $\mathcal{S}$ contains the irrationals which are approximable by rational numbers. In particular, by Euler-Lagrange theorem $\mathcal{S}$ contains all l_i/l_j, $1 \leq i \neq j \leq N$ such that l_i/l_j is an irrational quadratic number (i.e. satisfying a second degree equation with rational coefficients). According to [13], we have that $l_i/l_j \in \mathcal{S}$, $1 \leq i \neq j \leq N$, if and only if there exists a positive constant C such that: $|||\frac{l_i}{l_j} m||| := \min_{\frac{l_i}{l_j} m - x \in \mathbb{Z}} |x| \geq \frac{C}{m}$, $\forall \, m \in \mathbb{N}^*$.

COROLLARY 3.2. — 1. *If $l_i/l_j \in \mathcal{S}$, $\forall\, 1 \le i \ne j \le N$, there exists $\beta > 0$ such that for all $t \ge 0$ we have*

$$E(t) \le \frac{C}{(t+1)^{\frac{1}{\beta}}} ||(u^0, u^1)||^2_{\mathcal{D}(\mathcal{A}_d)}, \qquad \forall\, (u^0, u^1) \in \mathcal{D}(\mathcal{A}_d), \tag{3.32}$$

where $C > 0$ is a constant depending only on l_i, $i = 1, \ldots, N$.

2. *For all $\varepsilon > 0$ there exists a set $B_\varepsilon \subset \mathbb{R}$, such that the Lebesgue measure of $\mathbb{R} \setminus B_\varepsilon$ is equal to zero, and a constants $\beta, C_\varepsilon > 0$ for which, if $l_i/l_j \in B_\varepsilon$, $1 \le i \ne j \le N$, then for all $t \ge 0$*

$$E(t) \le \frac{C_\epsilon}{(t+1)^{\frac{1}{\beta+\varepsilon}}} ||(u^0, u^1)||^2_{\mathcal{D}(\mathcal{A}_d)}, \qquad \forall\, (u^0, u^1) \in \mathcal{D}(\mathcal{A}_d), \tag{3.33}$$

where $C_\varepsilon > 0$ is a constant depending only on l_i, $i = 1, \ldots, N$ and ε.

Proof. By a simple calculations we show that

$$\lambda B_1^*(\lambda^2 I + A_1)^{-1} B_1 = -\left(\sum_{i=1}^{N} \coth(\lambda l_i)\right)^{-1}, \forall\, \lambda \in \mathbb{C},\ Re\lambda > 0.$$

So the condition (3.17) is satisfied according to the following lemma:

LEMMA 3.3. — [3, Lemma 3.3]

Let $\gamma > 0$ be a fixed real number and $C_\gamma = \Big\{\lambda \in \mathbb{C} \mid Re(\lambda) = \gamma\Big\}$. Then, the function

$$f(\lambda) = \frac{1}{\displaystyle\sum_{i=1}^{N} \coth(\lambda l_i)}, \tag{3.34}$$

is bounded on C_γ.

For $k < 0$, we denote by $\mu_k = -\mu_{-k}$. Let $0 < \eta' \le \eta$ with $\eta' \le \frac{2\mu_1}{M}$. We claim that

$$\mu_{k+M} - \mu_k \ge \eta' M, \forall\, k \in \mathbb{Z}. \tag{3.35}$$

In fact, for $k > 0$ resp.($k + M < 0$) (3.35) follows from Corollary 2.4 resp.(Corollary 2.4 and the fact that $\mu_k = -\mu_{-k}$). For $k + M > 0$ and $k < 0$ we use that $\mu_{k+M} - \mu_k = \mu_{k+M} + \mu_{-k} \ge 2\,\mu_1 \ge M\eta'$.

We denote by A_j, $j = 1, \ldots, M$ the set of integers m satisfying :

$$\mu_m - \mu_{m-1} \ge \eta'$$
$$\mu_n - \mu_{n-1} < \eta', \forall\, m+1 \le n \le m+j-1$$
$$\mu_{m+j} - \mu_{m+j-1} \ge \eta'.$$

Then the $M(M+1)/2$ sets $A_j + k = \{n + k;\ n \in A_j\}$, $0 \le k < j \le M$ are disjoint and form a partition of the set $\mathbb{Z}$. Let us introduce for $m \in A_j$ the divided differences $e_m(t), \dots, e_{m+j-1}(t)$ of the exponential functions $e^{i\mu_n t}$, $n = m, \dots, m+j-1$. Since μ_k are simple (see [3]) then e_k (see [11]) is given by the following expression

$$e_k(t) = \sum_{p=m}^{k} \left[\prod_{q=m,\, q\neq p}^{k} (\mu_p - \mu_q) \right]^{-1} e^{i\mu_p t},\ \forall\ k = m, \dots, m+j-1.$$

For

$$(u_1^0, \dots, u_N^0, u_1^1, \dots, u_N^1)^t = \sum_{k\in\mathbb{Z}} a_k \left(\frac{1}{\mu_k}\, \phi_k^1, \dots, \frac{1}{\mu_k}\, \phi_k^N, \phi_k^1, \dots, \phi_k^N \right)^t,\ (a_k)_{k\in\mathbb{Z}} \in l^2,$$

we have

$$\frac{\partial\varphi_1}{\partial t}(0,t) = \sum_{k\in\mathbb{Z}} a_k e^{i\mu_k t}\, \phi_k^1(0) = \sum_{k\in\mathbb{Z}} b_k\, e_k(t),$$

where $\varphi = (\varphi_1, \dots, \varphi_N)^t$ is a solution of conservative system associated to (3.20)-(3.24). According to [11, Theorem 9.4] we have that for $T > \dfrac{2\pi}{\eta}$ there exists a constant $C_1 > 0$ such that

$$\int_0^T \left| \frac{\partial\varphi_1}{\partial t}(0,t) \right|^2 dt \geq C_1 \sum_{k\in\mathbb{Z}} |b_k|^2$$

and if $l_i/l_j \in \mathcal{S}$, $\forall\, 1 \leq i \neq j \leq N$, there exist $\beta, C_2 > 0$ such that we have:

$$\int_0^T \left| \frac{\partial\varphi_1}{\partial t}(0,t) \right|^2 dt \geq C_2 \sum_{k\in\mathbb{Z}} |\mu_k|^{-\beta}\, |a_k|^2 .$$

Which implies, according to Theorem 3.1, the estimate (3.32).

In order to prove (3.33) we use a well-known result (see [8] p. 120) asserting that for all $\varepsilon > 0$, there exists a set $B_\varepsilon \subset \mathbb{R}$, such that the Lebesgue measure of $\mathbb{R} \setminus B_\varepsilon$ is equal to zero, and a constant $C_\varepsilon > 0$ for which, if $\xi \in B_\varepsilon$, then $|||\xi m||| \geq \frac{C_\varepsilon}{m^{1+\varepsilon}}$.

Let us notice that by Roth's theorem (see [17]) B_ε contains all real numbers having the property that $\dfrac{l_i}{l_j}$ is an algebraic irrational (see [8] for details). If $l_i/l_j \in B_\varepsilon$, $1 \leq i \neq j \leq N$, there exists a constant $C > 0$ such that $|||m l_i/l_j||| \geq \frac{C}{m^{1+\varepsilon}}$, $\forall\, m \geq 1$. Then, as above we have for $T > \frac{2\pi}{\eta}$ that there exists a constant $C_3 > 0$ such that:

$$\int_0^T \left| \frac{\partial\varphi_1}{\partial t}(0,t) \right|^2 dt \geq C_3 \sum_{k\in\mathbb{Z}} |\mu_k|^{-\beta-\varepsilon}\, |a_k|^2 .$$

Which implies (3.33), according to Theorem 3.1. □

3.2. Application to a polynomial stabilization of a star-shaped network of Euler-Bernoulli beams. — We consider the following initial and boundary value problem :

$$\frac{\partial^2 u_i}{\partial t^2}(x,t) + \frac{\partial^4 u_i}{\partial x^4}(x,t) = 0, \quad 0 < x < l_i,\ t > 0, \tag{3.36}$$

$$u_i(l_i,t) = 0,\ \frac{\partial^2 u_i}{\partial x^2}(l_i,t) = 0, \quad t > 0, \tag{3.37}$$

$$u_i(0,t) = u_j(0,t),\ \sum_{i=1}^{N} \frac{\partial u_i}{\partial x}(0,t) = 0,\ \frac{\partial^2 u_i}{\partial x^2}(0,t) = \frac{\partial^2 u_j}{\partial x^2}(0,t), \quad t > 0, \tag{3.38}$$

$$\sum_{i=1}^{N} \frac{\partial^3 u_i}{\partial x^3}(0,t) = -\frac{\partial u_1}{\partial t}(0,t), \quad t > 0, \tag{3.39}$$

$$u_i(x,0) = u_i^0(x),\ \frac{\partial u_i}{\partial t}(x,0) = u_i^1(x), \quad 0 < x < l_i, \tag{3.40}$$

for $i,j = 1,\dots,N$, $2 \le N \in \mathbb{N}$ and where $u_i : [0,l_i] \times (0,+\infty) \to \mathbb{R}$, be the displacement of the beam of length l_i.

Let $H = \prod_{i=1}^{N} L^2(0,l_i)$, $A_1 = \frac{d^4}{dx^4}$,

$$\mathcal{D}(A_1) = \Bigg\{ ((u_i)_{i=1,\dots,N}) \in \prod_{i=1}^{N} H^4(0,l_i),\ u_i(0) = u_j(0),\ \frac{d^2 u_i}{dx^2}(l_i) = 0,$$

$$\frac{du_i}{dx}(0) = \frac{du_j}{dx}(0),\ \forall\, 1 \le i,j \le N,\ u_i(l_i) = 0,\ \sum_{i=1}^{N} \frac{d^3 u_i}{dx^3}(0) = 0 \Bigg\}$$

$\mathcal{D}(A_1^{\frac{1}{2}})$ is the completed of $\mathcal{D}(A_1)$ with respect to the norm

$$\|z\|_{\mathcal{D}(A_1^{\frac{1}{2}})} = \sqrt{(A_1 z, z)_H} = \sqrt{\sum_{i=1}^{N} \int_0^{l_i} \frac{d^4 z_i}{dx^4}\, z_i \, dx}.$$

Let $B_1 \in \mathcal{L}(\mathbb{R}, \mathcal{D}(A_1^{\frac{1}{2}})')$, $B_1 v = (A_1)_{-1} \mathcal{N} v$, $\forall\, v \in \mathbb{R}$, where $(A_1)_{-1} : \mathcal{D}(A_1^{\frac{1}{2}}) \to \mathcal{D}(A_1^{\frac{1}{2}})'$ is an extension of A_1 to $\mathcal{D}(A_1^{\frac{1}{2}})$, $\mathcal{N} \in \mathcal{L}(\mathbb{R}, \mathcal{D}(A_1^{\frac{1}{2}}))$ and $\mathcal{N} v$ is a solution of:

$$\frac{d^4 (\mathcal{N} v)_i}{dx^4} = 0, \quad 0 < x < l_i, \tag{3.41}$$

$$(\mathcal{N}v)_i(l_i) = 0, \ \frac{d^2(\mathcal{N}v)_i}{dx^2}(l_i) = 0, \tag{3.42}$$

$$(\mathcal{N}v)_i(0) = (\mathcal{N}v)_j(0), \ \frac{d(\mathcal{N}v)_i}{dx}(0) = \frac{d(\mathcal{N}v)_j}{dx}(0), \tag{3.43}$$

$$\sum_{i=1}^{N} \frac{d^3(\mathcal{N}v)_i}{dx^3}(0) = v, \tag{3.44}$$

for all $i, j = 1, \ldots, N$, and

$$B_1^* \psi = \psi_1(0), \ \forall \psi \in \mathcal{D}(A_1^{\frac{1}{2}}).$$

We denote by $\lambda_k = \mu_k^4$ the eigenvalues of A_1. In the case: $\dfrac{l_i}{l_j} \notin \mathbb{Q}$, $\forall 1 \leq i \neq j \leq N$, the eigenvalues λ_k are simple (see [1]) and the corresponding eigenfunctions are given by:

$$\phi_k^i(x) = \frac{\sin(\mu_k(x - l_i))}{\sin(\mu_k l_i) \left(\displaystyle\sum_{i=1}^{N} \frac{l_i}{\sin^2(\mu_k l_i)} \right)^{\frac{1}{2}}}, \ i = 1, \ldots, N.$$

We define the energy of u solution of (3.36)-(3.40) at instant t by

$$E(t) = \sum_{i=1}^{N} \frac{1}{2} \int_0^{l_i} \left(\left| \frac{\partial u_i}{\partial t}(x,t) \right|^2 + \left| \frac{\partial^2 u_i}{\partial x^2}(x,t) \right|^2 \right) dx. \tag{3.45}$$

The wellposedness space for (3.36)-(3.40) is $X = \mathcal{D}(A_1^{\frac{1}{2}}) \times \displaystyle\prod_{i=1}^{N} L^2(0, l_i)$.

Denote

$$\begin{aligned} \mathcal{D}(\mathcal{A}_d) = \Big\{ &(u,v) \in \Big[\mathcal{D}(A_1^{\frac{1}{2}}) \cap \prod_{i=1}^{N} H^4(0, l_i) \Big] \times \mathcal{D}(A_1^{\frac{1}{2}}), \ \frac{d^2 u_i}{dx^2}(l_i) = 0, \\ &\frac{d^2 u_i}{dx^2}(0) = \frac{d^2 u_j}{dx^2}(0), \ \forall 1 \leq i, j \leq N, \ \sum_{i=1}^{N} \frac{d^3 u_i}{dx^3}(0) = -v_1(0) \Big\}. \end{aligned} \tag{3.46}$$

The corresponding operator $\mathcal{A}_d$ is defined by

$$\mathcal{A}_d \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ -\dfrac{d^4 u}{dx^4} \end{pmatrix}, \ \forall (u,v) \in \mathcal{D}(\mathcal{A}_d).$$

If $(u^0, u^1) \in X$, then the problem (3.36)-(3.40) admits a unique solution $u \in C(0, +\infty; \mathcal{D}(A_1^{\frac{1}{2}})) \cap C^1(0, +\infty; \displaystyle\prod_{i=1}^{N} L^2(0, l_i))$ and we have: $\displaystyle\lim_{t \to +\infty} E(t) = 0$ holds true for any finite energy solution of (3.36)-(3.40) if and only if

$$\frac{l_i}{l_j} \notin \mathbb{Q}, \forall 1 \leq i \neq j \leq N. \tag{3.47}$$

COROLLARY 3.4. — 1. *Suppose that* $l_i/l_j \in \mathcal{S}, \forall 1 \leq i \neq j \leq N$. *There exists* $\gamma > 0$ *such that for all* $t \geq 0$ *we have*

$$E(t) \leq \frac{C}{(t+1)^{\frac{1}{\gamma}}} ||(u^0, u^1)||^2_{\mathcal{D}(\mathcal{A}_d)}, \qquad \forall (u^0, u^1) \in \mathcal{D}(\mathcal{A}_d), \tag{3.48}$$

where $C > 0$ *is a constant depending only on* l_i, $i = 1, \ldots, N$.

2. *For all* $\varepsilon > 0$ *there exists a set* $B_\varepsilon \subset \mathbb{R}$, *such that the Lebesgue measure of* $\mathbb{R} \setminus B_\varepsilon$ *is equal to zero, and a constant* $C_\varepsilon > 0$ *for which, if* $l_i/l_j \in B_\varepsilon, \forall 1 \leq i \neq j \leq N$, *then there exists* $\gamma > 0$ *suh that for all* $t \geq 0$ *we have*

$$E(t) \leq \frac{C_\varepsilon}{(t+1)^{\frac{1}{\gamma+\varepsilon}}} ||(u^0, u^1)||^2_{\mathcal{D}(\mathcal{A}_d)}, \qquad \forall (u^0, u^1) \in \mathcal{D}(\mathcal{A}_d), \tag{3.49}$$

where $C_\varepsilon > 0$ *is a constant depending only on* l_i, $i = 1, \ldots, N$ *and* ε.

Proof. — By a simple calculations we show that

$$\lambda B_1^*(\lambda^2 I + A_1)^{-1} B_1 = \frac{i}{2w} \left\{ \frac{1}{\sum_{j=1}^N cotg(wl_j)} - \frac{1}{\sum_{j=1}^N \coth(wl_j)} \right\},$$

forall $\lambda = iw^2 \in \mathbb{C}$, $w = re^{i\theta}$, $r > 0$, $\theta \in [-\frac{\pi}{2}, 0]$, $Re\lambda > 0$.

So The condition (3.17) is satisified according to the following lemma:

LEMMA 3.5. — [1, Lemma 3.3] *Let* $\delta > 0$ *be a fixed real number and* $C_\delta = \left\{ w \in \mathbb{C} \mid Re(w)Im(w) = -\frac{\delta}{2} \right\}$. *Then*

$$f(w) = \frac{i}{2w} \left\{ \frac{1}{\sum_{j=1}^N cotg(wl_j)} - \frac{1}{\sum_{j=1}^N \coth(wl_j)} \right\}, \tag{3.50}$$

is bounded on C_δ.

The remainder of the proof is completely similar to Corollary 3.2. □

4. Related question

A question related to the problem studied in this paper is a Weyl formula with second term of the same elastic networks [2].

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Appendix

The aim of this appendix is to prove the equality (2.1). In fact formula (2.1) can be obtained by applying Theorem 2 in [16], but for the reader convenience we give the proof. Let G be a finite network made of edges $i_l, l = 1, \dots, N$ and n vertices. We denote by $\bar{i}_l$ the length of i_l and $L = \bar{i}_1 + \cdots + \bar{i}_N$ the length of the graph G. We denote by $\mathscr{E}$ resp. $\mathscr{S}$ the set of edges (resp. vertices).

Fix a vertex s, and let $m(s)$ be the number of arcs descended from s. For two arcs i and j (containing s), we define the real number $\epsilon_{ij} = \frac{2}{m(s)}$, if $i \neq -j$ and $\epsilon_{ij} = \frac{2}{m(s)} - 1$, if $i = -j$.

Consider the wave equation on the graph G:

$$\partial_t^2 u(x,t) = \partial_x^2 u(x,t),\ \partial_t u(x,0) = 0, u(x,0) = f,$$

It is well known that:

$$u(x,t) = \sum_{C,y} \prod_{l=1}^{m} \epsilon_{i_l i_{l+1}} f(y), \quad t > 0.$$

where the sum is taken over all paths $(i_1, \dots, i_m)$ such that $y \in i_1$, $x \in i_m$ with

$$d(x, I(i_n)) + \bar{i}_2 + \cdots + \bar{i}_{m-1} + d(y, T(i_1)) = t, \text{ for } m \geq 2,$$

$$d(x,y) = t \text{ for } m = 1.$$

Here $I(i)$ (resp. $T(i)$) is the starting (resp. end) point of the oriented arc i. We recall that a path $(i_1, \dots, i_m)$ is a circuit such that $T(i_1) = I(i_2), \dots, T(i_{m-1}) = I(i_m)$.

For $t < 0$, we have $u(x,t) = u(x,-t)$ due to the fact that $\partial_t u(x,0) = 0$.

Now let $\theta \in C_0^\infty(\mathbb{R})$. A simple calculus gives

$$\int \theta(t)u(x,t)dt = \int_0^{+\infty} \theta(t)u(x,t)dt + \int_{-\infty}^0 \theta(t)u(x,t)dt$$
$$= \frac{1}{2}\sum_C \prod_{l=1}^m \epsilon_{i_l i_{l+1}} \int_{i_1} f(y)(\theta(d(y,T(i_1)) + \bar{i}_2 + \cdots + \bar{i}_{m-1} + d(x,I(i_m)))$$
$$+ \theta(-(d(y,T(i_1)) + \bar{i}_2 + \cdots + \bar{i}_{m-1} + d(x,I(i_m)))))dy$$
$$+ \frac{1}{2}\sum_{i\in\mathscr{C}} \int_i f(y)(\theta(d(y,x)) + \theta(-d(y,x)))dy.$$

On the other hand the spectral theorem yields

$$\int \theta(t)u(x,t)dt = \sum_n \int\int \cos(\sqrt{\lambda_n}t)\theta(t)\phi_n(x)f(y)\phi_n(y)dydt,$$

where $(\phi_n)_{n\geq 1}$ is an orthonormalized basis of eigenfunctions corresponding to the problem (1.4)-(1.5).

Identifying the two above equalities and integrating over x, we obtain

$$\sum_n \int\int \cos(\sqrt{\lambda_n}t)\theta(t)dt = \frac{1}{2}\sum_{j\in\mathscr{C}} \theta(0)(\int_j dx + \int_j dx)$$

$$+\frac{1}{2}\sum_C \prod_{l=1,m>1}^m \epsilon_{i_l i_{l+1}} \int_{i_1} (\theta(d(x,T(i_1)) + \bar{i}_2 + \cdots + \bar{i}_{m-1} + d(x,I(i_m)))+$$
$$\theta(-(d(x,T(i_1)) + \bar{i}_2 + \cdots + \bar{i}_{m-1} + d(x,I(i_m)))))dx.$$

Now let $\theta \in C_0^\infty(]-\epsilon,\epsilon[)$, with ϵ small enough such that $\theta(\bar{i}_l) = 0$ for all $l = 1,\ldots,N$. In particular, we have

$$\theta(\pm(d(x,T(i_1)) + \bar{i}_2 + \cdots + \bar{i}_{m-1} + d(x,I(i_m)))) = 0,$$

for all circuit $(i_1,\ldots,i_m) \neq (i_l,-i_l)$. Consequently

$$\sum_n \int\int \cos(\sqrt{\lambda_n}t)\theta(t)dt = L\theta(0)+$$

$$+\frac{1}{2}\sum_{i\in\mathscr{C}} \epsilon_{-ii} \int_i (\theta(2d(x,T(i)) + \theta(-2d(x,I(-i)))))dx.$$

The assumption on the support of θ implies that

$$\int_i \theta(2d(x,T(i))) + \theta(-2d(x,I(-i)))dx = \int_{\mathbb{R}} \theta(2x)dx.$$

Next, using the fact that $\epsilon_{-jj} = \frac{2}{m(s)} - 1$, we obtain

$$\sum_{i\in\mathscr{E}} \epsilon_{-ii} = \sum_{j\in\mathscr{E}} \frac{2}{m(I(j))} - 1 =$$

$$\sum_{s\in\mathscr{S}} \sum_{j;I(j)=s} \frac{2}{m(s)} - 1 = \sum_{s\in\mathscr{S}} (2 - m(s)) = 2(n - N).$$

We recall that n (resp. N) is the number of vertices (resp. of edges).

Summing up, we have proved that

$$\sum_n \int \cos(\sqrt{\lambda_n} t)\theta(t)dt = L\theta(0) + \frac{n-N}{2} \int_{\mathbb{R}} \theta(x)dx.$$

Applying the above equality to the function $e^{-i\lambda t}\theta(t)$ instead of $\theta(t)$, we obtain

LEMMA 4.1. — *Let $\theta \in C_0^\infty(]-\epsilon,\epsilon[)$ with ϵ small enough. We have*

$$\frac{1}{2}\sum_n \widehat{\theta}(\lambda - \sqrt{\lambda_n}) + \widehat{\theta}(\lambda + \sqrt{\lambda_n}) = \theta(0)L + \frac{n-N}{2}\widehat{\theta}(\lambda),$$

for all $\lambda \in \mathbb{R}$.

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