Bulletin
de la SOCIÉTÉ MATHÉMATIQUE DE FRANCE

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Tome 139
Fascicule 1

2011

SOCIÉTÉ MATHÉMATIQUE DE FRANCE
Publié avec le concours du Centre national de la recherche scientifique
pages 129-144

Bull. Soc. math. France
139 (1), 2011, p. 129–144

WHY JORDAN ALGEBRAS ARE NATURAL IN STATISTICS: QUADRATIC REGRESSION IMPLIES WISHART DISTRIBUTIONS

by G. Letac & J. Wesołowski

Abstract. — If the space Q of quadratic forms in $\mathbb{R}^n$ is splitted in a direct sum $Q_1 \oplus \cdots \oplus Q_k$ and if X and Y are independent random variables of $\mathbb{R}^n$, assume that there exist a real number a such that $E(X|X+Y) = a(X+Y)$ and real distinct numbers $b_1, ..., b_k$ such that $E(q(X)|X+Y) = b_i q(X+Y)$ for any q in Q_i. We prove that this happens only when $k=2$, when $\mathbb{R}^n$ can be structured in a Euclidean Jordan algebra and when X and Y have Wishart distributions corresponding to this structure.

Résumé (*Pourquoi les algèbres de Jordan sont-elles naturelles en statistiques? La régression quadratique implique la distribution de Wishart*)

Si l'espace Q des formes quadratiques sur $\mathbb{R}^n$ est décomposé en une somme directe $Q_1 \oplus \cdots \oplus Q_k$ et si X et Y sont des variables aléatoires indépendantes de $\mathbb{R}^n$, supposons qu'il existe un nombre réel a tel que $E(X|X+Y) = a(X+Y)$ ainsi que des nombres réels distincts $b_1, ..., b_k$ tels que $E(q(X)|X+Y) = b_i q(X+Y)$ pour tout q de Q_i. Nous montrons que cela n'arrive que pour $k=2$, que lorsque $\mathbb{R}^n$ peut être structuré en algèbre de Jordan euclidienne et que lorsque X et Y suivent des lois de Wishart correspondant à cette structure.

Texte reçu le 2 juin 2009, accepté le 6 novembre 2009

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2000 Mathematics Subject Classification. — 60H10, 62H05.

Key words and phrases. — Symmetric cones, random matrices, characterization of Wishart laws.

I. Introduction

Let S_r be the set of (r,r) real symmetric matrices and let X and Y be independent random variables valued in S_r such that they are Wishart distributed $\gamma_{p,\sigma}$ and $\gamma_{p',\sigma}$, which means that

$$\mathbb{E}(e^{-\mathrm{tr}\,\theta X}) = \det(I_r+\theta\sigma)^{-p} \tag{1.1}$$

where θ and σ are in the set P_r of the positive definite elements of S_r and p is in

$$\Lambda = \{\frac{1}{2},\ldots,\frac{r-1}{2}\}\cup(\frac{r-1}{2},\infty) \tag{1.2}$$

(In (1.1) tr means trace). Note that for $a = p/(p+p')$

$$\mathbb{E}(X|X+Y) = a(X+Y). \tag{1.3}$$

Assume furthermore that $p+p' > \frac{r-1}{2}$. This implies that $(X+Y)^{-1}$ exists. Then it is known that $Z = (X+Y)^{-1/2}X(X+Y)^{-1/2}$ and $X+Y$ are independent and that $Z \sim uZu^T$ for any orthogonal (r,r) matrix u. There are many consequences, nuances and characterizations of the Wishart distributions related to this result. One of these consequences is the following fact: for any $s\in S_r$ consider the two quadratic forms on S_r defined by

$$q_1^s(x) = \frac{1}{2}\mathrm{tr}^2(xs)+\mathrm{tr}\,(sxsx),\;\; q_2^s(x) = \mathrm{tr}^2(xs)-\mathrm{tr}\,(sxsx) \tag{1.4}$$

and the two numbers

$$b_1 = \frac{p}{p+p'}\;\frac{p+1}{p+p'+1},\;\; b_2 = \frac{p}{p+p'}\;\frac{p-\frac{1}{2}}{p+p'-\frac{1}{2}}.$$

Then for $i=1,2$ and for any s

$$\mathbb{E}(q_i^s(X)|X+Y) = b_i q_i^s(X+Y) \tag{1.5}$$

This is the particular case $d=1$ of Corollary 2.3 of Letac and Massam (1998). An important fact about this set $(q_1^s,q_2^s)_{s\in S_r}$ is that it spans the whole space of quadratic forms $\mathcal{Q}$ on S_r (since if $q^s(x)=\mathrm{tr}^2(xs)$ then $\{q^s\;;\;s\in S_r\}$ spans $\mathcal{Q}$). More specifically denote by $\mathcal{Q}_i$ the subspace of $\mathcal{Q}$ generated by $\{q_i^s\;;\;s\in S_r\}$. Then $\mathcal{Q}=\mathcal{Q}_1\oplus\mathcal{Q}_2$ (see for instance Theorem 5.2 below for a proof).

The aim of the paper is to prove a reciprocal statement of (1.3) and (1.5): Let V be a linear real finite dimensional space (instead of S_r) and denote by $\mathcal{Q}$ the space of all quadratic forms on V. Fix a decomposition $\mathcal{Q}=\mathcal{Q}_1\oplus\mathcal{Q}_2\oplus\cdots\oplus\mathcal{Q}_k$ with $k\geq 2$ as a direct sum of linear subspaces. Consider two independent random variables X and Y with exponential moments satisfying (1.3) for some a and $\mathbb{E}(q(X)|X+Y) = b_i q(X+Y)$ for all $q\in\mathcal{Q}_i$ and for some distinct real numbers $b_1,\ldots,b_k$. We show that under these circumstances, necessarily $k=2$ and X and Y are Wishart distributed in the following sense: there necessarily exists a structure of Euclidean Jordan algebra on V (like symmetric matrices, Hermitian matrices, or space with a Lorentz cone) such that X and Y are Wishart on the symmetric cone associated to it. Section 5 contains more detailed information about the two spaces Q_1 and Q_2 of quadratic forms on S_r (or more generally, on a Euclidean Jordan algebra)

II. Some history of the subject

WISHART DISTRIBUTIONS ON S_r. Wishart distributions have been introduced by J. Wishart (1928) as distributions of $Z_1Z_1^T + \cdots + Z_NZ_N^T \sim \gamma_{N/2,2\Sigma}$ where $Z_1, \ldots, Z_N$ are iid in $\mathbb{R}^r$ such that $Z_i \sim N(0, \Sigma)$. Elegant calculations about them are in Bartlett (1933) and the classical reference is Muirhead (1982). For the space S_r of (r, r) real symmetric matrices the extension of the definition of $\gamma_{p,\sigma}$ from a half integer p to the whole set Λ defined by (1.2) is made in the fundamental paper of Olkin and Rubin (1962). Proving that a distribution $\gamma_{p,\sigma}$ on the semi positive definite matrices such that (1.1) holds only if p is in Λ was considered as a challenge by statisticians (see Eaton (1983)) although the appendix of Olkin and Rubin contains an unnoticed proof of it (and unfortunately erroneous: see Casalis and Letac (1994)). This conjecture was independently proved by Shanbhag (1988) and Peddada and Richards (1989) by quite different means, although a solution already appeared in Gyndikin (1975) and seems to have been well known by analysts, who also call the set Λ and its extensions the Wallach set (see Lassalle (1987) for proofs and references).

Lukacs-Olkin-Rubin Theorem. — Wishart distributions on S_r are the most natural generalization of the gamma distributions on the positive line. Lukacs (1956) shows that if X and Y are positive, independent non Dirac random variables and if $Z = X/(X + Y)$, then Z and $X + Y$ are independent if and only if there exists $\sigma, p, p' > 0$ such that $X \sim \gamma_{p,\sigma}$ and $Y \sim \gamma_{p',\sigma}$. This was extended to S_r by Olkin and Rubin (1962) by a proper definition of Z such that Z is symmetric (for instance by choosing $Z = (X + Y)^{-1/2}X(X + Y)^{-1/2}$ or by choosing $Z = C^{-1}X(C^{-1})^T$ where C is the triangular matrix with positive diagonal elements coming from the Cholesky decomposition $CC^T = X + Y$). They show that if X and Y are independent non Dirac random semi positive definite matrices in S_r such that $X+Y$ is invertible and such that $Z \sim uZu^T$ for any orthogonal (r, r) matrix u then Z and $X + Y$ are independent if and only if there exists a positive definite matrix σ and p and p' in Λ with $p+p' > (r-1)/2$ such that $X \sim \gamma_{p,\sigma}$ and $Y \sim \gamma_{p',\sigma}$. If Z is defined as $(X+Y)^{-1/2}X(X+Y)^{-1/2}$, Bobecka and Wesołowski (2002) have shown that the invariance hypothesis for Z by the orthogonal group can be dropped provided one assumes that X and Y have smooth densities. Removing this assumption of density is still a challenge.

Wishart distributions on Hermitian matrices and on Euclidean Jordan algebras. — Since normal distributions on Hermitian spaces have been considered (see e.g., Goodman (1963)), therefore Wishart distributions on Hermitian matrices occur naturally. Actually physicists considered them quite early (see Mehta (2004)). Carter (1975) in an unpublished PhD thesis extends Olkin and Rubin to this case.

On the other hand, works on the classification of natural exponential families by their variance function have led to the observation that the exponential family $\{\gamma_{p,\sigma}; \sigma \in P_r\}$ of Wishart distributions on S_r with fixed shape parameter $p \in \Lambda$ has a variance function which is the map from S_r into itself $x \mapsto V(m)(x) = \frac{1}{p} mxm$ where m is in P_r. In other terms, this means that if κ is a cumulant function of $\gamma_{p,\sigma}$ then for all x in S_r we have

$$\kappa''(\theta)(x) = \frac{1}{p}\kappa'(\theta)x\kappa'(\theta).$$

Facts about multivariate distributions such that their corresponding variance functions are quadratic in the mean are collected in Letac (1989). In particular, Wishart distributions obtained from simple Euclidean Jordan algebras are described there. An indispensable reference for simple Euclidean Jordan algebras is Faraut and Koranyi (1994) always abreviated F.-K. below. Recall that simple Euclidean Jordan algebras are basically in one to one correspondence with the irreducible symmetric cones (self dual cones in Euclidean space such that the group of automorphisms of the cone acts transitively on it), in the way that S_r is linked to P_r. A quick definition of the Wishart distribution $\gamma_{p,\sigma}$ on the Jordan algebra V with rank r, Peirce constant d, cone $\overline{\Omega}$ of square elements, trace and determinant function tr and det can be done by its Laplace transform

$$\int_{\overline{\Omega}} e^{-\operatorname{tr}\theta x}\gamma_{p,\sigma}(dx) = \det(e+\theta\sigma)^{-p}$$

where σ is in the interior Ω of $\overline{\Omega}$ and where p is in the Gyndikin set of the Jordan algebra V defined by

$$\Lambda_V = \{\frac{d}{2}, d, \ldots, \frac{d}{2}(r-1)\} \cup (\frac{d}{2}(r-1), \infty). \tag{2.6}$$

While the definition of determinant is the standard one for S_r and for Hermitian matrices, it requires some care for the three other types of Jordan algebras: quaternionic Hermitian matrices, 27 dimensional Albert algebra and the algebra of the Lorentz cone.

Particular cases of use of Wishart distributions on Jordan algebras in statistics occurred earlier (Andersson (1975) for the Hermitian and quaternionic cases, and Jensen (1988) for the Lorentz cone, with its deep connexions to Clifford algebras). Jordan algebras are the natural framework for Wishart distributions: Casalis and Letac (1996) is a clarification and an extension to Jordan algebras of Olkin and Rubin (1962) and of Carter (1975); Carter follows step by step the difficult Olkin and Rubin's approach and his work was unknown to Casalis and Letac (1996).

Quadratic homogeneity and Wishart distributions. — A remarkable fact about the classical Wishart distributions on S_r is that the above variance function $m \mapsto V(m)$ is not only quadratic in m but homogeneous quadratic. This happens also to be true for Wishart distributions on any Euclidean Jordan algebra. This observation lead Casalis (1991) to prove the converse: any natural exponential family with a homogeneous quadratic variance function is a Wishart family, as conjectured in Letac (1989). Put in other words, if κ is a cumulant function of some random variable X valued in $\mathbb{R}^n$ such that $\kappa''(\theta) = V(\kappa'(\theta))$ where V is a homogeneous quadratic function, then $\mathbb{R}^n$ can be structured in a Jordan algebra such that X is Wishart for that structure.

Quadratic regression property. — A slight extension of Lukacs (1956) is to take two non Dirac independent rv X and Y on the positive line such that there exist positive a and b such that $\mathbb{E}(X|X+Y) = a(X+Y)$ and $\mathbb{E}(X^2|X+Y) = b(X+Y)^2$ and to prove that there exist positive p, p', σ such that $X \sim \gamma_{p,\sigma}$ and $Y \sim \gamma_{p',\sigma}$. To see this, just multiply these two equalities by $e^{\theta(X+Y)}$, take expectations and obtain two differential equations for the Laplace transforms of X and Y. This procedure is contained in Laha and Lukacs (1960). Bivariate regression version of Lukacs theorem based on conditions $E(X_i^2|X+Y) = b(X_i+Y_i)^2$, $i = 1, 2$, where $X = (X_1, X_2)$ and $Y = (Y_1, Y_2)$ are independent was obtained in Wang (1981). This result was generalized in Letac and Wesolowski (2008) by considering regressions of quadratic forms $E(q(X)|X+Y) = bq(X+Y)$ for all quadratic forms q orthogonal to an arbitrary fixed quadratic form q_0. That is in the setting of the present paper we required $k = 1$ and codimension of $\mathcal{Q}_1$ to be equal 1.

Letac and Massam (1998) use the quadratic regression approach to get a simpler proof of Olkin and Rubin theorem, as extended to Jordan algebras in Casalis and Letac (1996). It actually characterizes the Wishart distributions of independent X and Y in S_r (and more generally of a Jordan algebra) through the following properties: if for $i = 1, 2$, $s \in S_r$ and q_i^s are defined by (1.4), then (1.5) holds (with suitable analogues of q_i if the Jordan algebra is not S_r). Note that this regression perspective leads to a characterization of $\gamma_{p,\sigma}$, $\gamma_{p',\sigma}$ without the hypothesis of invertibility of $X+Y$ which was needed in the Olkin and Rubin characterization.

III. Main result

Let V be a real linear space with dimension $n > 1$, let V^* be its dual and consider the space $\mathcal{F} = L_s(V, V^*)$ of the symmetric linear maps from V to V^*. If $\theta \in V^*$ and $x \in V$ we write $\langle \theta, x\rangle$ for $\theta(x)$. Denote by $\mathcal{Q}$ the space of quadratic forms q on V, namely the set of real functions q on V such that $(x, y) \mapsto \frac{1}{2}(q(x + y) - q(x) - q(y))$ is bilinear on $V \times V$ and $q(\lambda x) = \lambda^2 q(x)$ when λ is a real number. The map from $\mathcal{F}$ to $\mathcal{Q}$ defined by $f \mapsto q_f$ where $x \mapsto q_f(x) = \langle f(x), x\rangle$ is one to one. More specifically:

$$\frac{1}{2}(q_f(x + y) - q_f(x) - q_f(y)) = \frac{1}{2}(\langle f(x), y\rangle + \langle f(y), x\rangle) = \langle f(x), y\rangle$$

For $q \in \mathcal{Q}$ we therefore define the inverse map $q \mapsto f_q$ of $f \mapsto q_f$ by

$$\frac{1}{2}(q(x + y) - q(x) - q(y)) = \langle f_q(x), y\rangle.$$

Let us also define here the concept of irreducibility for a probability measure μ on V. We say first that μ is reducible if there exists a direct sum $V_1 \oplus V_2 = V$ with $\dim V_i > 0$ for $i = 1, 2$, two probability measures μ_1 and μ_2 on V_1 and V_2 such that $\mu = \mu_1 \otimes \mu_2$. In other terms, if $X \sim \mu$ its projections X_1 on V_1 parallel to V_2 and X_2 on V_2 parallel to V_1 are independent. Suppose that furthermore X has a Laplace transform $L = e^{\kappa}$ defined on some open set $\Theta \subset V^* = V_1^* \oplus V_2^*$. In this case $\kappa(\theta) = \kappa_1(\theta_1) + \kappa_2(\theta_2)$ where θ_i is the projection of θ on V_i^* and κ_1 and κ_2 are the cumulant functions of X_1 and X_2. We say also that X and κ are reducible in that case. Finally, μ, X and κ are said to be irreducible if they are not reducible...

THEOREM III.1. — *Let $\mathcal{Q}_1 \oplus \mathcal{Q}_2 \oplus \cdots \oplus \mathcal{Q}_k = \mathcal{Q}$ be a direct sum decomposition of the space of quadratic forms on V with $k \geq 2$. Let X and Y be two independent irreducible random variables valued in V such that their Laplace transforms exist on an open set $\Theta \subset V^*$. We assume that*

1. *there exists a real number a such that $\mathbb{E}(X|X + Y) = a(X + Y)$;*
2. *there exist distinct numbers $b_1, \ldots, b_k$ such that for any $i = 1, \ldots, k$ and for any $q \in \mathcal{Q}_i$ we have*

$$\mathbb{E}(q(X)|X + Y) = b_i q(X + Y). \tag{3.7}$$

Under these circumstances $0 < a < 1$, $k = 2$ and there exists a simple Euclidean Jordan algebra structure on V such that X and Y are Wishart distributed on the positive cone of the algebra with the same scale parameter and respective shape parameters p and p' in Λ_V defined in (2.6). Moreover $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are spanned by

$$q_1^s(x) = \frac{d}{2} tr^2(xs) + tr(\mathbb{P}(x)(s)s), \quad q_2^s(x) = tr^2(xs) - tr(\mathbb{P}(x)(s)s) \tag{3.8}$$

where tr, $\mathbb{P}$ and d are respectively the trace, the quadratic map and the Peirce constant of the Jordan algebra and $s \in V$. In this case

$$a = \frac{p}{p+p'},\ b_1 = \frac{p}{p+p'}\ \frac{p+1}{p+p'+1},\ b_2 = \frac{p}{p+p'}\ \frac{p-\frac{d}{2}}{p+p'-\frac{d}{2}}. \tag{3.9}$$

Proof. — Denote by L_X and L_Y the Laplace transforms of X and Y. It is standard to prove that from condition 1) we have $L_X^{1-a} = L_Y^a$: just multiply both sides of $\mathbb{E}(X|X+Y) = a(X+Y)$ by $e^{\langle\theta, X+Y\rangle}$ where $\theta \in \Theta$ and take expectations of both sides to obtain the differential equation $(1-a)L'_X/L_X = aL'_Y/L_Y$. The fact that X and Y are irreducible implies that $a=0$ or $a=1$ is impossible. The fact that $\log L_X$ and $\log L_Y$ are convex implies that $a<0$ or $a>1$ are impossible. From now on we denote $e^\kappa = L_X = L_Y^{a/(1-a)}$.

In the sequel, we use the symbol Tr for the trace of an endomorphism. The symbol tr is reserved for the trace in a Jordan algebra. If q is a quadratic form on V we write

$$q(\frac{\partial}{\partial\theta})(\kappa)(\theta) = \mathrm{Tr}\,(f_q\kappa''(\theta)).$$

Since κ is a real twice differentiable function defined on an open subset of V^*, the second derivative $\kappa''(\theta)$ is an element of $L_s(V^*, V)$, the linear map f_q is an element of $L_s(V, V^*)$ and thus $f_q\kappa''(\theta)$ belongs to $L(V^*, V^*)$. It therefore makes sense to speak of the trace of this endomorphism of V^*. Note that $\langle f_q(x), x\rangle = \mathrm{Tr}\,(f_q(x\otimes x))$ and that $\frac{\partial}{\partial\theta}\otimes\frac{\partial}{\partial\theta}\kappa = \kappa''$. This explains the definition $q(\frac{\partial}{\partial\theta})(\kappa) = \mathrm{Tr}\,(f_q\kappa'')$. Also $q(\kappa')$ can be written in terms of f_q as $q(\kappa') = \mathrm{Tr}\,(f_q(\kappa'\otimes\kappa')) = \langle f_q(\kappa'), \kappa'\rangle$.

Calculations done in Letac and Wesołowski (2008) (2.9), show that for any $i = 1, \dots k$ and for all $q \in \mathcal{Q}_i$ we have

$$(1-\frac{b_i}{a})q(\frac{\partial}{\partial\theta})(\kappa) = (\frac{b_i}{a^2}-1)q(\kappa'). \tag{3.10}$$

(Again, to prove (3.10) just multiply (3.7) by $e^{\langle\theta, X+Y\rangle}$ and take expectations). Observe that $b_i = a$ is impossible, since it implies that $q(\kappa') = 0$ for any q in $\mathcal{Q}_i$. Since $\mathcal{Q}_i$ is not the zero space, there exists a non zero q with $q(\kappa') = 0$. Now $\{x \in V; q(x) = 0\}$ is a quadric of V and has an empty interior. On the other hand, since X is irreducible, this implies that X cannot be concentrated on some affine subspace of V. Therefore κ is strictly convex and the set $\kappa'(\Theta)$ is open and cannot be contained in a quadric. Thus $a = b_i$ is impossible, division by $(1-\frac{b_i}{a})$ is permitted and we rewrite (3.10) as

$$q(\frac{\partial}{\partial\theta})(\kappa) = p_i q(\kappa') \tag{3.11}$$

where $p_i = \frac{b_i - a^2}{a^2 - ab_i}$.

Now let us fix $\theta \in V^*$ and consider the element $\theta \otimes \theta$ of $\mathcal{F}$ defined by $(\theta\otimes\theta)(x) = \langle\theta, x\rangle\theta$. Denote by $\mathcal{F}_i$ the image of Q_i by the isomorphism $q \mapsto f_q$. Obviously we have

$$\mathcal{F}_1 \oplus \mathcal{F}_2 \oplus \cdots \oplus \mathcal{F}_k = \mathcal{F}.$$

Therefore there exist elements $f_i \in \mathcal{F}_i$ such that $f_1 + \cdots + f_k = \theta \otimes \theta$. Since $f_1, \ldots, f_k$ depend actually on θ we rather write $f_i(\theta, x)$ instead of $f_i(x)$ for $x \in V$. Thus $x \mapsto f_i(\theta, x)$ is a linear map from V to V^*. We rewrite the equality $\theta \otimes \theta = f_1 + \cdots + f_k$ as

$$\langle\theta, x\rangle^2 = \langle f_1(\theta, x), x\rangle + \cdots + \langle f_k(\theta, x), x\rangle$$

for any x in V. We now fix $\theta = \theta_0$ in this equality and we recall that $q(\frac{\partial}{\partial\theta})(\kappa)(\theta)$ means $\mathrm{Tr}\,(f_q\kappa''(\theta))$. Thus we get

$$\mathrm{Tr}\,((\theta_0 \otimes \theta_0)\ \kappa''(\theta)) = \sum_{i=1}^{k} \mathrm{Tr}\,[f_i(\theta_0, \cdot)\kappa''(\theta)].$$

We now use the fact that $x \mapsto \langle f_i(\theta_0, x), x\rangle = q(x)$ is a quadratic form belonging to Q_i to which we apply (3.11). Therefore we obtain

$$\mathrm{Tr}\,((\theta_0 \otimes \theta_0)\ \kappa''(\theta)) = \sum_{i=1}^{k} p_i \mathrm{Tr}\,(f_i(\theta_0, \cdot)\kappa'(\theta) \otimes \kappa'(\theta)). \tag{3.12}$$

Since this is true for any θ_0 in V^* this is enough to claim that κ'' is a quadratic homogeneous function of κ'.

We now apply the Casalis' theorem (1991) which says that if κ is irreducible and if κ'' is a quadratic homogeneous function of κ', then there exists a simple Euclidean Jordan algebra structure on V related to X in a way that we explain now. Let Ω be the open cone of the squares of V, let tr and det be the trace and determinant functions on the Jordan algebra, let d and r be the Peirce and rank constants of V. Then there exists $p \in \Lambda_V$ defined by (2.6) and $\sigma \in \Omega$ such X has the Wishart distribution $\gamma_{p,\sigma}$ on $\overline{\Omega}$ defined by its Laplace transform $\mathbb{E}(e^{-\mathrm{tr}\,\theta X}) = \det(I_r + \theta\sigma)^{-p}$ for all $\theta \in \Omega$.

To complete the proof, denote for a while by $\widetilde{Q}_1$ and $\widetilde{Q}_2$ the spaces of quadratic forms spanned by $(q_1^s)_{s\in V}$ and $(q_2^s)_{s\in V}$ as defined in (3.8). Denote also

$$\widetilde{b}_1 = \frac{p}{p+p'}\ \frac{p+1}{p+p'+1},\ \widetilde{b}_2 = \frac{p}{p+p'}\ \frac{p-\frac{d}{2}}{p+p'-\frac{d}{2}}.$$

Recall that we want to prove that $k = 2$ and that $\{\widetilde{Q}_1, \widetilde{Q}_2\} = \{Q_1, Q_2\}$. Let now $q \in Q_i$. Therefore $\mathbb{E}(q(X)|X+Y) = b_i q(X+Y)$. We now write $q = q_1 + q_2$ with $q_i \in \widetilde{Q}_i$ which is possible since $\widetilde{Q}_1 \oplus \widetilde{Q}_2 = Q$. Recall that since X and Y

have distributions $\gamma_{p,\sigma}$ and $\gamma_{p',\sigma}$ we can write $\mathbb{E}(q_i(X)|X+Y) = \widetilde{b}_i q_i(X+Y)$. Thus

$$(\widetilde{b}_1 - b_i)q_1(X+Y) = (b_i - \widetilde{b}_2)q_2(X+Y).$$

Since $X+Y$ is valued in the open set Ω this implies $(\widetilde{b}_1 - b_i)q_1 = (b_i - \widetilde{b}_2)q_2$. Thus the two sides of this equality are zero: either $b_i = \widetilde{b}_1$ and $q_2 = 0$ or the reverse statement holds. Since we have assumed that $b_1, \dots, b_k$ are distinct, this ends the proof. □

IV. Comments

1. Surprisingly enough, while starting from a linear space V without any additional algebraic structure, the regression conditions on X and Y of the theorem impose by themselves a Euclidean Jordan algebra structure on V.
2. The three numbers a, b_1 and b_2 together with the dimension of V determine uniquely the structure of Jordan algebra on V in the following sense: we can see from the equations (3.9) that $b_2 < a^2 < b_1 < a$. Moreover these equations give the Peirce constant d of V by

$$d = 2\frac{a-b_1}{b_1-a^2}\,\frac{a^2-b_2}{a-b_2}.$$

   Since the rank r satisfies $\dim V = r + \frac{d}{2}r(r-1)$ the type of the Jordan algebra is completely known.
3. In the theorem, $k=1$ would lead to X and Y concentrated on a line $\mathbb{R}v$ of V. If $X = X_1 v$ and $Y = Y_1 v$ then X_1 and Y_1 would be one dimensional gamma distributed and X would not be irreducible since we have assumed $\dim V > 1$. Furthermore if in the theorem we do not assume that $b_1, \dots, b_k$ are distinct, then either they are all equal to one b and this sends us back to the trivial case $k=1$ or they are not and if $k' \geq 2$ is the number of distinct b_i's, then the theorem gives $k' = 2$.
4. Some comments about irreducibility are in order. If L_Y is a power of L_X, then Y is irreducible if and only if X is. Therefore irreducibility can be assumed in the theorem for X only. If irreducibility is not assumed, we have an artificial generality. For instance suppose that $(X_1, X_2, X_3, Y_1, Y_2, Y_3)$ are independent real rv with $X_i \sim \gamma_{\alpha_i,\sigma}$ and $Y_i \sim \gamma_{\beta_i,\sigma}$. Then for $i \neq j$ we have

$$\mathbb{E}(X_i X_j|X+Y) = \frac{\alpha_i\alpha_j}{(\alpha_i+\beta_i)(\alpha_j+\beta_j)}(X_i+Y_i)(X_j+Y_j),$$
$$\mathbb{E}(X_i^2|X+Y) = \frac{\alpha_i(\alpha_i+1)}{(\alpha_i+\beta_i)(\alpha_i+\beta_i+1)}(X_i+Y_i)^2.$$

This implies that $k = 6$ corresponding to six independent quadratic forms on $V = \mathbb{R}^3$ defined by $q_{ij}(x) = x_i x_j$ for $i \leq j$.

V. The spaces $\mathcal{Q}_1$ and $\mathcal{Q}_2$: the operator Ψ

If V is a simple Euclidean Jordan algebra with rank r and Peirce constant $d = 2d'$, denote by $\mathcal{F} = L_s(V)$ the space of symmetric linear operators on V. The dimension of V is $n = r + dr(r-1)/2$. Given $y \in V$, important examples of elements of $\mathcal{F}$ are respectively $\mathbb{L}(y)$ defined by $x \mapsto yx$ where yx is the Jordan product, and

$$\mathbb{P}(y) = 2(\mathbb{L}(y))^2 - \mathbb{L}(y^2)$$

as defined in F.-K. page 32. If a and b are in V we denote by $a \otimes b$ the endomorphism $x \mapsto a\,\mathrm{tr}\,(bx)$ of V. The endomorphism $a \otimes b + b \otimes a$ belongs to $\mathcal{F}$. Denote by $\mathcal{F}_1$ and $\mathcal{F}_2$ the linear subspaces of $\mathcal{F}$ respectively generated by and $\{d'y \otimes y + \mathbb{P}(y); y \in V\}$ and $\{y \otimes y - \mathbb{P}(y); y \in V\}$. From (3.8) $\mathcal{F}_1$ and $\mathcal{F}_2$ are canonically isomorphic to $\mathcal{Q}_1$ and $\mathcal{Q}_2$ by $q \mapsto f_q$ where $q(x) = \langle f_q(x), x\rangle$. We endow $\mathcal{F}$ with the Euclidean structure defined by $\mathrm{Tr}\,(ab)$. Here again we distinguish the trace tr of the Jordan algebra V from the trace Tr of the endomorphisms on the linear space V. Here is a list of various traces:

PROPOSITION V.1. — 1. $Tr\,(a \otimes b) = tr\,(ab)$, $Tr\,[(a \otimes b)(c \otimes d)] = tr\,(ad)tr\,(bc)$,

$$Tr\,((a_1 \otimes b_1)\cdots(a_k \otimes b_k)) = tr\,(a_1 b_k)tr\,(a_2 b_1)\cdots tr\,(a_k b_{k-1}).$$

2. $Tr\,[\mathbb{L}(a)\mathbb{L}(b)(c \otimes d)] = tr\,[(a(bc))d]$
3. $Tr\,(\mathbb{P}(a)(b \otimes c)) = tr\,[(\mathbb{P}(a)b)c]$

Proof. — (1) is standard since it only involves the Euclidean structure of V and not its Jordan algebra structure. (2) is a consequence of (1). Applying the definition of $\mathbb{P}(a)$, (3) is a consequence of (2).

In the theorem below, we consider an endomorphism Ψ of $\mathcal{F}$ such that $\Psi(y \otimes y) = \mathbb{P}(y)$ for all $y \in V$. It is an essential tool of the two papers Casalis and Letac (1996) and Letac and Massam (1998). The theorem shows that $\mathcal{F}_1$ and $\mathcal{F}_2$ are its two eigenspaces and uses this fact to give the dimensions of the spaces of quadratic forms $\mathcal{Q}_1$ and $\mathcal{Q}_2$ defined in Th. 3.1 above. □

THEOREM V.2. — 1. *There exists a symmetric endomorphism Ψ of $\mathcal{F}$ such that $\Psi(y \otimes y) = \mathbb{P}(y)$ for all $y \in V$. It satisfies*

$$\Psi(\mathbb{P}(y)) = d'y \otimes y + (1 - d')\mathbb{P}(y) \tag{5.13}$$

2. *The spaces $\mathcal{F}_1$ and $\mathcal{F}_2$ are orthogonal and $\mathcal{F} = \mathcal{F}_1 \oplus \mathcal{F}_2$*
3. *The spaces $\mathcal{F}_1$ and $\mathcal{F}_2$ are the two eigenspaces of Ψ corresponding to the two eigenvalues 1 and $-d'$ respectively.*

4. *The dimensions of* $\mathscr{F}_1$ *and* $\mathscr{F}_2$ *are given by*

$$\frac{n(n+1)}{2} - \dim \mathscr{F}_1 = \dim \mathscr{F}_2 = \frac{r(r-1)}{2} \times \frac{1 + d'(2r-3) + d'^2(r-1)(r-2)}{1+d'}$$

Examples. — For the Jordan algebra associated to the Lorentz cone where $r = 2$ we get $\dim \mathscr{F}_2 = 1$. More specifically, if E is a Euclidean space with scalar product $\vec{x}.\vec{y}$ consider the Jordan algebra $V = \mathbb{R} \times E$ endowed with the Jordan product between $x = (x_0, \vec{x})$ and $y = (y_0, \vec{y})$ defined by

$$xy = (x_0 y_0 + \vec{x}.\vec{y}\,,\ x_0\vec{y} + y_0\vec{x}).$$

Here the Lorentz cone is $\{(x_0, \vec{x}) \in V\ ;\ x_0 > \|\vec{x}\|\}$, the trace is $\mathrm{tr}\,(x_0, \vec{x}) = 2x_0$ and the Peirce constant is $d = \dim E - 1$. In this case $\mathscr{F}_2$ is spanned by the symmetry S defined by $(x_0, \vec{x}) \mapsto (x_0, -\vec{x})$. To see this observe that if $e = (1, \vec{0})$ then $S = e \otimes e - \mathbb{P}(e)$ is in $\mathscr{F}_2$ and use $\dim \mathscr{F}_2 = 1$. As a consequence if $\begin{bmatrix} a & b \\ b^* & c \end{bmatrix}$ represents a symmetric endomorphism of V (where a is real, c is a symmetric endomorphism of E and b is a linear form on E) then $\begin{bmatrix} a & b \\ b^* & c \end{bmatrix}$ is in $\mathscr{F}_1$ if and only if it is orthogonal to

$$S = \begin{bmatrix} 1 & 0 \\ 0 & -\mathrm{id}_E \end{bmatrix},$$

that is if and only if $a = \mathrm{Tr}\, c$. The dimension of $\mathscr{F}_1$ is $\frac{1}{2}(n-1)(n+2)$.

For the Jordan algebra S_r of symmetric real matrices where $d = 1$, we get $\dim \mathscr{F}_2 = \frac{r^2}{12}(r-1)(r+1)$ and $\dim \mathscr{F}_1 = \frac{r}{24}(r+1)(r^2+5r+6)$. For the Jordan algebra of Hermitian matrices where $d = 2$, we get

$$\dim \mathscr{F}_1 = \left(\frac{r(r+1)}{2}\right)^2,\ \dim \mathscr{F}_2 = \left(\frac{r(r-1)}{2}\right)^2,$$

and since $d' = 1$, Ψ is an orthogonal symmetry with respect to $\mathscr{F}_2$. For the Jordan algebra of Hermitian quaternionic matrices where $d = 4$, we get $\dim \mathscr{F}_2 = 4r\frac{r(r-1)(r-2)}{6} + \frac{r(r-1)}{2}$ and $\dim \mathscr{F}_1 = \frac{r^2}{3}(4r^2 - 1)$. For the Albert algebra where $d = 8$ and $r = 3$ we get $\dim \mathscr{F}_2 = 27,\ \dim \mathscr{F}_1 = 351 = 27 \times 13$.

Proof. — (1) The existence of Ψ is proved in Casalis and Letac (1996) (Lemma 6.3) and (5.13) is proved in Letac and Massam (1998) (Proposition 3.1). For proving that Ψ is symmetric, enough is to see that $\mathrm{Tr}\,[\Psi(x \otimes x)(y \otimes y)]$ is symmetric in x and y in V since $\{y \otimes y\ ;\ y \in V\}$ spans $\mathscr{F}$. Equivalently we have to see that $\mathrm{Tr}\,[\mathbb{P}(x)(y \otimes y)]$ is symmetric. From Proposition 3.1 part 3, we have to show that $\mathrm{tr}\,[(\mathbb{P}(x)y)y)]$ is symmetric. Applying the definition of $\mathbb{P}$, we get

$$\mathrm{tr}\,[(\mathbb{P}(x)y)y)] = \mathrm{tr}\,[(2(x(xy) - x^2y)y].$$

Let us now use Proposition II.1.1, (iii) in F.-K. which says

$$\mathbb{L}(x^2y)-\mathbb{L}(x^2)\mathbb{L}(y)=2\mathbb{L}(xy)\mathbb{L}(x)-2\mathbb{L}(x)\mathbb{L}(y)\mathbb{L}(x).$$

Applying this equality to y we get $(x^2y)y-x^2y^2=2(xy)^2-2x(y(xy))$ that we rewrite as $2(xy)^2+x^2y^2=2x(y(xy))+(x^2y)y$. Since the left hand side is symmetric in (x,y) this proves $2x(y(xy))+(x^2y)y=2y(x(xy))+(y^2x)x$ which implies in turn that $(2(x(xy)-x^2y)y$ is symmetric in (x,y) and shows that Ψ is symmetric.

(2) and (3) Since $\{y\otimes y\ ;\ y\in V\}$ spans $\mathscr{F}$ and since

$$y\otimes y=\frac{1}{1+d'}(d'y\otimes y+\mathbb{P}(y))+\frac{1}{1+d'}(y\otimes y-\mathbb{P}(y)),$$

clearly $\mathscr{F}=\mathscr{F}_1+\mathscr{F}_2$. From the formula (5.13) and the definition of Ψ we get easily that $\mathscr{F}_1$ and $\mathscr{F}_2$ are made of eigenvectors of Ψ respectively for the eigenvalues 1 and $-d'$. In particular $\mathscr{F}_1\cap\mathscr{F}_2=\{0\}$. Therefore $\mathscr{F}=\mathscr{F}_1\oplus\mathscr{F}_2$ and thus the endomorphism Ψ has no other eigenvalues. From the fact that Ψ is symmetric, $\mathscr{F}_1$ and $\mathscr{F}_2$ are orthogonal.

(4) It is the difficult point. We have $\dim\mathscr{F}_1+\dim\mathscr{F}_2=\frac{n(n+1)}{2}$ where n is the dimension of V. An other linear equation for $(\dim\mathscr{F}_1,\ \dim\mathscr{F}_2)$ is $\operatorname{trace}(\Psi)=\dim\mathscr{F}_1-d'\dim\mathscr{F}_2$ leading to

$$\dim\mathscr{F}_2=\frac{1}{1+d'}\left(\frac{n(n+1)}{2}-\operatorname{trace}(\Psi)\right). \tag{5.14}$$

We embark for a calculation of trace(Ψ) by selecting an orthonormal basis $f=(f_\ell)_{\ell=1}^{n(n+1)/2}$ of $\mathscr{F}$ and by computing $\operatorname{Tr}[\Psi(f_\ell)f_\ell]$ in order to get

$$\operatorname{trace}(\Psi)=\sum_{\ell=1}^{n(n+1)/2}\operatorname{Tr}[\Psi(f_\ell)f_\ell].$$

The basis f is chosen as follows. We start from a Jordan frame $(c_1,\dots,c_r)$ of V (see F.-K. page 44). Recall that $c_s^2=c_s$ and $c_sc_t=0$ for $s\neq t$. We denote by $V(c,\lambda)$ the eigenspace of V of $\mathbb{L}(c)$ for the eigenvalue λ. For $1\leq s<t\leq r$ we denote

$$V_{st}=V(c_s,\frac{1}{2})\cap V(c_t,\frac{1}{2}),\ V_{ss}=V(c_s,1).$$

Recall that $V=\bigoplus_{1\leq s\leq t\leq r}V_{st}$, that the dimension of V_{st} is d for $s<t$ and 1 for $s=t$ and that these spaces are orthogonal (F.-K. Th. IV 2.1, (i)). Let $(c^1_{s,t},\dots,c^d_{s,t})$ be an orthonormal basis of the space $V_{s,t}$ for $s<t$. The space V_{ss} is spanned by c_s. For simplicity denote also by $e=(e_1,\dots,e_n)$ the orthonormal basis of V defined by the c_s's and the c^k_{st}'s. Finally the basis f of $\mathscr{F}$ consists of the elements of the form $f_\ell=e_i\otimes e_i$ for $i=1,\dots,n$, or $f_\ell=(e_i\otimes e_j+e_j\otimes e_i)/\sqrt{2}$ for $1\leq i<j\leq n$. Since e is an orthonormal basis of the Euclidean space V it

is standard to see that f is an orthonormal basis of the space $\mathscr{T}$ of symmetric endomorphisms of V.

We now compute $\mathrm{Tr}\,[\Psi(f_\ell)f_\ell] = C_\ell$ for all possible choices of f_ℓ in the basis f.

1. CASE A: $f_\ell = e_i \otimes e_i$. From Proposition 5.1, part 5 we have for all $x \in V$:

$$\mathrm{Tr}\,(\mathbb{P}(x)(x \otimes x)) = \mathrm{tr}\, x^4 \tag{5.15}$$

Case A1: $e_i = c_s$. Thus inserting $x = c_s$ in (5.15) we get $C_\ell = \mathrm{tr}\, c_s = 1$.

Case A2: $e_i = c^k_{st}$. We use the fact that $x^2 = \frac{\|x\|^2}{2}(c_s + c_t)$ when $x \in V_{st}$ (see F.-K. Proposition IV. 1.4 (i)) and apply (5.15) to $x = c^k_{st}$. We get

$$C_\ell = \mathrm{tr}\,(c^k_{st})^4 = \frac{1}{4}\mathrm{tr}\,[(c_t + c_s)^2] = \frac{1}{2}.$$

2. CASE B: $f_\ell = (e_i \otimes e_j + e_j \otimes e_i)/\sqrt{2}$. We use the following calculation:

$$\Psi(x \otimes y + y \otimes x) = \mathbb{P}(x+y) - \mathbb{P}(x) - \mathbb{P}(y) = 2[\mathbb{L}(x)\mathbb{L}(y) + \mathbb{L}(y)\mathbb{L}(x) - \mathbb{L}(xy)]$$

(F.-K. page 32) and, using Proposition 5.1 part 2:

$$\mathrm{Tr}\,[(\mathbb{L}(x)\mathbb{L}(y) + \mathbb{L}(y)\mathbb{L}(x) - \mathbb{L}(xy))(x \otimes y + y \otimes x)] = \mathrm{tr}\,[(yx^2)y + (xy^2)x].$$

Thus

$$C_\ell = \mathrm{tr}\,[(e_j e_i^2)e_j + (e_i e_j^2)e_i]. \tag{5.16}$$

Case B1: $e_i = c_s,\ e_j = c_t$ with $s < t$. From (5.16):

$$C_\ell = \mathrm{tr}\,[(c_t c_s^2)c_t + (c_s c_t^2)c_s] = 0.$$

Case B2: $e_i = c_s,\ e_j = c^k_{uv}$ with $1 \le u < v \le r$ with $s \in \{u, v\}$. By the definition of V_{uv} we have $c^k_{uv}c_s = \frac{1}{2}c^k_{uv}$ and thus from (5.16):

$$\begin{aligned} C_\ell &= \mathrm{tr}\,[(c^k_{uv}c_s^2)c^k_{uv} + (c_s(c^k_{uv})^2)c_s] = \mathrm{tr}\,[(c^k_{uv}c_s)c^k_{uv} + (c_s\frac{1}{2}(c_u + c_v))c_s] \\ &= \mathrm{tr}\,[\frac{1}{4}(c_u + c_v) + (c_s\frac{1}{2}(c_u + c_v))c_s] = 1. \end{aligned}$$

Case B3: $e_i = c_s,\ e_j = c^k_{uv}$ with $1 \le u < v \le r$ with $s \notin \{u, v\}$. Here we have $c^k_{uv}c_s = 0$ from F.-K. page 68 last line. A calculation similar to B2 gives $C_\ell = 0$.

Case B4: $e_i = c^k_{uv},\ e_j = c^m_{uv}$ with $1 \le u < v \le r$ and $1 \le k < m \le d$. Note that if x and y have norm 1 in V_{uv} then

$$(yx^2)y + (xy^2)x = (y\frac{1}{2}(c_u + c_v))y + (x\frac{1}{2}(c_u + c_v))x = \frac{1}{2}(y^2 + x^2) = \frac{1}{2}(c_u + c_v)$$

Applying this to $x = c^k_{uv}$ and $y = c^m_{uv}$ we get $C_\ell = 1$ through (5.16).

Case B5: $e_i = c^k_{st}$, $e_j = c^m_{uv}$ with $1 \leq s < t \leq r$, with $1 \leq u < v \leq r$, with $1 \leq k, m \leq d$ and with $\{s,t\} \cap \{u,v\}$ reduced to one point, say $u = s$. Note that if x and y have norm 1 in $x \in V_{st}$ and $y \in V_{sv}$ then

$$(yx^2)y+(xy^2)x = (y\frac{1}{2}(c_s+c_t))y+(x\frac{1}{2}(c_s+c_v))x = \frac{1}{4}(y^2+x^2) = \frac{1}{8}(2c_s+c_t+c_v)$$

Applying this to $x = c^k_{st}$ and $y = c^m_{sv}$ we get $C_\ell = 1/2$ through (5.16).

Case B6: $e_i = c^k_{st}$, $e_j = c^m_{uv}$ with $1 \leq s < t \leq r$ and with $1 \leq u < v \leq r$ with $\{s,t\} \cap \{u,v\} = \varnothing$. Using F.-K. page 68 last line we see that $(yx^2)y + (xy^2)x = 0$ when $x \in V_{st}$ and $y \in V_{uv}$. Therefore $C_\ell = 0$.

We are now in position to compute the trace of Ψ. We adopt the obvious notation $C(A_1) = \sum_{\ell \in A_1} C_\ell$. Thus

$$\mathrm{trace}(\Psi) = C(A_1) + C(A_2) + C(B_2) + C(B_4) + C(B_5).$$

Since C_ℓ is constant on each of these five sets A_1, A_2, B_2, B_4, B_5 we first count the number of their elements:

$$N(A_1) = r,\ N(A_2) = r(r-1)d',\ N(B_2) = 2r(r-1)d',$$

$$N(B_4) = r(r-1)d'(d' - \frac{1}{2}),\ N(B_5) = 2r(r-1)(r-2)d'^2.$$

We get finally

$$\mathrm{trace}(\Psi) = r + r(r-1)d'\,[2 + (r-1)d']$$

which leads to the result through (5.14). □

Comments. — Observe that $\mathrm{Tr}\,[\Psi(f)f] = \mathrm{Tr}\,f^2$ if and only if f is in $\mathscr{F}_1$ (write $f = f_1 + f_2$ with $f_i \in \mathscr{F}_i$ and $\mathrm{Tr}\,[\Psi(f)f] - Trf^2 = -(d'+1)\mathrm{Tr}\,f_2^2$ to see this). Thus in the above orthonormal basis $(f_\ell)_1^{n(n+1)/2}$ of $\mathscr{F}$ we have $f_\ell \in \mathscr{F}_1$ if and only if $C_\ell = 1$, which happens only in the cases A1, B2 and B4. This is a set of size $N(A_1) + N(B_2) + N(B_4) < \dim \mathscr{F}_1$. Similarly $\mathrm{Tr}\,[\Psi(f)f] = -d'\mathrm{Tr}\,f^2$ if and only if f is in $\mathscr{F}_2$, and this shows that no f_ℓ is in $\mathscr{F}_2$ since $C_\ell \geq 0$ for all ℓ.

BIBLIOGRAPHY

[1] S. Andersson – “Invariant normal models”, *Ann. Statist.* **3** (1975), p. 132–154.

[2] D. J. Bartlett – “On the theory of the statistical regression”, *Proc. Royal Soc. Edinburgh* **53** (1933), p. 260–283.

[3] K. Bobecka & J. Wesołowski – “The Lukacs-Olkin-Rubin theorem without invariance of the “quotient””, *Studia Math.* **152** (2002), p. 147–160.

[4] E. M. Carter – "Characterization and testing problems in the complex Wishart distribution", Ph.D. Thesis, University of Toronto, 1975.

[5] M. Casalis – "Les familles exponentielles à variance quadratique homogène sont des lois de Wishart sur un cône symétrique", *C. R. Acad. Sci. Paris Sér. I Math.* **312** (1991), p. 537–540.

[6] M. Casalis & G. Letac – "Characterization of the Jørgensen set in generalized linear models", *Test* **3** (1994), p. 145–162.

[7] ______, "The Lukacs-Olkin-Rubin characterization of Wishart distributions on symmetric cones", *Ann. Statist.* **24** (1996), p. 763–786.

[8] M. L. Eaton – *Multivariate statistics*, Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics, John Wiley & Sons Inc., 1983.

[9] J. Faraut & A. Korányi – *Analysis on symmetric cones*, Oxford Mathematical Monographs, The Clarendon Press Oxford Univ. Press, 1994, Oxford Science Publications.

[10] N. R. Goodman – "Statistical analysis based on a certain multivariate complex Gaussian distribution. (An introduction)", *Ann. Math. Statist.* **34** (1963), p. 152–177.

[11] S. Gyndikin – "Invariant generalized functions in homogeneous spaces", *J. Funct. Anal. Appl.* **9** (1975), p. 50–52.

[12] S. T. Jensen – "Covariance hypotheses which are linear in both the covariance and the inverse covariance", *Ann. Statist.* **16** (1988), p. 302–322.

[13] R. G. Laha & E. Lukacs – "On a problem connected with quadratic regression", *Biometrika* **47** (1960), p. 335–343.

[14] M. Lassalle – "Algèbre de Jordan et ensemble de Wallach", *Invent. Math.* **89** (1987), p. 375–393.

[15] G. Letac – "Le problème de la classification des familles exponentielles naturelles de $\mathbf{R}^d$ ayant une fonction variance quadratique", in *Probability measures on groups, IX (Oberwolfach, 1988)*, Lecture Notes in Math., vol. 1379, Springer, 1989, p. 192–216.

[16] G. Letac & H. Massam – "Quadratic and inverse regressions for Wishart distributions", *Ann. Statist.* **26** (1998), p. 573–595.

[17] G. Letac & J. Wesołowski – "Laplace transforms which are negative powers of quadratic polynomials", *Trans. Amer. Math. Soc.* **360** (2008), p. 6475–6496.

[18] E. Lukacs – "A characterization of the gamma distribution", *Ann. Math. Statist.* **26** (1955), p. 319–324.

[19] M. L. Mehta – *Random matrices*, third ed., Pure and Applied Mathematics (Amsterdam), vol. 142, Elsevier/Academic Press, Amsterdam, 2004.

[20] R. J. Muirhead – *Aspects of multivariate statistical theory*, John Wiley & Sons Inc., 1982.

[21] I. Olkin & H. Rubin – "A characterization of the Wishart distribution", *Ann. Math. Statist.* **33** (1962), p. 1272–1280.

[22] S. D. Peddada & D. S. P. Richards – "Proof of a conjecture of M. L. Eaton on the characteristic function of the Wishart distribution", *Ann. Probab.* **19** (1991), p. 868–874.

[23] D. N. Shanbhag – "The Davidson–Kendall problem and related results on the structure of the Wishart distribution", *Austr. J. Statist.* **30A** (1988), p. 272–280.

[24] Y. H. Wang – "Extensions of Lukacs' characterization of the gamma distribution", in *Analytical methods in probability theory (Oberwolfach, 1980)*, Lecture Notes in Math., vol. 861, Springer, 1981, p. 166–177.

[25] J. Wishart – "The generalised product moment distribution in samples from a normal multivariate population", *Biometrika* **20A** (1928), p. 32–52.