

Bulletin

de la SOCIÉTÉ MATHÉMATIQUE DE FRANCE

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Tome 140

Fascicule 4

2012

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Publié avec le concours du Centre national de la recherche scientifique

pages 485-532

KHINCHIN TYPE CONDITION FOR TRANSLATION SURFACES AND ASYMPTOTIC LAWS FOR THE TEICHMÜLLER FLOW

BY LUCA MARCHESE

ABSTRACT. — We study a diophantine property for translation surfaces, defined in terms of saddle connections and inspired by classical Khinchin condition. We prove that the same dichotomy holds as in Khinchin theorem, then we deduce a sharp estimate on how fast the typical Teichmüller geodesic wanders towards infinity in the moduli space of translation surfaces. Finally we prove some stronger result in genus one.

RÉSUMÉ (*Conditions de type Khinchin pour les surfaces de translation et lois asymptotiques pour le flot de Teichmüller*)

On étudie une propriété diophantienne pour les surfaces de translation, définie en termes de connexions de selles et inspirée par la condition de Khinchin classique. On prouve la même dichotomie du théorème de Khinchin et on en déduit une estimation sur la vitesse des excursions à l'infini pour une géodésique de Teichmüller typique dans l'espace des modules des surfaces de translation. Enfin on prouve un résultat plus fort en genre un.

1. Introduction

The moduli space of flat tori is identified with the *modular surface*, that is the quotient of the hyperbolic half plane \mathbb{H} by the action of $\mathrm{PSL}(2, \mathbb{Z})$ via *Moebius transformations*, which is homeomorphic to a punctured sphere and has finite

Texte reçu le 9 juillet 2010, révisé le 7 mars 2011, accepté le 23 juin 2011.

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2010 Mathematics Subject Classification. — 37D40, 37A20; 11J70, 11K50.

Key words and phrases. — Translation surfaces, Teichmüller flow, Khinchin condition, interval exchange transformations.

area. A neighborhood of the cusp corresponds to those flat tori with a very short closed geodesic, or equivalently to points in the standard fundamental domain with big imaginary part. The geodesic flow g_t acts ergodically on the unitary tangent bundle of $\mathbb{H}/\mathrm{PSL}(2, \mathbb{Z})$, therefore a generic geodesic makes infinitely many excursions to the cusp. The asymptotic rate of this phenomenon is measured by the so-called *logarithmic law*.

THEOREM. — *For any point z in $\mathbb{H}/\mathrm{PSL}(2, \mathbb{Z})$ and almost any unitary tangent vector v at z , if g_t is the geodesic at z in the direction of v and d denotes the Poincaré distance, we have*

$$\limsup_{t \rightarrow \infty} \frac{d(g_t(z), z)}{\log t} = 1/2.$$

The logarithmic law has been generalized in many settings, in particular Sullivan proved it for the geodesic flow on manifolds with negative curvature(see [13])and Masur proved a logarithmic law for *Teichmüller geodesics* on the moduli space of complex curves of any genus(see [11]).

The geodesic flow on $\mathbb{H}/\mathrm{PSL}(2, \mathbb{Z})$ has a well know relation with the *continued fraction algorithm*, which have been described in [12], and in general with arithmetics. In particular the logarithmic law is strictly related to Khinchin Theorem(see [5]), which concerns the general diophantine condition on a real number α defined in terms of the set of integers n such that:

$$\{n\alpha\} < \varphi(n),$$

where $\{\cdot\}$ denotes the fractionary part and $\varphi : \mathbb{N} \rightarrow \mathbb{R}_+$ is a positive (vanishing) sequence.

THEOREM (Khinchin). — *Let $\varphi : \mathbb{N} \rightarrow \mathbb{R}_+$ be a positive sequence.*

- *If $\sum_{n=1}^{+\infty} \varphi(n) < +\infty$ then for almost any α there are just finitely many $n \in \mathbb{N}$ such that $\{n\alpha\} < \varphi(n)$.*
- *If $n\varphi(n)$ is monotone decreasing and $\sum_{n=1}^{+\infty} \varphi(n) = +\infty$ then for almost any α there are infinitely many $n \in \mathbb{N}$ such that $\{n\alpha\} < \varphi(n)$.*

Translation surfaces. — The natural generalization to higher genus of a flat torus brings to the notion of *translation surface*, that is a compact, orientable and boundary-less flat surface X , with conical singularities whose angle is a multiple of 2π . If g is the genus of the surface, $\Sigma = \{p_1, \dots, p_r\}$ is the set of conical singularities and k_1, \dots, k_r are positive integers such that for any $i = 1, \dots, r$ the angle at p_i is $2k_i\pi$, then we have the relation $k_1 + \dots + k_r = 2g + r - 2$. Flat neighborhoods in X are naturally identified with open sets in \mathbb{C} , that is they admit a local coordinate z , thus $X \setminus \Sigma$ inherits the structure of Riemann surface and it is easy to see that the structure extends to X . Since the angles at conical singularities are multiples of 2π then dz defines a complex one form

on $X \setminus \Sigma$, which extends by $z^{k_i-1}dz$ at any point $p_i \in \Sigma$, that is it has a zero of order $k_i - 1$. The datum of a translation surface is therefore equivalent to the datum of a Riemann surface together with a complex one form. The *moduli space* of translation surfaces of fixed genus g admits a *stratification*, where a *stratum* is the set $\mathcal{H}(k_1, \dots, k_r)$ of those translation surfaces with r conical singularities with fixed values $2k_1\pi, \dots, 2k_r\pi$ for the angles. We assume that the singularities are labeled. Any $\mathcal{H}(k_1, \dots, k_r)$ is a complex orbifold with complex dimension $2g + r - 1$, that is it is locally homeomorphic to the quotient of \mathbb{C}^{2g+r-1} by the action of a finite group, orbifold points occurring at those translation surfaces whose underlying complex structure admits non-trivial automorphisms. In general strata are neither compact nor connected and their connected components have been classified in [6].

A *saddle connection* for a translation surface X is a geodesic $\gamma : (0, T) \rightarrow X$ for the flat metric such that $\gamma^{-1}(\Sigma) = \{0, T\}$, that is γ starts and ends in Σ and it does not contain any other conical singularity in its interior. If γ is a saddle connection for X we define a complex number $\text{Hol}(\gamma) := \int_\gamma w_X$, which is called the *holonomy* of γ , where w_X is the holomorphic one form associated to X . We call $\text{Hol}(X)$ the set of complex numbers $v = \text{Hol}(\gamma)$, where γ varies among the saddle connections of X . It is possible to see that $\text{Hol}(X)$ contains pure imaginary elements only for translation surfaces X lying in a codimension one subset of $\mathcal{H}(k_1, \dots, k_r)$, thus in particular a zero measure subset. Nevertheless $\text{Hol}(X)$ is always dense in $\mathbb{P}\mathbb{R}^2$ and in particular it accumulates to the imaginary axis. We define a diophantine condition comparing the deviation from the imaginary axis of elements of $\text{Hol}(X)$ with their norm. For a monotone decreasing function $\varphi : (0, +\infty) \rightarrow (0, +\infty)$ and for an element v in $\text{Hol}(X)$ we consider the condition

$$(1.1) \quad |\Re(v)| < \varphi(|v|).$$

For any translation surface X the associated one form w_X induces a pair of parallel vector fields ∂_x and ∂_y on $X \setminus \Sigma$ defined by $w_X(\partial_x) = 1$ and $w_X(\partial_y) = \sqrt{-1}$ and called respectively *horizontal* and *vertical* vector field. They are not complete, since their trajectories stop if they arrive at a point of Σ . In particular any point $p_i \in \Sigma$ is the starting point of exactly k_i half-trajectories of ∂_x , which are called *horizontal separatrices*. A *frame* for a translation surface X is the datum of r different horizontal separatrices (S_1, \dots, S_r) , such that S_i starts at p_i for any $i \in \{1, \dots, r\}$. Any X admits $\prod_{i=1}^r k_i$ different choices of a frame. We denote \widehat{X} the datum (X, S_1, \dots, S_r) of a translation surface with frame and we call $\widehat{\mathcal{H}}(k_1, \dots, k_r)$ the stratum of the moduli space of translation surfaces with frame, which is a covering space of $\mathcal{H}(k_1, \dots, k_r)$ with natural projection $\widehat{X} \mapsto X$. It is possible to show that the covering is non-trivial, that is it does not admit a continuous inverse $\mathcal{H}(k_1, \dots, k_r) \rightarrow \widehat{\mathcal{H}}(k_1, \dots, k_r)$, thus a continuous

choice of a frame is not possible on $\mathcal{H}(k_1, \dots, k_r)$ but only on $\widehat{\mathcal{H}}(k_1, \dots, k_r)$. Conceptually the construction is the same as that of the orientable double covering of a non-orientable manifold. Note that the space $\widehat{\mathcal{H}}(k_1, \dots, k_r)$ has also been considered by C. Boissy in [1], where the author computes the number of connected components of strata of translation surfaces with frame.

Let \widehat{X} be a translation surface with frame whose frame is (S_1, \dots, S_r) . Let p_j and p_i be any two points in Σ (possibly the same) and let (m, l) be a pair of integers with $1 \leq m \leq k_j$ and $1 \leq l \leq k_i$. We define the *bundle* $\mathcal{C}^{(p_j, p_i, m, l)}(\widehat{X})$ as the set of those saddle connections γ for X which start at p_j , end in p_i and satisfy

$$2(m - 1)\pi \leq \text{angle}(\gamma, S_j) < 2m\pi \quad \text{and} \quad 2(l - 1)\pi \leq \text{angle}(\gamma, S_i) < 2l\pi,$$

where S_j and S_i are the horizontal separatrices in the frame starting respectively at p_j and p_i . A choice of a frame for X therefore induces a decomposition of $\text{Hol}(X)$ into subsets

$$\text{Hol}^{(p_j, p_i, m, l)}(\widehat{X}) := \{\text{Hol}(\gamma); \gamma \in \mathcal{C}^{(p_j, p_i, m, l)}(\widehat{X})\}.$$

We prove the following dichotomy.

THEOREM 1.1. — *Let $\varphi : (0, +\infty) \rightarrow (0, +\infty)$ be a positive function.*

1. *If $\varphi(t)$ is decreasing monotone and $\int_1^{+\infty} \varphi(t)dt < +\infty$ then $\text{Hol}(X)$ contains just finitely many solutions of Equation (1.1) for almost any X in $\mathcal{H}(k_1, \dots, k_r)$.*
2. *If $t\varphi(t)$ is monotone decreasing and $\int_1^{+\infty} \varphi(t)dt = +\infty$ then for almost any \widehat{X} in $\widehat{\mathcal{H}}(k_1, \dots, k_r)$, for any pair of points p_j, p_i in Σ and for any pair of integers (m, l) with $1 \leq m \leq k_j$ and $1 \leq l \leq k_i$ the set $\text{Hol}^{(p_j, p_i, m, l)}(\widehat{X})$ contains infinitely many solutions of Equation (1.1).*

We note that the function φ in Theorem 1.1 is defined on $(0, +\infty)$, anyway we consider its integral just over $(1, +\infty)$ because we are just interested by convergence or divergence at infinity.

Translation surfaces are strictly linked to *interval exchange transformations* (i.e.t.s in the following), a class of maps of the interval which has been largely studied, for example in [14], [9], [17]. In particular the second part of Theorem 1.1 (divergent integral) is a consequence of a generalization of Khinchin Theorem to i.e.t.s which is proved in [7]. On the other hand the first part of Theorem 1.1 (convergent integral) admits a stronger statement, which is proved independently from i.e.t.s with an easier argument. Such argument is based on a natural construction, which consists in fixing a translation surface X and rotating its vertical direction. If θ is the rotation angle, we call X_θ the rotated translation surface. The image of the application $\theta \mapsto X_\theta$ is an

embedded circle in the stratum $\mathcal{H}(k_1, \dots, k_r)$ of X , except for orbifold points, where the image has a finite number of self-intersections. Globally we get a smooth orbit foliation of $\mathcal{H}(k_1, \dots, k_r)$ under the action of $\mathrm{SO}(2, \mathbb{R})$, thus the convergent part of Theorem 1.1 is an immediate consequence of the following Proposition.

PROPOSITION 1.2. — *Let $\varphi : (0, +\infty) \rightarrow (0, +\infty)$ be a monotone decreasing function with $\int_1^{+\infty} \varphi(t) dt < +\infty$. Then for any X in $\mathcal{H}(k_1, \dots, k_r)$ and almost any $\theta \in [0, 2\pi)$ the set $\mathrm{Hol}(X_\theta)$ contains just finitely many solutions of Equation (1.1).*

Teichmüller flow. — For any translation surface X and any element $G \in \mathrm{SL}(2, \mathbb{R})$ we can define a new translation surface GX whose local charts are the composition of local charts of X with G (here our local charts map open sets of X to \mathbb{C}). We have therefore an action of $\mathrm{SL}(2, \mathbb{R})$ on $\mathcal{H}(k_1, \dots, k_r)$, the *Teichmüller flow* \mathcal{F}_t is the action of the diagonal elements of $\mathrm{SL}(2, \mathbb{R})$ with traces $2 \cosh t$. Since \mathcal{F}_t preserves horizontal and vertical directions, then it acts naturally on frames of saddle connections, that is the Teichmüller flow \mathcal{F}_t lifts to the covering $\widehat{\mathcal{H}}(k_1, \dots, k_r)$ (but not the whole action of $\mathrm{SL}(2, \mathbb{R})$).

Any translation surface X admits an area form $dz \wedge d\bar{z}$ on $X \setminus \Sigma$, which defines a smooth function $X \mapsto \mathrm{Area}(X)$ on strata. We call $\mathcal{H}^{(1)}(k_1, \dots, k_r)$ the smooth hyperboloid in $\mathcal{H}(k_1, \dots, k_r)$ of area one translation surfaces and $\widehat{\mathcal{H}}^{(1)}(k_1, \dots, k_r)$ the corresponding hyperboloid in the covering space of translation surfaces with frame. The Teichmüller flow preserves the hyperboloids. A fundamental result of Masur and Veech (see [9] and [14]) says that \mathcal{F}_t preserves an unique smooth probability measure $\mu^{(1)}$ on $\mathcal{H}^{(1)}(k_1, \dots, k_r)$ and its restriction to any connected component of the stratum is ergodic.

Strata are non compact and area one hyperboloids are not compact too, therefore by ergodicity a generic orbit of \mathcal{F}_t makes excursions *to infinity*. A criterion to describe non-compact part of strata can be stated in terms of the *systole function* $X \mapsto \mathrm{Sys}(X)$, which assigns to X the length of its shortest saddle connection. The main property is that a sequence X_n in $\mathcal{H}(k_1, \dots, k_r)$ (or in the area-one hyperboloid) diverges, that is it escapes from any compact set, if and only if $\mathrm{Sys}(X_n) \rightarrow 0$. In general the non-compact part of strata is not a simple cusp, like for flat tori, and the vanishing of the systole does not give any information on the part of it towards the sequence X_n is diverging to. In order to keep track of this information, for any \widehat{X} in $\widehat{\mathcal{H}}(k_1, \dots, k_r)$, for any pair of points $p_j, p_i \in \Sigma$ and any pair of integers (m, l) with $1 \leq m \leq k_j$ and $1 \leq l \leq k_i$ we set

$$\mathrm{Sys}^{(p_j, p_i, m, l)}(\mathbf{K}) := \min\{|v|; v \in \mathrm{Hol}^{(p_j, p_i, m, l)}(\mathbf{K})\},$$

that is the length of the shortest saddle connection in the bundle $\mathcal{C}^{(p_j, p_i, m, l)}(\widehat{X})$. By ergodicity a generic orbit is neither divergent nor bounded, thus for any datum (p_j, p_i, m, l) as before the quantity $\text{Sys}^{(p_j, p_i, m, l)}(\mathcal{F}_t \mathbf{X})$ stays bounded away from 0 for the most of the time, but there are arbitrarily big instants t_n such that it becomes small. The following result gives a sharp quantitative description of this phenomenon.

THEOREM 1.3. — *Let $\psi : [0, +\infty) \rightarrow (0, +\infty)$ be a monotone decreasing function.*

1. *If $\int_0^{+\infty} \psi(t)dt < +\infty$ then for almost any $X \in \mathcal{H}^{(1)}(k_1, \dots, k_r)$ we have:*

$$\lim_{t \rightarrow \infty} \frac{\text{Sys}(\mathcal{F}_t X)}{\sqrt{\psi(t)}} = \infty.$$

2. *If $\int_0^{+\infty} \psi(t)dt = +\infty$ then for almost any $\widehat{X} \in \widehat{\mathcal{H}}^{(1)}(k_1, \dots, k_r)$, for any pair of points $p_j, p_i \in \Sigma$ and any pair of integers (m, l) with $1 \leq m \leq k_j$ and $1 \leq l \leq k_i$ we have*

$$\liminf_{t \rightarrow \infty} \frac{\text{Sys}^{(p_j, p_i, m, l)}(\mathcal{F}_t \mathbf{X})}{\sqrt{\psi(t)}} = 0.$$

For the parameter family $\psi_r(t) := \min\{1, t^{-r}\}$ with $r \geq 1$ a straightforward application of both items of Theorem 1.3 implies the following Corollary.

COROLLARY 1.4. — *For any pair of points $p_i, p_j \in \Sigma$, for any pair of integers (m, l) with $1 \leq m \leq k_j$ and $1 \leq l \leq k_i$ and for almost any $\widehat{X} \in \widehat{\mathcal{H}}^{(1)}(k_1, \dots, k_r)$ we have:*

$$\limsup_{t \rightarrow \infty} \frac{-\log \circ \text{Sys}^{(p_j, p_i, m, l)}(\mathcal{F}_t \mathbf{X})}{\log t} = \frac{1}{2}.$$

It is natural to compare the estimate in Corollary 1.4 to Masur’s logarithm law(see [11]). Indeed it says that for any datum (p_j, p_i, m, l) and for a generic \widehat{X} the quantity $-\log \circ \text{Sys}^{(p_j, p_i, m, l)}(\mathcal{F}_t \mathbf{X})$ has the same asymptotic behavior as $\text{dist}([X], [\mathcal{F}_t X])$, that is its maximal excursion up to time t has amplitude $\log \sqrt{t}$, where $[X]$ and $[\mathcal{F}_t X]$ are the Riemann surfaces underlying respectively to X and $\mathcal{F}_t X$ and where dist denotes the *Teichmüller’s distance* in the moduli space of Riemann surfaces of genus g .

QUESTION 1.5. — Theorem 1.3 is a consequence of Theorem 1.1, which is based itself on a generalization of Khinchin Theorem for i.e.t.s, namely Theorem 3.7 in paragraph 3.4.1. I.e.t.s have been generalized with *linear involutions* by Danthony and Nogueira(see [3])and the latter have been related to half-translation surfaces by Boissy and Lanneau(see [2]). We ask if it is possible to extend Theorem 3.7 to linear involutions(more precisely to the subclass of them

which is relevant for half-translation surfaces, as it is explained in [2]), which would bring to an extension of Theorem 1.3 to the setting of half-translation surfaces.

Punctured tori. — Since the orbit foliation under $\mathrm{SO}(2, \mathbb{R})$ is smooth, the divergent part of Theorem 1.1 implies that for almost any translation surface X the statement in the Theorem holds for almost any rotated surface X_θ . We ask what we can prove for any leaf of the foliation and not just for a generic one. We have some partial result in this direction for a flat torus X with r punctures. Since we consider a flat torus we do not have true conical singularities, that is any puncture has an euclidian neighborhood, anyway their position determines a subset $\Sigma = \{p_1, \dots, p_r\}$ of X with a non-trivial geometric information. The moduli space of pairs (X, Σ) is a fibre bundle over the modular surface. Any point of Σ is the starting point of exactly one horizontal separatrix, thus a frame is intrinsically defined and a bundle of saddle connections is simply determined by the pair of points p_j, p_i in Σ where the saddle connections γ respectively start and end, therefore we denote it simply $\mathcal{C}^{(p_j, p_i)}(X)$ and we write $\mathrm{Hol}^{(p_j, p_i)}(X)$ for the set of the corresponding holonomy vectors. We remark that even if points in Σ have a flat neighborhood, we want saddle connections not have them as interior points.

THEOREM 1.6. — *Consider function $\varphi : (0, +\infty) \rightarrow (0, +\infty)$ such that $t\varphi(t)$ is decreasing monotone and $\int_1^{+\infty} \varphi(t)dt < +\infty$. Then for any flat torus X with r punctures p_1, \dots, p_r and for almost any $\theta \in [0, 2\pi)$ there are at least r different pairs of punctures (p_j, p_i) such that the corresponding sets $\mathrm{Hol}^{(p_j, p_i)}(X_\theta)$ contain infinitely many solutions of Equation (1.1).*

We observe that for an arbitrary set of punctures $\Sigma = \{p_1, \dots, p_r\}$ we cannot expect to have the same result of Theorem 1.6 for any pair of points. This is evident for example if X is the flat torus $\mathbb{C}/(2\mathbb{Z})^2$ with four punctures corresponding to the quotient $\mathbb{Z}^2/(2\mathbb{Z})^2$: indeed in this case $\mathrm{Hol}^{(p_i, p_i)}(X)$ is empty for any $p_i \in \Sigma$. We observe also that we do not know if our result is optimal, that is neither we are able to find generically more than r different pairs (p_j, p_i) as in the statement, nor we can find counterexamples.

Content of this paper. — *Section 2* is a brief survey of the background theory of translation surfaces and interval exchange transformations. In §2.1 we give rigorous definitions for translation surfaces and their moduli space. In §2.2 we introduce the class of interval exchange transformations and their parameter space, which is union of euclidian spaces. Then we describe *Veech's construction*, which is a procedure to get a translation surface as *suspension* of an i.e.t..

Section 3 is devoted to the proof of Theorem 1.1. In §3.1 we prove Proposition 1.2 and according to the discussion above this proves also the convergent case of Theorem 1.1. The argument is a simple application of the Borel-Cantelli Lemma on any leaf of the $SO(2, \mathbb{R})$ -foliation. In §3.2 we define translation surfaces with frame, whose moduli space admits bundles of saddle connections invariant under the action of the Teichmüller flow. In §3.3 we give a combinatorial construction of families of saddle connection in a required bundle. We work with Veech's construction and our construction is based on the combinatorial notion of *reduced triple* for an i.e.t., which is introduced in Definition 3.5. The main result of the paragraph is Proposition 3.6. In §3.4 we give the proof of the divergent part of Theorem 1.1: we apply Theorem 3.7, which is a generalization of Khinchin Theorem for i.e.t.s and provides us with infinitely many reduced triples satisfying Equation (3.2), a diophantine condition related to Equation (1.1). Then we apply the construction of §3.3.

Section 4 contains the proof of Theorem 1.3. We first observe that Theorem 1.1 also holds on the hyperboloid $\widehat{\mathcal{H}}^{(1)}(k_1, \dots, k_r)$ of area one translation surfaces with frame with respect to the associated smooth measure $\mu^{(1)}$. In §4.1 we prove some preliminary estimates on the dilatation of lengths of saddle connection under the Teichmüller flow. In §4.2 we complete the proof of Theorem 1.3. The proof of Corollary 1.4 is omitted since it is straightforward.

Section 5 contains the proof of Theorem 1.6. In §5.1 we recall that a flat torus X with a set Σ of r punctures whose saddle connections are never vertical can be represented as a suspension of a *rotational* i.e.t. T with $r + 1$ intervals, that is an i.e.t. with just one true singularity. Then we study how the parameters describing the associated rotational i.e.t. change when the vertical direction of X is rotated. The main result in the paragraph is Lemma 5.1. In §5.2 we prove Theorem 1.6 applying the construction of §5.1 and using an arithmetic result, namely Theorem A.1, which is a generalization of Khinchin Theorem for a diophantine condition introduced by Equation (A.1).

Appendix A contains the proof of Theorem A.1. In §A.1 we recall some very classical results for the *continued fraction algorithm*, our presentation of this algorithm follows a geometrical interpretation. In §A.2 we define a family of approximations for an irrational number called *twisted approximation*. The proof of the Theorem is completed in §A.3 and §A.4.

Acknowledgements. — I would like to thank Jean-Christophe Yoccoz for many inspiring discussions. I am also grateful to Correntin Boissy, Giovanni Forni, Pascal Hubert, Erwan Lanneau, Stefano Marmi and anonymous referee for many precious remarks.

2. Background

2.1. Translation surfaces. — Let g be a positive integer and consider r positive integers k_1, \dots, k_r such that $k_1 + \dots + k_r = 2g + r - 2$. A *translation surface* is given by the following data

1. A compact, boundaryless, orientable topological surface X of genus g with a finite subset $\Sigma = \{p_1, \dots, p_r\}$ of r points.
2. An atlas on $X \setminus \Sigma$ such that the changes of charts are translations. The charts map open subset of X to \mathbb{C} , and if ρ and ρ' are two elements of the atlas then $\rho' \circ \rho^{-1}(z) = z + \text{const.}$ Such charts are called *translation charts*.
3. For any $p_i \in \Sigma$ a neighborhood V_i of p_i , a neighborhood W of 0 in \mathbb{C} and a ramified covering $(V_i, p_i) \rightarrow (W, 0)$ of degree k_i whose branches are translation charts of the atlas.

We call $\text{Transl}(k_1, \dots, k_r)$ the set of translation structures as above. Any element X in $\text{Transl}(k_1, \dots, k_r)$ provides us with the following structures

- A Riemann surface structure on X (not only on $X \setminus \Sigma$).
- An holomorphic 1-form w_X on X , which in translation charts is given by dz . The zeros of w_X are exactly the points of Σ , and in a chart around any $p_i \in \Sigma$ the form w_X is given by $z^{k_i-1} dz$, that is it has a zero of order $k_i - 1$.
- A flat metric $g_X := |dz|^2$ defined on $X \setminus \Sigma$, with cone singularities of total angle $2k_i\pi$ at p_i in Σ .
- Two vertical and horizontal vector fields ∂_y and ∂_x on $X \setminus \Sigma$ defined by $w_X(\partial_x) = 1$ and $w_X(\partial_y) = \sqrt{-1}$ and called respectively *horizontal* and *vertical* vector field. These fields are not complete since their trajectories stop at points in Σ . We call *horizontal separatrices* the trajectories of ∂_x starting from a point $p_i \in \Sigma$. A singularity p_i of order k_i is the starting point of exactly k_i horizontal separatrices.
- An area form $\frac{\sqrt{-1}}{2} w_X \wedge \overline{w_X}$ on $X \setminus \Sigma$.

We recall from the introduction the notion of *saddle connection* for a translation surface X : it is a geodesic path $\gamma : (0, T) \rightarrow X$ for the flat metric such that $\gamma^{-1}(\Sigma) = \{0, T\}$, that is it starts and ends at points in Σ and does not contain other such points in its interior.

2.1.1. Moduli space. — Let us consider two elements X and X' in $\text{Transl}(k_1, \dots, k_r)$ whose sets of singular points are respectively $\Sigma = \{p_1, \dots, p_r\}$ and $\Sigma' = \{p'_1, \dots, p'_r\}$. We define an equivalence relation on $\text{Transl}(k_1, \dots, k_r)$ saying that $X \sim X'$ if there exists a diffeomorphism $f : X \rightarrow X'$ such that

- $f(p_i) = p'_i$ for all $i = 1, \dots, r$

- ρ is a translation chart of X if and only if $\rho \circ f$ is a translation chart of X' .

The quotient is the *stratum* of the so called *moduli space* of translation surfaces with cone singularities of orders k_1, \dots, k_r , and is denoted $\mathcal{H}(k_1, \dots, k_r)$.

2.1.2. *Teichmüller space.* — We fix a reference topological pair (M^*, Σ^*) , where M^* is homeomorphic to any surface in $\text{Transl}(k_1, \dots, k_r)$ and $\Sigma^* = \{p_1^*, \dots, p_r^*\}$ is any subset of M^* with r points. We denote $\text{Diff}^+(M^*, \Sigma^*)$ the group of diffeomorphisms of M^* which preserve the orientation and are the identity on Σ^* and we call $\text{Diff}_0^+(M^*, \Sigma^*)$ the subgroup of those diffeomorphisms which are isotopic to the identity. A *marking* of a translation surface X is a diffeomorphism of pairs

$$\phi : (M^*, \Sigma^*) \rightarrow (X, \Sigma)$$

such that $\phi(p_i^*) = p_i$ for all $i = 1, \dots, r$. We denote (X, ϕ) the datum of a translation surface with marking and we denote $\text{Transl}^*(k_1, \dots, k_r)$ the set of translation surfaces with marking. We define an equivalence relation saying that $(X, \phi) \sim (X', \phi')$ if there exists a diffeomorphism $f : X \rightarrow X'$ as in §2.1.1 such that

$$\phi'^{-1} \circ f \circ \phi \in \text{Diff}_0^+(M^*, \Sigma^*).$$

The quotient of $\text{Transl}^*(k_1, \dots, k_r)$ by \sim is a stratum of the so-called *Teichmüller space* of translation surfaces with cone singularities of orders k_1, \dots, k_r , and is denoted $\mathcal{T}(k_1, \dots, k_r)$.

2.1.3. *Period map and local coordinates.* — We fix a basis of cycles $\{\gamma_1^*, \dots, \gamma_d^*\}$ for $H_1(M^*, \Sigma^*, \mathbb{Z})$, where $d = 2g + r - 1$. The so-called *period map* $\Theta : \mathcal{T}(k_1, \dots, k_r) \rightarrow \mathbb{C}^d$ associates to a translation structure with marking (X, ϕ) a complex vector z in \mathbb{C}^d by the formula

$$z_i := \int_{\phi(\gamma_i^*)} w_X.$$

We observe that the integral above is an isotopy invariant, so Θ is well defined. The Teichmüller space $\mathcal{T}(k_1, \dots, k_r)$ inherits its topology from \mathbb{C}^d via Θ .

PROPOSITION. — *The map Θ is a local homeomorphism.*

Proof. — The map Θ is continuous by definition, so it is enough to prove that it is open and that its restriction to small open sets is injective. It is known that any translation surface X admits a triangulation whose vertices are the points in Σ and whose edges are saddle connections(see [4] for example). If (X, ϕ) is a translation surface with marking, the pre-images under ϕ of the edges of the triangulation are elements in $H_1(M^*, \Sigma^*, \mathbb{Z})$ and the triangulation can be represented combinatorially as a(non planar)graph, whose vertices are the points

in Σ^* and whose edges are elements in $H_1(M^*, \Sigma^*, \mathbb{Z})$. Once the combinatorics of the triangulation is fixed, the values z_1, \dots, z_d of the periods on the basis $\{\gamma_1^*, \dots, \gamma_d^*\}$ of $H_1(M^*, \Sigma^*, \mathbb{Z})$ determine the triangulation, and therefore the translation surface. In particular the values z_1, \dots, z_d for the periods which define a triangulation are determined by some system of linear inequalities, which give an open condition in \mathbb{C}^d , therefore Θ is open. On the other hand, we can take open sets \mathcal{U} in $\mathcal{T}(k_1, \dots, k_r)$ small enough in order to have that any (X, ϕ) in \mathcal{U} admits a triangulation whose combinatorial representation is the same. From the discussion above it follows that any such (X, ϕ) is determined by the values $z_i = \int_{\phi(\gamma_i^*)} w_X$ of its periods, therefore $\Theta|_{\mathcal{U}}$ is injective. The proposition is proved. \square

Since Θ is a local homeomorphism, $\mathcal{T}(k_1, \dots, k_r)$ inherits from \mathbb{C}^{2g+r-1} the structure of complex manifold, whose complex dimension is of course $2g+r-1$. The group $\text{Diff}^+(M^*, \Sigma^*)$ acts (on the right) on $\mathcal{T}(k_1, \dots, k_r)$ by

$$(X, \phi) \mapsto (X, \phi \circ f).$$

Moreover the action of the subgroup $\text{Diff}_0^+(M^*, \Sigma^*)$ is trivial, since the Teichmüller space is defined modulo isotopies. It follows that we have an action (on the right) on $\mathcal{T}(k_1, \dots, k_r)$ of the so called *mapping class group*

$$\text{Mod}(g, r) := \text{Diff}^+(M^*, \Sigma^*) / \text{Diff}_0^+(M^*, \Sigma^*).$$

The action is faithful and proper, but it has non-trivial stabilizer at surfaces admitting complex automorphisms. The quotient is exactly the moduli space $\mathcal{H}(k_1, \dots, k_r)$, which inherits therefore the structure of a complex orbifold with complex dimension $2g+r-1$.

2.1.4. Smooth measure and Teichmüller flow. — For any $G \in \text{GL}(2, \mathbb{R})$ and any $X \in \text{Transl}(k_1, \dots, k_r)$ we define a new translation surface GX whose local charts have the form $G \circ \rho$, where ρ is a local chart of X . Since the translation subgroup is normal in the group of affine automorphisms of \mathbb{R}^2 , then GX is still a translation surface. Thus $\text{GL}(2, \mathbb{R})$ acts on the left on $\text{Transl}(k_1, \dots, k_r)$. Since $\text{Diff}^+(M^*, \Sigma^*)$ acts on the right on $\text{Transl}(k_1, \dots, k_r)$, then the two actions commute and we get an action (on the left) of $\text{GL}(2, \mathbb{R})$ on $\mathcal{H}(k_1, \dots, k_r)$

Let us consider the standard volume form $d\text{Leb}$ on \mathbb{C}^{2g+r-1} , normalized in order to give co-volume one to the integer lattice $(\mathbb{Z} \oplus i\mathbb{Z})^{2g+r-1}$. Using the period map Θ we pull back the standard form and we get a smooth form $dm := \Theta^* d\text{Leb}$ on the stratum $\mathcal{T}(k_1, \dots, k_r)$, whose associated smooth measure is denoted m . The action of any element f in $\text{Mod}(g, r)$ on $H_1(M^*, \Sigma^*, \mathbb{Z})$ is represented by some A_f in $\text{SL}(d, \mathbb{Z})$ which satisfies $\Theta \circ f = A_f \circ \Theta$, thus $\text{Mod}(g, r)$ preserves the form dm on $\mathcal{T}(k_1, \dots, k_r)$, and therefore the measure

m. It follows that the projection to the quotient induces a well defined volume element $d\mu$ on $\mathcal{H}(k_1, \dots, k_r)$, whose associated smooth measure is denoted μ .

For any $X \in \text{Transl}(k_1, \dots, k_r)$ the corresponding holomorphic 1-form w_X induces an area form $\frac{\sqrt{-1}}{2} w_X \wedge \bar{w}_X$ on $X \setminus \Sigma$. The area of X is given by

$$\text{Area}(X) := \frac{\sqrt{-1}}{2} \int_{X \setminus \Sigma} w_X \wedge \bar{w}_X.$$

The integral above is obviously invariant under the action of $\text{Diff}^+(M^*, \Sigma^*)$ and in particular of $\text{Diff}_0^+(M^*, \Sigma^*)$, therefore it defines a function on $\mathcal{F}(k_1, \dots, k_r)$ which is invariant under $\text{Mod}(g, r)$. This amounts to say that we have an *area function* $X \mapsto \text{Area}(X)$ on the moduli space $\mathcal{H}(k_1, \dots, k_r)$. For any choice of a marking (X, ϕ) in $\mathcal{F}(k_1, \dots, k_r)$ the well known *Riemann's bilinear relation* expresses $\text{Area}(X)$ as a real analytic function of $z = \Theta(X, \phi)$, it follows that the area function is smooth on $\mathcal{F}(k_1, \dots, k_r)$ and we have a sub-manifold of real codimension one

$$\mathcal{F}^{(1)}(k_1, \dots, k_r) := \{(X, \phi) \in \mathcal{F}(k_1, \dots, k_r); \text{Area}(X) = 1\}.$$

The moduli space of area one translation surfaces is the smooth hyperboloid

$$\mathcal{H}^{(1)}(k_1, \dots, k_r) := \mathcal{F}^{(1)}(k_1, \dots, k_r) / \text{Mod}(g, r).$$

It is possible to see that it is non-compact and in general non-connected (see [6]). We have an homeomorphism from $\mathbb{R}_+ \times \mathcal{H}^{(1)}(k_1, \dots, k_r)$ to $\mathcal{H}(k_1, \dots, k_r)$ sending (λ, X) to λX , where the multiplication by λ is given by the action of $\text{GL}(2, \mathbb{R})$. It follows that the volume element $d\mu$ decomposes as

$$d\mu = \lambda^n d\lambda \wedge d\mu^{(1)},$$

where $n = 4g + 2r - 3$ and $d\mu^{(1)}$ is a volume form on $\mathcal{H}^{(1)}(k_1, \dots, k_r)$, whose associated smooth measure is denoted $\mu^{(1)}$.

$\text{SL}(2, \mathbb{R})$ acts on $\mathcal{H}^{(1)}(k_1, \dots, k_r)$ as subgroup of $\text{GL}(2, \mathbb{R})$ and it is easy to see that it preserves $\mu^{(1)}$. The *Teichmüller flow* \mathcal{F}_t on $\mathcal{H}^{(1)}(k_1, \dots, k_r)$ is the action of the diagonal subgroup of $\text{SL}(2, \mathbb{R})$, that if for any $t \in \mathbb{R}$:

$$\mathcal{F}_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}.$$

We recall the following fundamental result (see [9] and [14]).

THEOREM. — *The smooth measure $\mu^{(1)}$ gives finite volume to $\mathcal{H}^{(1)}(k_1, \dots, k_r)$ and its restriction to any connected component is ergodic with respect to \mathcal{F}_t .*

2.2. Interval exchange transformations and Veech's construction

2.2.1. *Interval exchange transformations.* — Let X be a translation surface and let I be an horizontal segment in X , that is a finite segment of a trajectory of the horizontal field ∂_x . For $x \in I$ we denote $T(x)$ the first intersection of I with the positive trajectory of the vertical field ∂_y starting from x , in other words we consider the *first return map* to I of the vertical flow. Since ∂_y preserves the area element on X , then T is defined almost everywhere on I . Moreover T preserves the length element dx on I , therefore it acts locally as a translation in the coordinate x . It follows that the domain of T is union of open sub-intervals of I , whose endpoints lie on positive trajectories of ∂_y ending in points of Σ without passing from I . There are just finitely many such trajectories, thus the domain of T is union of a finite number of intervals and the effect of T is to rearrange them in I by translations. A map like T is called an *interval exchange transformation*.

DEFINITION 2.1. — Let $d \geq 2$ and let $\lambda = (\lambda_1, \dots, \lambda_d)$ be a vector in \mathbb{R}_+^d and π be a permutation in the symmetric group S_d . An interval exchange transformation (we write i.e.t.) with d intervals is a map T from an interval I to itself defined by the data (π, λ) as follows.

- The interval I admits two partitions into open sub-intervals $I = \bigsqcup_{i=1}^d I_i^t$ and $I = \bigsqcup_{i=1}^d I_i^b$. For any $i \in \{1, \dots, d\}$, if we start counting from the left, I_i^t is in i -th position in the first partition and I_i^b is in i -th position in the second partition.
- For any $i \in \{1, \dots, d\}$ we have that $|I_i^t| = |I_{\pi(i)}^b| = \lambda_i$ and the restriction of T to I_i^t is the translation onto $I_{\pi(i)}^b$.

The data π and λ are called respectively the combinatorial datum and the length datum of T .

For any combinatorial datum π let us call $\Delta_\pi := \{\pi\} \times \mathbb{R}_+^d$ the set of all i.e.t.s with combinatorial datum π . For an i.e.t. T with d intervals and for any $i \in \{1, \dots, d\}$ we call u_i^t the left endpoint of I_i^t and u_i^b the left endpoint of I_i^b . Any u_i^t is a discontinuity of T and any u_i^b is a discontinuity of T^{-1} . We say that the combinatorial datum π is *admissible* if for any integer k with $1 \leq k < d$ we have $\pi\{1, \dots, k\} \neq \{1, \dots, k\}$. A *connection* for T is a triple (q, p, n) with $1 < q \leq d$, $1 < p \leq d$ and $n \in \mathbb{N}$ such that $T^n u_q^b = u_p^t$. In particular, if T has no connection, then its combinatorial datum is admissible.

2.2.2. *Veech's construction.* — In this paragraph we describe a construction given by Veech in [14] and known as *zippered rectangles construction*. We follow the presentation of [8], even if our notation is slightly different. Let $d \geq 2$. For an admissible combinatorial datum π in S_d let Θ_π be the open convex polyhedral cone in \mathbb{R}^d of those τ such that for any $1 \leq i \leq d - 1$ we have

$$\sum_{j \leq i} \tau_j > 0 \text{ and } \sum_{\pi(j) < i} \tau_j < 0.$$

The cone Θ_π is never empty since it contains the vector τ with coordinates $\tau_i := \pi(i) - i$. We say that a vector τ in Θ_π is a *suspension datum* for π .

Consider a pair (λ, τ) with $\lambda \in \Delta_\pi$ and $\tau \in \Theta_\pi$. We introduce the complex vector $\zeta = \lambda + \sqrt{-1}\tau \in \mathbb{C}^d$, then for any $i \in \{1, \dots, d\}$ we define two complex numbers

$$\xi_i^t := \sum_{j < i} \zeta_j \text{ and } \xi_i^b := \sum_{\pi(j) < i} \zeta_j.$$

Observe that $\xi_1^t = \xi_1^b = 0$. Condition $\tau \in \Theta_\pi$ means that for any i with $1 < i \leq d$ we have $\Im(\xi_i^t) > 0$ and $\Im(\xi_i^b) < 0$. We also define the *translation datum* $\theta \in \mathbb{C}^d$ setting $\theta_i := \xi_{\pi(i)}^b - \xi_i^t$ for any i , then we decompose it as $\theta = \delta - \sqrt{-1}h$ with $\delta, h \in \mathbb{R}^d$. Since π is admissible, we have $h_i > 0$ for any $i \in \{1, \dots, d\}$. Finally we set $\xi_* = \sum_{i=1}^d \xi_i$, $\lambda_* := \sum_{i=1}^d \lambda_i$ and $\tau_* = \sum_{i=1}^d \tau_i$.

The data (π, λ) define an i.e.t. T , that we may assume acting on the interval $I = (0, \lambda_*)$. If u_i^t and u_i^b denote the singularities for T and T^{-1} we have

$$u_i^t := \sum_{j < i} \lambda_j \text{ and } u_i^b := \sum_{\pi(j) < i} \lambda_j.$$

Observe that $u_i^t = \Re(\xi_i^t)$ and $u_i^b = \Re(\xi_i^b)$ for all $i = 1, \dots, d$. We construct a translation surface $X = X(\pi, \lambda, \tau)$ containing I as horizontal segment and such that the construction in paragraph 2.2.1 gives exactly T as first return to I of the vertical flow on X . We embed I in the complex plane, identifying it with $(0, \lambda_*) \times \{0\}$ and accordingly we denote elements in $\mathbb{C} \cap I$ just with their real coordinate. We define $2d$ open rectangles in \mathbb{C} setting for any $i \in \{1, \dots, d\}$

$$R_i^t := (u_i^t, u_i^t + \lambda_i) \times (0, h_i)$$

$$R_i^b := (u_i^b, u_i^b + \lambda_{\pi^{-1}(i)}) \times (-h_{\pi^{-1}(i)}, 0).$$

In order to get a surface we zip together these rectangles with the identifications described below.

1. For each $i \in \{1, \dots, d\}$ the rectangle R_i^t is equivalent to the rectangle $R_{\pi(i)}^b$ under the translation by the translation datum θ_i .
2. For each i with $1 < i \leq d$ we paste together R_i^t and R_{i-1}^t along the common vertical open segment connecting u_i^t to ξ_i^t .

3. For each i with $1 < i \leq d$ we paste together R_i^b and R_{i-1}^b along the vertical open segment connecting ξ_i^b to u_i^b .
4. For any $i \in \{1, \dots, d\}$ we paste R_i^t to I along its lower horizontal boundary segment $(u_i^t, u_i^t + \lambda_i) \times \{0\}$.
5. For any $i \in \{1, \dots, d\}$ we paste R_i^b to I along its upper horizontal boundary segment $(u_i^b, u_i^b + \lambda_{\pi^{-1}(i)}) \times \{0\}$.
6. We observe that $h_{\pi^{-1}(d)} = \Im(\xi_{\pi^{-1}(d)}^t) - \Im(\xi_d^b) = \Im(\xi_{\pi^{-1}(d)+1}^t) - \tau_*$ and similarly $h_d = \tau_* - \Im(\xi_{\pi(d)+1}^b)$. If $\tau_* \geq 0$ then $h_{\pi^{-1}(d)} \geq \Im(\xi_{\pi^{-1}(d)+1}^t)$ and the translation by $\theta_{\pi^{-1}(d)}$ induces an identification between the vertical segment connecting $u_{\pi^{-1}(d)+1}^t + \sqrt{-1}h_{\pi^{-1}(d)}$ to $\xi_{\pi^{-1}(d)+1}^t$ and the vertical segment connecting λ_* to ξ_* . If $\tau_* < 0$ then $-h_d > \Im(\xi_{\pi(d)+1}^b)$ and the translation by $-\theta_d$ induces an identification between the vertical segment connecting $\xi_{\pi(d)+1}^b$ to $u_{\pi(d)+1}^b - \sqrt{-1}h_d$ and the vertical segment connecting ξ_* to λ_* .
7. Finally we add the origin 0 of \mathbb{C} , the points ξ_i^t with $1 < i \leq d$, the points ξ_i^b with $1 < \pi(i) \leq d$ and the point ξ_*

DEFINITION 2.2. — *For any admissible combinatorial datum π in S_d and for any pair of length-suspension data (λ, τ) for π we call $X(\pi, \lambda, \tau)$ the translation surface which is obtained following the procedure above. A translation surface X such that there exists a triple of data (π, λ, τ) as above satisfying $X = X(\pi, \lambda, \tau)$ is said representable with the Veech construction.*

Now we determine the stratum of the moduli space where the surface $X(\pi, \lambda, \tau)$ lies. Fix a small positive number ϵ and for $i \in \{2, \dots, d\}$ define two half-sectors $D_i^t := B(\xi_i^t, \epsilon) \cap R_i^t$ and $D_i^b := B(\xi_i^b, \epsilon) \cap R_i^b$ in the complex plane, where $B(z, \epsilon)$ denotes the euclidian ball with center z and radius ϵ . Our notation suggests that these half-sectors look like the letter D. Any D_i^t is equivalent to $D_{\pi(i)}^b$ under the translation by θ_i , therefore they are identified to the same half-sector in $X(\pi, \lambda, \tau)$. Similarly for $i \in \{1, \dots, d-1\}$ define two half-sectors $C_i^t := B(\xi_i^t, \epsilon) \cap R_{i+1}^t$ and $C_i^b := B(\xi_i^b, \epsilon) \cap R_{i+1}^b$, here the notation suggests that these half-sectors look like the letter C. As before C_i^t is equivalent to $C_{\pi(i)}^b$ under the translation by θ_i and they are identified to the same half-sector in $X(\pi, \lambda, \tau)$. We introduce the set $HS := \{(i, D)\}_{i \in \{2, \dots, d\}} \cup \{(i, C)\}_{i \in \{1, \dots, d-1\}}$, which has $2(d-1)$ elements, that we use as labels for the half-sectors introduced above. The labeling obviously associates the symbol (i, D) to any half-sector D_i^t and the symbol (i, C) to any half-sector C_i^t . Finally we introduce a bijection $\widehat{\pi}$ of the elements of HS defined by

$$\widehat{\pi}(i, D) := (\pi(i-1), C) \text{ and } \widehat{\pi}(i, C) := (\pi^{-1}(i+1), D).$$

Consider any singular point $p \in \Sigma$ for the surface $X(\pi, \lambda, \tau)$ and any half-sector D_i^t in p . Turning in counterclockwise sense around p we pass from D_i^t to C_{i-1}^t , which is identified to $C_{\pi(i-1)}^b$. Then turning in the same sense we pass from $C_{\pi(i-1)}^b$ to $D_{\pi(i-1)+1}^b$, which is equivalent to $D_{\pi^{-1}(j+1)}^t$, where $j = \pi(i - 1)$. It follows that for any $p \in \Sigma$, the number of half-sectors that we meet turning around p equals to the length of a cycle of $\widehat{\pi}$. We have proved the following Lemma.

LEMMA. — *For any triple of data (π, λ, τ) the translation surface $X(\pi, \lambda, \tau)$ lies in a stratum $\mathcal{H}(k_1, \dots, k_r)$ which depends only from the combinatorial datum π .*

We also recall the following well known result(see [16]).

LEMMA. — *For any X in $\mathcal{H}(k_1, \dots, k_r)$ without vertical saddle connections there exists an admissible $\pi \in S_{2g+r-1}$ and a pair of data $(\lambda, \tau) \in \Delta_\pi \times \Theta_\pi$ such that $X = X(\pi, \lambda, \tau)$, anyway the triple of data (π, λ, τ) representing X is not unique. In particular almost any X in $\mathcal{H}(k_1, \dots, k_r)$ is representable with the Veech construction.*

2.2.3. *Veech's local charts.* — Recall from §2.1.2 that in the definition of the Teichmüller space we fix a reference topological pair (M^*, Σ^*) . Moreover in order to define the period map we fix also a basis $\{\gamma_1^*, \dots, \gamma_d^*\}$ for $H_1(M^*, \Sigma^*, \mathbb{Z})$ (see §2.1.3).

According to the construction in §2.2.2, for $d \geq 2$ and for any admissible $\pi \in S_d$ we have a well defined map

$$(2.1) \quad \begin{aligned} \mathcal{I}_\pi : \Delta_\pi \times \Theta_\pi &\rightarrow \mathcal{H}(k_1, \dots, k_r) \\ (\lambda, \tau) &\mapsto X(\pi, \lambda, \tau). \end{aligned}$$

Consider any $(\lambda, \tau) \in \Delta_\pi \times \Theta_\pi$ and the associated translation surface $X = X(\pi, \lambda, \tau)$. Using the notation of §2.2.2, for any $i \in \{1, \dots, d\}$ we define $\widehat{\zeta}_i^t$ as the segment in \mathbb{C} connecting ξ_i^t to $\xi_i^t + \zeta_i$ and $\widehat{\zeta}_i^b$ as the segment in \mathbb{C} connecting ξ_i^b to $\xi_i^b + \zeta_{\pi^{-1}(i)}$. It is easy to see that for any $i \in \{1, \dots, d\}$ either $\widehat{\zeta}_i^t$ or $\widehat{\zeta}_i^b$ projects to a saddle connection in $X(\pi, \lambda, \tau)$, that we denote $\widehat{\zeta}_i$. It is also easy to see that the set of curves $\{\widehat{\zeta}_1, \dots, \widehat{\zeta}_d\}$ is a basis for $H_1(X, \Sigma, \mathbb{Z})$. Therefore we can define naturally an isotopy class $\phi(\pi, \lambda, \tau)$ of maps of pairs $\phi : (M^*, \Sigma^*) \rightarrow (X, \Sigma)$ such that for any $i \in \{1, \dots, d\}$ we have

$$\phi(\pi, \lambda, \tau)(\gamma_i^*) = \widehat{\zeta}_i,$$

where the equal means that the two curves are *isotopic*(not just *homologous*). It follows that the zippered rectangle construction induces naturally a marking $\phi(\pi, \lambda, \tau)$ of $X(\pi, \lambda, \tau)$, that is we have a map

$$(2.2) \quad \begin{aligned} \mathcal{J}_\pi^\mathcal{J} : \Delta_\pi \times \Theta_\pi &\rightarrow \mathcal{T}(k_1, \dots, k_r) \\ (\lambda, \tau) &\mapsto (X(\pi, \lambda, \tau), \phi(\pi, \lambda, \tau)). \end{aligned}$$

LEMMA 2.3. — *For any admissible π the map $\mathcal{J}_\pi^\mathcal{J}$ in (2.2) is an homeomorphism onto its image. Moreover the push-forward $\mathcal{J}_{\pi*}^\mathcal{J}\text{Leb}$ of the lebesgue measure on $\Delta_\pi \times \Theta_\pi$ is equivalent to the restriction of the smooth measure m to the image of $\mathcal{J}_\pi^\mathcal{J}$.*

Proof. — Since $\phi(\pi, \lambda, \tau)(\gamma_i^*) = \widehat{\zeta}_i$ for any $i \in \{1, \dots, d\}$, then $\Theta \circ \mathcal{J}_\pi^\mathcal{J}$ is the natural immersion of $\Delta_\pi \times \Theta_\pi$ in \mathbb{C}^{2g+r-1} , where g and r are determined by π and where Θ is the period map on the corresponding stratum of the Teichmüller space. Since Θ is a local homeomorphism, then also $\mathcal{J}_\pi^\mathcal{J}$ is, moreover it is injective, therefore is an homeomorphism. Since $\mathcal{J}_\pi^\mathcal{J}$ is a local inverse of Θ the second part of the statement is obvious. \square

COROLLARY 2.4. — *For any admissible π the map \mathcal{J}_π in (2.1) is continuous and open. Moreover the push-forward $\mathcal{J}_{\pi*}\text{Leb}$ of the lebesgue measure on $\Delta_\pi \times \Theta_\pi$ is equivalent to the restriction of the smooth measure μ to the image of \mathcal{J}_π .*

3. Dichotomy for Khinchin type condition

3.0.4. *A remark on decreasing monotone functions.* — Consider a decreasing monotone function $\varphi : (0, +\infty) \rightarrow (0, \infty)$. We can restrict φ to positive integers, obtaining a sequence $(\varphi(n))_{n \in \mathbb{N}^*}$. Since $\varphi(t)$ is monotone, then $\sum_{n=1}^{+\infty} \varphi(n) < +\infty$ if and only if $\int_1^{+\infty} \varphi(t) dt < +\infty$. For any real number $\theta > 1$ we can also consider the sequence $\psi_n := \theta^n \varphi(\theta^n)$ and it is an easy exercise in calculus to see that since $\varphi(t)$ is decreasing monotone than $\sum_{n=0}^{+\infty} \psi_n < +\infty$ if and only if $\int_1^{+\infty} \varphi(t) dt < +\infty$.

3.1. Proof of the convergent case of Theorem 1.1. — In this paragraph we prove Proposition 1.2 and therefore the convergent part of Theorem 1.1 too. Let X be any translation surface. $\text{SO}(2, \mathbb{R})$ acts on the stratum of X as subgroup of $\text{SL}(2, \mathbb{R})$ and we call X_θ the image of X under

$$R_\theta = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}.$$

Let $\varphi : (0, +\infty) \rightarrow (0, +\infty)$ be decreasing monotone with $\int_1^{+\infty} \varphi(t) dt < +\infty$. We show that for almost any $\theta \in [0, 2\pi)$ the set $\text{Hol}(X_\theta)$ contains just finitely many solutions of Equation (1.1). Since the flat metric of X_θ is the same as the flat metric of X , then the set of all saddle connections of X is also the set of

saddle connections of X_θ for any θ . It follows that $\text{Hol}(X_\theta) = R_\theta \text{Hol}(X)$ and in particular the length $|\text{Hol}(\gamma)|$ of a saddle connection γ is invariant under rotations. For any $v \in \text{Hol}(X)$ we define the set

$$I(v) := \{\theta \in [0, 2\pi); |\Re(R_\theta v)| \leq \varphi(|v|)\},$$

that is the set of those θ such that $\text{Hol}(X_\theta)$ contains $R_\theta v$ as solution of Equation (1.1). Our strategy is to prove that $\sum_{v \in \text{Hol}(X)} \text{Leb}(I(v)) < \infty$, then the classical Borel-Cantelli Lemma implies that almost any $\theta \in [0, 2\pi)$ is contained into a finite number of sets $I(v)$, which is equivalent to say that $\text{Hol}(X_\theta)$ contains just finitely many solutions of Equation (1.1) for almost any θ .

In [10] Masur proves that for any translation surface X the number $N(X, L)$ of elements v of $\text{Hol}(X)$ with norm $|v| \leq L$ has quadratic growth with L , that is there are two positive constants $c < C$ such that for any L big enough we have

$$cL^2 \leq N(X, L) \leq CL^2.$$

We write

$$\sum_{v \in \text{Hol}(X)} \text{Leb}(I(v)) = \sum_{|v| \leq 1} \text{Leb}(I(v)) + \sum_{k \in \mathbb{N}^*} \left(\sum_{2^{k-1} < |v| \leq 2^k} \text{Leb}(I(v)) \right).$$

We observe that $\Re(R_\theta v)/|v|$ is the sinus of the angle between $R_\theta v$ and the imaginary axis. Since φ is bounded, then for any $\epsilon > 0$ and for $|v|$ big enough we have $\text{Leb}(I(v)) < (2 + \epsilon)\varphi(|v|)/|v|$. Recalling that $t\varphi(t)$ is decreasing monotone and applying Masur’s estimate for $N(X, 2^k)$ with k big enough, we get

$$\sum_{2^{k-1} < |v| \leq 2^k} \text{Leb}(I(v)) \leq C(2 + \epsilon)2^{2k} \frac{\varphi(2^{k-1})}{2^{k-1}}.$$

It follows that the tail of the sum $\sum_{v \in \text{Hol}(X)} \text{Leb}(I(v))$ is controlled by the series $\sum_{k=1}^\infty 2^k \varphi(2^k)$, modulo a multiplicative constant. According to Remark 3.0.4, condition $\int_1^{+\infty} \varphi(t)dt < +\infty$ implies $\sum_{v \in \text{Hol}(X)} \text{Leb}(I(v)) < +\infty$. Proposition 1.2 is proved and the convergent part of Theorem 1.1 too.

3.2. Bundles of saddle connections

3.2.1. *Translation surfaces with frame.* — A *frame* for a translation surface X is the datum of r horizontal separatrices (S_1, \dots, S_r) such that S_i starts in $p_i \in \Sigma$ for any $i \in \{1, \dots, r\}$. We denote \widehat{X} the datum (X, S_1, \dots, S_r) of a translation surface with frame and $\widehat{\text{Transl}}(k_1, \dots, k_r)$ the set of translation surfaces with frame whose singularities have orders k_1, \dots, k_r . An equivalence relation is defined on this set setting $\widehat{X} \sim \widehat{X}'$ if there exists a diffeomorphism

$f : X \rightarrow X'$ as in the definition of the moduli space in §2.1.1 such that for any $i = 1, \dots, r$ we have

$$f(S_i) = S'_i,$$

where (S_1, \dots, S_r) and (S'_1, \dots, S'_r) are respectively the frame of \widehat{X} and the frame of \widehat{X}' . We denote $\widehat{\mathcal{H}}(k_1, \dots, k_r)$ the quotient, that we call moduli space of translation surfaces with frame. The map

$$\begin{array}{ccc} \widehat{\mathcal{H}}(k_1, \dots, k_r) & \rightarrow & \mathcal{H}(k_1, \dots, k_r) \\ \widehat{X} & \mapsto & X \end{array}$$

is a covering map of degree $\prod_{i=1}^r k_i$, therefore $\widehat{\mathcal{H}}(k_1, \dots, k_r)$ inherits the structure of complex orbifold of complex dimension $2g + r - 1$. Local inverses for the covering are described in §3.2.2.

3.2.2. *Local inverses for $\widehat{\mathcal{H}}(k_1, \dots, k_r)$.* — For any translation surface X let us consider a triangulation whose vertices are the points in Σ and whose edges are saddle connections. Let $\{T_1, \dots, T_N\}$ be the family of triangles in such triangulation. Any T_i admits an embedding in \mathbb{C} , thus cutting X along a proper subset of edges we can represent it as a polygon \mathcal{P} , or a finite union of polygons (still denoted \mathcal{P}), with sides in $\partial\mathcal{P}$ identified in pairs, identified sides being of course parallel and with the same length and orientation. We call \mathcal{P} a *planar development* of X . We observe that \mathcal{P} inherits from the triangulation of X the structure of a planar graph, whose faces are exactly the triangles $\{T_1, \dots, T_N\}$. To any $p_i \in \Sigma$ with $k_i > 1$ correspond k_i vertices in $\partial\mathcal{P}$ (and not in the interior of \mathcal{P}). Let X be any translation surface with a triangulation $\{T_1, \dots, T_N\}$ as above. It is easy to see that any such triangulated X admits a planar development \mathcal{P} without horizontal sides in $\partial\mathcal{P}$. It is also easy to see that for any X there exists an open neighborhood \mathcal{U} in the correspondent stratum of the moduli space such that any X' in \mathcal{U} admits a triangulation which is combinatorially the same as the triangulation of X . In particular any X' in \mathcal{U} admits a planar development \mathcal{P}' which is combinatorially identical to \mathcal{P} as a (planar) graph, moreover, if \mathcal{U} is sufficiently small, we can also assume that $\partial\mathcal{P}'$ does not contain horizontal sides.

Let X a translation surface with triangulation $\{T_1, \dots, T_N\}$ and with planar development \mathcal{P} without horizontal sides in its boundary and let \mathcal{U} be an open neighborhood of X as above. Finally let (S_1, \dots, S_r) be a frame for X . Consider X' in \mathcal{U} , with induced triangulation $\{T'_1, \dots, T'_N\}$, and any conical singularity p'_i for X' with order k_i . If $k_i = 1$ then there exists a unique horizontal separatrix S'_i for X' starting at p_i . On the other hand, if $k_i > 1$ then all vertices of \mathcal{P} corresponding to p_i belong to $\partial\mathcal{P}$, which does not contain horizontal sides, hence there exists a unique triangle $T_{l(i)}$ in the triangulation of X which contains in its interior the initial segment of S_i . Then we

define S'_i as the (unique) horizontal separatrix for X' beginning in the interior of the triangle $T'_{l(i)}$ in the triangulation of X' . We have a continuous choice of frame $X' \mapsto (S'_1, \dots, S'_r)$ for $X' \in \mathcal{U}$ and thus a continuous local inverse of the covering map $\mathcal{H}(k_1, \dots, k_r) \rightarrow \mathcal{H}(k_1, \dots, k_r)$

3.2.3. *Teichmüller flow on $\widehat{\mathcal{H}}(k_1, \dots, k_r)$.* — Let X be a translation surface and (S_1, \dots, S_r) be a frame for X . For any $t \in \mathbb{R}$ we define a frame for $\mathcal{F}_t X$ as follows. Consider a triangulation $\{T_1, \dots, T_N\}$ for X and a planar development \mathcal{P} without horizontal sides in its boundary. Let $\{T^t_1, \dots, T^t_N\}$ be the corresponding triangulation for $\mathcal{F}_t X$ with the same combinatorics as $\{T_1, \dots, T_N\}$ and where any triangle T^t_i is the affine image of T_i under the diagonal element of $SL(2, \mathbb{R})$ with trace $2 \cosh t$. Let \mathcal{P}^t be the corresponding planar development with the same combinatorics as \mathcal{P} . As before, for points $p_i \in \Sigma$ with $k_i = 1$ there is a unique horizontal separatrix S^t_i for $\mathcal{F}_t X$ starting at p_i . On the other hand, if $k_i > 1$ then all vertices of \mathcal{P} corresponding to p_i belong to $\partial\mathcal{P}$, which does not contain horizontal segments. Thus the beginning segment of S_i belongs to the interior of a triangle $T_{l(i)}$ in X and we define S^t_i as the horizontal separatrix for $\mathcal{F}_t X$ whose beginning segment belongs to the corresponding triangle $T^t_{l(i)}$. We get a frame (S^t_1, \dots, S^t_r) for $\mathcal{F}_t X$ and we observe that such frame does not depend from the choice of the triangulation $\{T_1, \dots, T_N\}$ and of the planar development \mathcal{P} . Thus we put $\mathcal{F}_t \widehat{X} := (\mathcal{F}_t X, S^t_1, \dots, S^t_r)$, which defines the Teichmüller flow \mathcal{F}_t on $\widehat{\mathcal{H}}(k_1, \dots, k_r)$.

The volume form on $\mathcal{H}(k_1, \dots, k_r)$ lifts to $\widehat{\mathcal{H}}(k_1, \dots, k_r)$, defining a smooth volume, that we still call μ . The hyperboloid $\widehat{\mathcal{H}}^{(1)}(k_1, \dots, k_r)$ of area-one translation surfaces with frame is obviously defined, together with the volume form $\mu^{(1)}$ induced by the decomposition of μ . Obviously \mathcal{F}_t preserves the measure μ on $\widehat{\mathcal{H}}(k_1, \dots, k_r)$, moreover it leaves invariant $\widehat{\mathcal{H}}^{(1)}(k_1, \dots, k_r)$ and $\mu^{(1)}$ is invariant under the restriction of \mathcal{F}_t to the area-one hyperboloid.

The definition of $\widehat{\mathcal{H}}(k_1, \dots, k_r)$ is conceptually the same as the definition of the double-orientable cover of a non-orientable manifold. The introduction of the covering is necessary because the choice of a frame is not invariant under \mathcal{F}_t for elements of $\mathcal{H}(k_1, \dots, k_r)$. For example in $\mathcal{H}(k_1, \dots, k_r)$ there are elements X admitting a *pseudo-Anosov* diffeomorphism such that, if T is the period of the associated closed orbit of \mathcal{F}_t and (S_1, \dots, S_r) is any frame on X , we have $(\mathcal{F}_T S_1, \dots, \mathcal{F}_T S_r) \neq (S_1, \dots, S_r)$.

3.2.4. *Bundles of saddle connections.* — Let \widehat{X} any element in $\widehat{\mathcal{H}}(k_1, \dots, k_r)$, let p_i, p_j be any pair of point in the singular set Σ of X and let $l \in \{1, \dots, k_i\}$ and $m \in \{1, \dots, k_j\}$.

DEFINITION 3.1. — A bundle of saddle connections is a set $\mathcal{C}^{(p_j, p_i, m, l)}(\widehat{X})$ of those saddle connections γ for the translation surface X which start in p_j , end in p_i and satisfy

$$\begin{aligned} 2(m-1)\pi &\leq \text{angle}(S_j, \gamma) < 2m\pi \\ 2(l-1)\pi &\leq \text{angle}(S_i, \gamma) < 2l\pi, \end{aligned}$$

where (S_1, \dots, S_r) is the frame for X corresponding to \widehat{X} . We denote $\text{Hol}^{(p_j, p_i, m, l)}(\widehat{X})$ the set of complex vectors $v = \int_\gamma w_X$ for γ in $\mathcal{C}^{(p_j, p_i, m, l)}(\widehat{X})$.

LEMMA 3.2. — Let \widehat{X} be any element in $\widehat{\mathcal{H}}(k_1, \dots, k_r)$. Let p_j, p_i be any pair of points in Σ and (m, l) be a pair of integers with $1 \leq m \leq k_j$ and $1 \leq l \leq k_i$. Then for any $t \in \mathbb{R}$ we have

$$\text{Hol}^{(p_j, p_i, m, l)}(\mathcal{F}_t \widehat{X}) = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \text{Hol}^{(p_j, p_i, m, l)}(\widehat{X}).$$

Proof. — The Teichmüller flow on $\mathcal{H}(k_1, \dots, k_r)$ obviously preserves saddle connections and the names of singular points in Σ . The statement says that the action of \mathcal{F}_t on $\widehat{\mathcal{H}}(k_1, \dots, k_r)$ preserves any bundle $\mathcal{C}^{(p_j, p_i, m, l)}(\widehat{X})$. Let \mathcal{P} be a planar development of X and let \mathcal{P}^t be the corresponding planar development for $\mathcal{F}_t X$. The datum of a frame (S_1, \dots, S_r) is equivalent to the datum of a set of r vertices of \mathcal{P} (as planar graph), any of them corresponding to a unique point in Σ . A saddle connection γ for X is represented in the planar development by a family of finitely many parallel segments in \mathcal{P} , the first and the last respectively starting and ending in two vertices of \mathcal{P} and all the others belonging to $\partial\mathcal{P}$ and therefore identified in pairs according to the identifications in $\partial\mathcal{P}$. A bundle $\mathcal{C}^{(p_j, p_i, m, l)}(\widehat{X})$ thus corresponds to those saddle connections which in the planar development start in v and end in v' , where (v, v') is a pair of vertices of \mathcal{P} which depends only from the datum (p_j, p_i, m, l) . Since the affine map between \mathcal{P} and \mathcal{P}^t preserves vertical and horizontal directions and the combinatorics of the two developments as planar graphs, the statement is evident. \square

3.2.5. *Relation with Veech's construction.* — Let us fix $d \geq 2$ and an admissible combinatorial datum $\pi \in S_d$. For (λ, τ) in $\Delta_\pi \times \Theta_\pi$ consider the translation surface $X(\pi, \lambda, \tau)$. We fix a small positive real number ϵ and for any $i \in \{2, \dots, d\}$ we define an horizontal segment $S_i^\epsilon : (0, \epsilon) \rightarrow \mathbb{C}$ in the complex plane by

$$S_i^\epsilon(t) := \xi_i^t + t(1, 0).$$

The projection to $X(\pi, \lambda, \tau)$ of the segment S_i^ϵ is the beginning segment of an horizontal separatrix in $X(\pi, \lambda, \tau)$, that we call S_i^{Veech} . There are exactly $d-1$

horizontal separatrices as above and they are all distinct. The orders of the singularities of $X(\pi, \lambda, \tau)$ satisfy $k_1 + \dots + k_r = 2g + r - 2 = d - 1$, therefore

$$\{S_2^{\text{Veech}}, \dots, S_d^{\text{Veech}}\}$$

is the set of all the horizontal separatrix for $X(\pi, \lambda, \tau)$.

DEFINITION 3.3. — *For a pair of integers (q, p) with $1 < q \leq d$ and $1 < p \leq d$ we denote $\mathcal{V}^{(q,p)}(\pi, \lambda, \tau)$ the set of those saddle connections γ for $X(\pi, \lambda, \tau)$ which start in the point of Σ where S_q^{Veech} starts, end in the point of Σ where S_p^{Veech} starts, and satisfy*

$$\begin{aligned} 0 &\leq \text{angle}(S_q^{\text{Veech}}, \gamma) < 2\pi \\ 0 &\leq \text{angle}(S_p^{\text{Veech}}, \gamma) < 2\pi. \end{aligned}$$

LEMMA 3.4. — *Let (λ, τ) in $\Delta_\pi \times \Theta_\pi$, consider $X = X(\pi, \lambda, \tau)$ in $\mathcal{H}(k_1, \dots, k_r)$ and take any pre-image \widehat{X} of X in $\widehat{\mathcal{H}}(k_1, \dots, k_r)$. Then for any pair of points p_j, p_i in Σ and any pair of integers (m, l) with $1 \leq m \leq k_j$ and $1 \leq l \leq k_i$ there exists a unique pair (q, p) in $\{2, \dots, d\}^2$ such that*

$$\mathcal{C}^{(p_j, p_i, m, l)}(\widehat{X}) = \mathcal{V}^{(q,p)}(\pi, \lambda, \tau).$$

Proof. — Let (S_1, \dots, S_r) be the frame on $X = X(\pi, \lambda, \tau)$ carried by \widehat{X} . Since $(S_2^{\text{Veech}}, \dots, S_d^{\text{Veech}})$ are all the horizontal separatrices of X , then for any pair of points p_j, p_i in Σ and any pair of integers (m, l) with $1 \leq m \leq k_j$ and $1 \leq l \leq k_i$ there exist a unique pair (q, p) in $\{2, \dots, d\}^2$ such that S_q^{Veech} starts in p_j (where S_j starts), S_p^{Veech} starts in p_i (where S_i starts) and we have

$$\text{angle}(S_q^{\text{Veech}}, S_j) = 2(m - 1)\pi \text{ and } \text{angle}(S_p^{\text{Veech}}, S_i) = 2(l - 1)\pi.$$

The statement in the Lemma is therefore evident. □

3.3. Combinatorially defined saddle connections. — Let $\pi \in S_d$ be an admissible combinatorial datum and (λ, τ) be a pair of length-suspension data in $\Delta_\pi \times \Theta_\pi$. We recall that if $T : I \rightarrow I$ is the i.e.t. defined by the data (π, λ) , then the interval I embeds naturally in the translation surface $X(\pi, \lambda, \tau)$ as an horizontal segment and T is the first return map to I of the vertical flow on $X(\pi, \lambda, \tau)$.

3.3.1. *Combinatorially defined homology classes.* — In the following we use the notation of Veech's construction for $X = X(\pi, \lambda, \tau)$ introduced in §2.2.2. We recall that to any complex vector $\zeta_i = \lambda_i + \sqrt{-1}\tau_i$ is naturally associated a saddle connection $\widehat{\zeta}_i$, which in particular defines an element $[\widehat{\zeta}_i]$ in $H_1(X, \Sigma, \mathbb{Z})$. The formulae

$$\xi_i^t = \sum_{j < i} \zeta_j ; \xi_i^b = \sum_{\pi(j) < \pi(i)} \zeta_j ; \theta_i = \xi_{\pi(i)}^b - \xi_i^t$$

extend formally on relative(or absolute)homology classes

$$[\xi_i^t] := \sum_{j < i} [\widehat{\zeta}_j] ; [\xi_i^b] := \sum_{\pi(j) < \pi(i)} [\widehat{\zeta}_j] ; [\theta_i] := [\xi_{\pi(i)}^b] - [\xi_i^t].$$

The homology classes above have a representant which is concatenation of saddle connections for the flat structure $X = X(\pi, \lambda, \tau)$. Anyway in general it is not possible to find a representant which is a saddle connection itself.

We define a piecewise constant map $[\theta] : I \rightarrow H_1(X, \mathbb{Z})$ by setting $[\theta](x) = [\theta_i]$ if $x \in I_i^t$. Then we consider the Birkhoff sum $S_n[\theta]$ of the function $[\theta]$ over T , that is

$$S_n[\theta](x) = [\theta](x) + \dots + [\theta](T^{n-1}x).$$

We fix a pair of indexes q, p with $1 < q \leq d$ and $1 < p \leq d$. If T has no connection we can iterate T on u_q^b infinitely many times and we get a sequence of elements in $H_1(X, \Sigma, \mathbb{Z})$ defined by

$$[\gamma]_{q,p,n} := [\xi_p^t] - [\xi_q^b] - S_n[\theta](u_q^b).$$

We can consider the translation datum θ as a piecewise constant function $\theta : I \rightarrow \mathbb{C}$ by setting $\theta(x) = \theta_i$ if $x \in I_i^t$. We call $S_n\theta$ the Birkhoff sum over T of the function θ . If $\gamma_{q,p,n}$ is a saddle connection for $X(\pi, \lambda, \tau)$ in the homology class $[\gamma]_{q,p,n}$ then we obviously have

$$(3.1) \quad \text{Hol}(\gamma_{q,p,n}) = \xi_p^t - \xi_q^b - S_n\theta(u_q^b).$$

3.3.2. *Reduced triples and saddle connections.* — Let $T : I \rightarrow I$ be an i.e.t. without connections and q, p be a pair with $1 < q \leq d$ and $1 < p \leq d$. For any $n \in \mathbb{N}$ we call $I(q, p, n)$ the open subinterval of I whose endpoints are $T^n(u_q^b)$ and u_p^t (their reciprocal order does not matter).

DEFINITION 3.5. — *Let T and (q, p, n) be respectively an i.e.t. and a triple as above. We say that (q, p, n) is a reduced triple for T if for any k in $\{0, \dots, n\}$ the pre-image $T^{-k}(I(q, p, n))$ of $I(q, p, n)$ does not contain in its interior any singularity u_p^t , for T or any singularity u_q^b , for T^{-1} .*

PROPOSITION 3.6. — *Let T be an i.e.t. without connections defined by the data (π, λ) (therefore π is admissible). Let τ be any suspension datum for π and let $X = X(\pi, \lambda, \tau)$ be the associated translation surface. For any triple (q, p, n) reduced for T there exists a saddle connection $\gamma_{q,p,n}$ for $X(\pi, \lambda, \tau)$ which belongs to the set $\mathcal{V}^{(q,p)}(\pi, \lambda, \tau)$ and satisfies (3.1).*

Proof. — We can suppose without losing in generality that $T^n u_q^b < u_p^t$. Since (q, p, n) is reduced for T , this is equivalent to say that $T^k u_q^b < T^{k-n} u_p^t$ for all $k = 0, \dots, n$. Call $I^{(k)}$ the open interval $(T^k u_q^b, T^{k-n} u_p^t)$, so in particular $I(q, p, n) = I^{(n)}$ and all the $I^{(k)}$ have the same length. The fact that (q, p, n) is reduced for T means that there exists a sequence $\alpha(0), \dots, \alpha(n)$ of indexes in $\{1, \dots, d\}$ such that $I^{(k)} \subset I_{\alpha(k)}^t$ for $k = 0, \dots, n$. In particular we observe that for any $k \in \{1, \dots, n\}$ we have

$$S_k \theta(u_q^b) = S_k \theta(T^{-n} u_p^t) = \theta_{\alpha(0)} + \dots + \theta_{\alpha(k-1)}.$$

We set $S_0 \theta := 0$ and for $k \in \{1, \dots, n\}$ we introduce the simplified notation $S_k \theta := \theta_{\alpha(0)} + \dots + \theta_{\alpha(k-1)}$ for the sum above.

Embed in \mathbb{C} the interval I where T acts identifying it with $(0, \lambda_*) \times \{0\}$, that is points in I are identified with complex numbers with zero imaginary part. For any $i \in \{1, \dots, d\}$, if R_i^t and R_i^b are the open rectangles introduced in §2.2.2, we have $\overline{R_i^t} \cap \mathbb{R} = I_i^t$ and $\overline{R_i^b} \cap \mathbb{R} = I_i^b$. We set $v := \xi_p^t - \xi_q^b - S_n \theta$, then we define $\widehat{\gamma}$ as the half line in the complex plane starting at ξ_q^b in the direction of v , that is $\widehat{\gamma}(t) := \xi_q^b + tv$. We observe that $\Re(v) = |I(q, p, n)|$. The path $t \mapsto \widehat{\gamma}(t)$ projects to a geodesic path $t \mapsto \gamma(t)$ in $X(\pi, \lambda, \tau)$ with $\gamma(0) \in \Sigma$. A priori it is possible that γ can be extended to \mathbb{R}_+ , but we will show that its maximal interval of definition is $(0, 1)$ with $\gamma(1) \in \Sigma$, that is γ is a saddle connection.

For any $i = 1, \dots, d$ let us call $\Pi_i : R_i^b \rightarrow X$ the projection from R_i^b to X , whose image is denoted \mathcal{R}_i . For $1 \leq i \leq d$ the rectangles \mathcal{R}_i are disjoint each other and intersect along their boundary. Let $T > 0$ be a positive real number such that γ is defined on $(0, T)$. Let $\mathcal{R}_{i(0)}, \dots, \mathcal{R}_{i(k)}$ be the ordered sequence of the open rectangles defined above that $\gamma(t)$ meets for $t \in (0, T)$. We observe that $\widehat{\gamma}$ starts in R_q^b , so $i(0) = q$ and for small t we have $\gamma(t) = \Pi_{i(0)}(\widehat{\gamma}(t))$. We may also assume that $\gamma(T)$ belongs to $\partial \mathcal{R}_{i(k)}$, since otherwise we can extend it till to the boundary. Let $0 = t_0 < \dots < t_k < t_{k+1} = T$ be the sequence of instants such that for all $l \in \{0, \dots, k\}$ we have

$$\gamma(t) \in \mathcal{R}_{i(l)} \text{ for } t_l < t < t_{l+1} \text{ and } \gamma(t_{l+1}) \in \partial \mathcal{R}_{i(l)}.$$

LEMMA. — *Let us suppose that the sequence $0 = t_0 < \dots < t_k < t_{k+1} = T$ satisfies $k \leq n$. Then for all $l \in \{0, \dots, k\}$ the following conditions hold.*

a(l): For $t_l < t < t_{l+1}$ we have $\widehat{\gamma}(t) + S_l\theta \in R_{i(l)}^b$ and

$$\gamma(t) = \Pi_{i(l)}(\widehat{\gamma}(t) + S_l\theta).$$

b(l): $t_{l+1} < 1$.

c(l): $\widehat{\gamma}(t_{l+1}) + S_l\theta \in I^{(l)} \subset R_{\alpha(l)}^t \cap \mathbb{R}$ and there exists some $t_{l+2} > t_{l+1}$ such that $\widehat{\gamma}(t) + S_l\theta \in R_{\alpha(l)}^t$ for $t_{l+1} < t < t_{l+2}$.

In particular we have $i(l) = \pi(\alpha(l - 1))$ for all $l \in \{1, \dots, k - 1\}$.

Proof. — The three conditions above are satisfied for $l = 0$. In particular a(0) is obvious, moreover $\widehat{\gamma}(1) = \xi_p^t - S_n\theta$ does not belong to the closure of $R_{i(0)}^b$, so $t_1 < 1$ and b(0) also holds. Then we have $\Re(\widehat{\gamma}(t_1) - \widehat{\gamma}(0)) = t_1\Re(v) < |I^{(0)}|$, and since $I^{(0)}$ is a subinterval of $I_{i(0)}^b$, condition $\gamma(t_1) \in \partial\mathcal{R}_{i(0)}$ implies $\widehat{\gamma}(t_1) \in \partial R_{i(0)}^b \cap \mathbb{R}$. In particular we have $|\widehat{\gamma}(t_1) - u_{i(0)}^b| < |I^{(0)}|$ and therefore $\widehat{\gamma}(t_1)$ belongs to $I^{(0)}$. Since $I^{(0)} \subset I_{\alpha(0)}^t = R_{\pi(\alpha(0))}^t \cap \mathbb{R}$ then $\widehat{\gamma}(t) \in R_{\alpha(0)}^t$ for $t_1 < t < t_2$ and c(0) follows.

Now we consider $l \in \{1, \dots, k\}$ and we suppose that a(j), b(j) and c(j) hold for all $j \in \{1, \dots, l - 1\}$. The rectangle $R_{\alpha(l-1)}^t$ is equivalent to $R_{\pi(\alpha(l-1))}^b$ via the translation by $\theta_{\alpha(l-1)}$, so condition c(l - 1) implies condition a(l) and we necessarily have $i(l) = \pi(\alpha(l - 1))$. Moreover $\widehat{\gamma}(1) + S_l\theta = \xi_p^t - (S_n\theta - S_l\theta)$, which does not belong to the closure of $R_{i(l)}^b$, thus $t_{l+1} < 1$ and b(l) follows too. Finally we have

$$\Re(\widehat{\gamma}(t_{l+1}) + S_l\theta) = \Re(\xi_p^b + S_l\theta) + t_{l+1}\Re(v) = T^l u_{i(0)}^b + t_{l+1}|I^{(l)}|.$$

Condition $t_{l+1} < 1$ implies that $\Re(\widehat{\gamma}(t_{l+1}) + S_l\theta) \in I^{(l)} \subset I_{i(l)}^b$, where the inclusion of $I^{(l)}$ into $I_{i(l)}^b$ follows from the fact that (q, p, n) is reduced. Therefore condition $\gamma(t_{l+1}) \in \partial\mathcal{R}_{i(l)}$ implies $\widehat{\gamma}(t_{l+1}) + S_l\theta \in \partial R_{i(l)}^b \cap \mathbb{R}$ and in particular $\widehat{\gamma}(t_{l+1}) + S_l\theta \in I^{(l)}$. Since $I_{\alpha(l)}^t = R_{\alpha(l)}^t \cap \mathbb{R}$ contains $I^{(l)}$ then c(l + 1) holds.

The last part of the statement is proved in the inductive proof of a(l). The lemma is proved. □

The lemma implies that $\gamma(t)$ passes through the rectangles $\mathcal{R}_{i(0)}, \dots, \mathcal{R}_{i(n)}$ in X , where $i(0) = j$ and $i(l) = \pi(\alpha(l - 1))$ for all $l \in \{1, \dots, n\}$. Then we have a sequence $0 = t_0 < \dots < t_{n+1} < 1$ of instants such that for all $l \in \{0, \dots, n\}$ we have

$$\gamma(t) = \Pi_{i(l)}(\widehat{\gamma}(t) + S_l\theta) \text{ and } \gamma(t_{l+1}) \in \partial\mathcal{R}_{i(l)}.$$

Condition c(n) says that $\widehat{\gamma}(t_{n+1}) + S_n\theta$ belongs to $\partial R_{\alpha(n)}^t$, but $\gamma(t_{n+1}) \notin \Sigma$. Since $\widehat{\gamma}$ is a straight line in the plane with $\widehat{\gamma}(1) + S_n\theta = \xi_p^t \in \partial R_{\alpha(n)}^t$, then the instant t_{n+2} in c(n) equals to 1. It follows that $\gamma(t)$ can be extended for $t_{n+1} < t < 1$ as the projection from $R_{\alpha(n)}^t$ to X of $\widehat{\gamma}(t) + S_n\theta$. Then we evidently have $\gamma(1) \in \Sigma$, thus γ is a saddle connection. It is also evident that

$\text{Hol}(\gamma) = v = \xi_p^t - \xi_q^b - S_n\theta$ and that γ belongs to the set $\mathcal{V}^{(q,p)}(\pi, \lambda, \tau)$ introduced in Definition 3.3. The proposition is proved. \square

3.4. Proof of the divergent case of Theorem 1.1. — Consider a function $\varphi : (0, +\infty) \rightarrow (0, +\infty)$ such that $t\varphi(t)$ is decreasing monotone and $\int_1^{+\infty} \varphi(t)dt = +\infty$. According to Remark 3.0.4 the second condition is equivalent to $\sum_{n=1}^{\infty} \varphi(n) = +\infty$.

3.4.1. *Approximation of connections for i.e.t.s.* — Let T be an i.e.t. without connections. We look at triples (q, p, n) with $1 < q \leq d, 1 < p \leq d$ and $n \in \mathbb{N}$ such that

$$(3.2) \quad \left| T^n(u_q^b) - u_p^t \right| < \varphi(n).$$

Equation (3.2) defines a diophantine condition for i.e.t.s inspired by classical Khinchin condition for real numbers. Such condition has been studied in [7], where we prove the following result.

THEOREM 3.7. — *Let $\varphi : \mathbb{N} \rightarrow (0, +\infty)$ be a positive sequence. For any admissible combinatorial datum $\pi \in S_d$ we have the following dichotomy:*

- (a) *If $\varphi(n)$ is decreasing monotone and $\sum_{n=1}^{+\infty} \varphi(n) < +\infty$ then Equation (3.2) has just finitely many solutions for almost any i.e.t. T in Δ_π .*
- (b) *If $n\varphi(n)$ is decreasing monotone and $\sum_{n=1}^{+\infty} \varphi(n) = +\infty$ then for any pair (q, p) with $1 < q \leq d$ and $1 < p \leq d$ and for almost any i.e.t. T in Δ_π there exists infinitely many triples (q, p, n) reduced for T which are solution of Equation (3.2).*

3.4.2. *Approximation of vertical saddle connections.* — Let T be an i.e.t. with d intervals and without connections defined by the data (π, λ) . Let τ be a suspension datum for π and $X = X(\pi, \lambda, \tau)$ be the associated translation structure. Recall from §2.2.2 the vector h in \mathbb{R}_+^d defined by $h_i = \Im(\xi_i^t - \xi_{\pi(i)}^b)$. We consider h as a piecewise constant function $h : I \rightarrow \mathbb{R}_+$ defined by $h(x) = h_i$ if $x \in I_i^t$ and we denote $S_n h$ its Birkhoff sum.

For a pair of integers (q, p) with $1 < q \leq d$ and $1 < p \leq d$ we consider a reduced triple (q, p, n) for T and the associated saddle connection $\gamma_{q,p,n}$ for the surface $X = X(\pi, \lambda, \tau)$ given by Proposition 3.6. Since $\text{Hol}(\gamma_{q,p,n}) = \xi_p^t - \xi_q^b - S_n\theta(u_q^b)$ then

$$(3.3) \quad \Re(\text{Hol}(\gamma_{q,p,n})) = T^n u_q^b - u_p^t$$

and we observe that $|T^n u_q^b - u_p^t| \leq \lambda_*$ for all $n \in \mathbb{N}$. On the other hand

$$\Im(\text{Hol}(\gamma_{q,p,n})) = S_n h(u_q^b) + \Im \xi_p^t - \Im \xi_q^b$$

and $S_n h(u_q^b) \geq n h_{min}$ where $h_{min} := \min_{k=1, \dots, d} h_k > 0$. Therefore for any $\epsilon > 0$ and any $n \in \mathbb{N}$ big enough we have

$$(3.4) \quad (1 - \epsilon) |\gamma_{q,p,n}| \leq S_n h(u_q^b) \leq |\gamma_{q,p,n}|.$$

LEMMA 3.8. — Consider a function φ as above with $\int_1^\infty \varphi(t) dt = +\infty$. Let T be an i.e.t. without connections and uniquely ergodic, defined by the data (π, λ) . Let (q, p) be a pair in $\{2, \dots, d\}^2$ and suppose that there exist infinitely many triples (q, p, n) reduced for T which are solutions of Equation (3.2) with respect to φ . Then for any suspension datum τ in Θ_π and any $\epsilon > 0$ there are infinitely many saddle connections γ for the translation surface $X = X(\pi, \lambda, \tau)$ which belong to the set $\mathcal{V}^{(q,p)}(\pi, \lambda, \tau)$ and satisfy

$$|\Re(\text{Hol}(\gamma))| < \varphi \left(\frac{1 - \epsilon |\text{Hol}(\gamma)|}{1 + \epsilon \text{Area}(X)} \right).$$

Proof. — According to the hypothesis we have infinitely many reduced triples (q, p, n) for T such that $|T^n u_q^b - u_p^t| < \varphi(n)$. Therefore Proposition 3.6 gives infinitely many saddle connections $\gamma_{q,p,n}$ in $\mathcal{V}^{(q,p)}(\pi, \lambda, \tau)$ for the translation surface $X(\pi, \lambda, \tau)$. Since T is uniquely ergodic then the Birkhoff average $\frac{1}{n} S_n h(u_q^b)$ converges to $\int_I h = \text{Area}(X)$ for $n \rightarrow \infty$, thus for any $\epsilon > 0$ and any $n \in \mathbb{N}$ big enough we have

$$(1 - \epsilon)n \text{Area}(X) < S_n h(u_q^b) < (1 + \epsilon)n \text{Area}(X).$$

Recall that $\varphi(t)$ is decreasing monotone (because $t\varphi(t)$ is). Thus the assertion follows applying Equation (3.3) and Equation (3.4). The proposition is proved. □

PROPOSITION 3.9. — Let φ be a function as above with $\int_1^\infty \varphi(t) dt = +\infty$. Let π be any admissible combinatorial datum in S_d and let (λ_0, τ_0) be any pair of length and suspension data in $\Delta_\pi \times \Theta_\pi$. Then there exists a neighborhood $U \subset \Delta_\pi \times \Theta_\pi$ of (λ_0, τ_0) such that for any $1 < q \leq d$ and $1 < p \leq d$ and for almost any $(\lambda, \tau) \in U$ the set $\mathcal{V}^{(q,p)}(\pi, \lambda, \tau)$ contains infinitely many saddle connections γ for the surface $X(\pi, \lambda, \tau)$ such that

$$|\Re(\text{Hol}(\gamma))| < \varphi(|\text{Hol}(\gamma)|).$$

Proof. — Let us fix any $\epsilon > 0$ and let us call $A_0 := \text{Area}(X(\pi, \lambda_0, \tau_0))$. Then we consider the function

$$\varphi_0(t) := \varphi \left(\frac{1 + \epsilon}{1 - \epsilon} 2A_0 t \right).$$

Our hypothesis on φ is obviously equivalent to $\int_1^\infty \varphi_0(t)dt = +\infty$. Then we consider a neighborhood $\Delta \subset \Delta_\pi$ of λ_0 and a neighborhood $\Theta \subset \Theta_\pi$ such that for $\lambda \in \Delta$ and $\tau \in \Theta$ we have

$$A_0/2 < \text{Area}(X(\pi, \lambda, \tau)) < 2A_0.$$

The product $U := \Delta \times \Theta$ is obviously a neighborhood of (λ_0, τ_0) contained in $\Delta_\pi \times \Theta_\pi$ and is the neighborhood where we will prove the Proposition. Since $t\varphi(t)$ is decreasing monotone and therefore $\varphi(t)$ also is, for $(\lambda, \tau) \in U$ and for $X = X(\pi, \lambda, \tau)$ we have

$$\varphi_0 \left(\frac{1 - \epsilon}{1 + \epsilon} \frac{t}{\text{Area}(X)} \right) < \varphi(t).$$

According to Theorem 3.7 there exists a subset $\Delta' \subset \Delta$ of full measure such that for almost any $\lambda \in \Delta'$ and for all $1 < q \leq d$ and $1 < p \leq d$ there are infinitely many triples (q, p, n) which are reduced for the i.e.t. $T = (\pi, \lambda)$ and satisfy

$$|T^m u_q^b - u_p^t| < \varphi_0(n).$$

Moreover, according to the celebrated result of Masur and Veech([9] and [14]), almost any T is uniquely ergodic, therefore we can assume that $T = (\pi, \lambda)$ is uniquely ergodic for all $\lambda \in \Delta'$. Lemma 3.8 therefore applies and we have that for all $\lambda \in \Delta'$ and all τ in Θ the set $\mathcal{V}^{(q,p)}(\pi, \lambda, \tau)$ contains infinitely many saddle connections γ for the translation surface $X = X(\pi, \lambda, \tau)$ such that

$$\Re(|\text{Hol}(\gamma)|) < \varphi_0 \left(\frac{1 - \epsilon}{1 + \epsilon} \frac{|\text{Hol}(\gamma)|}{\text{Area}(X)} \right) < \varphi(|\text{Hol}(\gamma)|).$$

We conclude observing that according to the Fubini argument $\Delta' \times \Theta$ has full measure in U . The proposition is proved. □

3.4.3. *End of the proof.* — Let $\widehat{X}_0 = (X_0, S_1, \dots, S_r)$ be a translation surface with frame in $\widehat{\mathcal{H}}(k_1, \dots, k_r)$ such that X_0 has no vertical connection. As we explained in §2.2.2 this condition is satisfied by almost any X_0 in $\mathcal{H}(k_1, \dots, k_r)$ and therefore by almost any \widehat{X}_0 . Since X_0 has no vertical connections, there exist an admissible combinatorial datum π in S_d , where $d - 1 = k_1 + \dots + k_r$, and a pair (λ_0, τ_0) in $\Delta_\pi \times \Theta_\pi$ such that $X_0 = X(\pi, \lambda_0, \tau_0)$.

Consider the map $\mathcal{J}_\pi : \Delta_\pi \times \Theta_\pi \rightarrow \mathcal{H}(k_1, \dots, k_r)$ introduced in §2.2.3. Since $\widehat{\mathcal{H}}(k_1, \dots, k_r)$ is a covering space of $\mathcal{H}(k_1, \dots, k_r)$, then there exist an unique continuous map

$$\widehat{\mathcal{J}}_\pi : \Delta_\pi \times \Theta_\pi \rightarrow \widehat{\mathcal{H}}(k_1, \dots, k_r)$$

which lifts \mathcal{J}_π and such that $\widehat{\mathcal{J}}_\pi(\lambda_0, \tau_0) = \widehat{X}_0$. We consider the open neighborhood U of (λ_0, τ_0) given by Proposition 3.9. Let us consider any $(\lambda, \tau) \in U$ such that for any $1 < q \leq d$ and $1 < p \leq d$, the set $\mathcal{V}^{(q,p)}(\pi, \lambda, \tau)$ contains infinitely

many saddle connections γ whose holonomy $v = \text{Hol}(\gamma)$ satisfies Equation (1.1). According to Lemma 3.4, for the corresponding $\widehat{X} = \widehat{\mathcal{J}}_\pi(\lambda, \tau)$, for any pair of singular points p_i and p_j in Σ and any $1 \leq l \leq k_i$ and $1 \leq m \leq k_j$, the set $\text{Hol}^{(p_j, p_i, m, l)}(\widehat{X})$ contains infinitely many solutions of Equation (1.1). Corollary 2.4, at the level of the lift, says that $\Omega := \widehat{\mathcal{J}}_\pi(U)$ is an open neighborhood of \widehat{X}_0 . According to Proposition 3.9, the set of good $(\lambda, \tau) \in U$ has full measure, therefore the set of the corresponding good \widehat{X} in Ω has full measure too. We proved that a generic \widehat{X}_0 is contained in some open neighborhood $\Omega \subset \widehat{\mathcal{H}}(k_1, \dots, k_r)$ where the statement of the diverging case of Theorem 1.1 holds, therefore the statement holds almost everywhere on $\widehat{\mathcal{H}}(k_1, \dots, k_r)$ and the divergent case of Theorem 1.1 is proved.

4. Asymptotic laws

In this chapter we develop the dynamical consequences of Theorem 1.1 for the Teichmüller flow $\mathcal{G}_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$.

We first recall from §2.1.4 that the group of homotheties acts on $\widehat{\mathcal{H}}(k_1, \dots, k_r)$ as a subgroup of $\text{GL}(2, \mathbb{R})$ by $\widehat{X} \mapsto \rho\widehat{X}$ for $\rho \in \mathbb{R}_+$. The volume element on $\widehat{\mathcal{H}}(k_1, \dots, k_r)$ therefore decomposes as $d\mu = \rho^n d\rho \wedge d\mu^{(1)}$, where $n = 4g + 2r - 3$ and $d\mu^{(1)}$ is the volume element on the smooth hyperboloid $\widehat{\mathcal{H}}^{(1)}(k_1, \dots, k_r)$ of area one translation surfaces with frame.

Equation (1.1) is not covariant under homotheties, indeed if γ is a saddle connection for a translation surface X , changing X with ρX condition $|\Re(\text{Hol}(\gamma))| < \varphi(|\text{Hol}(\gamma)|)$ takes the form

$$|\Re(\text{Hol}(\gamma))| < \frac{\varphi(|\rho\text{Hol}(\gamma)|)}{\rho}.$$

Anyway for any $\rho \in \mathbb{R}_+$ we have $\int_1^{+\infty} \rho^{-1} \varphi(\rho t) dt = +\infty$ if and only if $\int_1^{+\infty} \varphi(t) dt = +\infty$. Since the divergence or the convergence of the integral is the only discriminant condition in Theorem 1.1 and as we have seen homotheties does not affect the condition, then Theorem 1.1 also holds on the smooth hyperboloid $\widehat{\mathcal{H}}^{(1)}(k_1, \dots, k_r)$ with respect to the measure $\mu^{(1)}$. Since the Teichmüller flow preserves the area of translation surfaces, it is more meaningful to study it on $\widehat{\mathcal{H}}^{(1)}(k_1, \dots, k_r)$.

4.1. Preliminary estimates. — In this paragraph we prove some useful estimates for saddle connections which do not involve any information about the bundle they belong to, therefore in all the paragraph we will work with simple translation surfaces without any choice of a frame. Let us consider any X in $\mathcal{H}(k_1, \dots, k_r)$ and a saddle connection γ for X . As we have seen, γ is a saddle connection for $\mathcal{F}_t X$ for any $t \in \mathbb{R}$. For $t \in \mathbb{R}$ let us denote $\text{Hol}(\gamma, t)$ the holonomy of γ with respect to the flat structure $\mathcal{F}_t X$. Then we set

$$\Re(\gamma, t) := \Re(\text{Hol}(\gamma, t)) ; \Im(\gamma, t) := \Im(\text{Hol}(\gamma, t)) ; |\gamma|_t := |\text{Hol}(\gamma, t)|.$$

For simplicity we write $|\gamma|$ for $|\gamma|_0$. We also define

$$\text{Area}(\gamma, t) := |\Re(\gamma, t)| \cdot |\Im(\gamma, t)| \text{ and } \cot(\gamma, t) := \frac{\Re(\gamma, t)}{\Im(\gamma, t)}.$$

Since $\Re(\gamma, t) = e^t \Re(\gamma, 0)$ and $\Im(\gamma, t) = e^{-t} \Im(\gamma, 0)$, then $\text{Area}(\gamma, t)$ is constant in t , and we denote it simply $\text{Area}(\gamma)$. On the other hand we have $\cot(\gamma, t) = e^{2t} \cot(\gamma, 0)$. It is easy to see that for any t these two quantities satisfy the relation

$$(4.1) \quad \text{Area}(\gamma) = |\gamma|_t^2 \frac{|\cot(\gamma, t)|}{1 + \cot^2(\gamma, t)}.$$

LEMMA 4.1. — *Let X be any translation surface and γ any saddle connection for X . If for some instant $t \geq 0$ we have $|\gamma|_t < 1$ then $t > \log |\gamma|$.*

Proof. — Since $\cot(\gamma, t) = e^{2t} \cot(\gamma, 0)$, then Equation (4.1) implies that for any t we have

$$|\gamma|^2 \frac{1}{1 + \cot^2(\gamma, 0)} = |\gamma|_t^2 \frac{e^{2t}}{1 + \cot^2(\gamma, t)}.$$

We observe that $|\cot(\gamma, t)| > |\cot(\gamma, 0)|$, since $t \geq 0$. Then $|\gamma|_t < 1$ implies that $e^{2t} > |\gamma|^2$ and the Lemma follows. □

Let us fix any X in $\mathcal{H}(k_1, \dots, k_r)$ and any saddle connection γ for X which is neither horizontal nor vertical. Let us denote $\tau := \tau(X, \gamma) \in \mathbb{R}$ the instant defined by

$$|\gamma|_\tau := \min\{|\gamma|_t; t \in \mathbb{R}\}.$$

Since the geometry of a translation surface is locally euclidian, the length $|\gamma|_t$ is minimal when $|\Re(\gamma, t)| = |\Im(\gamma, t)|$, that is $|\cot(\gamma, t)| = 1$ and we have

$$(4.2) \quad \tau(X, \gamma) = -\frac{1}{2} \log |\cot(\gamma, 0)|.$$

Since by definition $|\cot(\gamma, \tau)| = 1$, Equation (4.1) implies

$$(4.3) \quad |\gamma|_\tau^2 = 2\text{Area}(\gamma) \text{ and } |\gamma|_t^2 > 2\text{Area}(\gamma) \text{ for } t \neq \tau.$$

LEMMA 4.2. — *Let ϵ be any positive real number. For almost any X in $\mathcal{H}(k_1, \dots, k_r)$ and for all but finitely many saddle connection γ for X , the instant $\tau(X, \gamma)$ satisfies:*

$$\tau(X, \gamma) \leq (1 + \epsilon) \log |\gamma|.$$

Proof. — Since the function $t \mapsto t^{-(1+2\epsilon)}$ has convergent tail, the convergent case of Theorem 1.1 implies that for almost any X in $\mathcal{H}(k_1, \dots, k_r)$ and for all but finitely many saddle connections γ for X we have $|\Re(\text{Hol}(\gamma))| \geq |\gamma|^{-(1+2\epsilon)}$, which implies

$$|\cot(\gamma, 0)| \geq \frac{1}{|\Im(\gamma, 0)| \cdot |\gamma|^{1+2\epsilon}} \geq |\gamma|^{-2(1+\epsilon)}.$$

Then Equation (4.2) implies that for such X and for such γ the condition in the statement holds. The lemma is proved. \square

4.2. Proof of Theorem 1.3. — In this section we prove Theorem 1.3. Before entering into details we state the following useful Lemma, whose proof is a trivial exercise in calculus.

LEMMA 4.3. — *Let $\varphi : [1, +\infty) \rightarrow (0, +\infty)$ be a function such that $t\varphi(t)$ is decreasing monotone. The function $\widehat{\varphi} : [0, +\infty) \rightarrow (0, +\infty)$ defined by $\widehat{\varphi}(s) := e^s \varphi(e^s)$ is decreasing monotone. On the other hand, for any decreasing monotone function $\psi : [0, +\infty) \rightarrow (0, +\infty)$ there exists a unique function $\varphi : [1, +\infty) \rightarrow (0, +\infty)$ such that $t\varphi(t)$ is decreasing monotone and such that $\psi = \widehat{\varphi}$. The function φ is given by $\varphi(t) = \psi(\log t)/t$ and we have*

$$\int_0^\infty \psi(t) dt = \int_1^\infty \varphi(s) ds.$$

4.2.1. *Convergent case.* — Here we consider the case $\int_0^\infty \psi(t) dt < +\infty$. Our aim is to prove that for generic X we have

$$\lim_{t \rightarrow \infty} \frac{\text{Sys}(\mathcal{F}_t X)}{\sqrt{\psi(t)}} = \infty.$$

Our strategy is first to prove that for a generic X we have

$$(4.4) \quad \liminf_{t \rightarrow \infty} \frac{\text{Sys}(\mathcal{F}_t X)}{\sqrt{\psi(t)}} > 1.$$

Then, once (4.4) is proved, we observe that for any positive constant $C > 1$ the function $C^2\psi$ still has finite integral, thus we can change ψ with $C^2\psi$ and (4.4) still holds for $C^2\psi$, which is equivalent to say that $\liminf_{t \rightarrow \infty} \text{Sys}(\mathcal{F}_t X)/\sqrt{\psi(t)} > C$. We get that $\lim_{t \rightarrow \infty} \text{Sys}(\mathcal{F}_t X)/\sqrt{\psi(t)}$ exists and is equal to $+\infty$.

We pass to the proof of Equation (4.4). Let us suppose that there exists a positive measure subset \mathcal{J} of $\mathcal{H}^{(1)}(k_1, \dots, k_r)$ such that for any $X \in \mathcal{J}$ we have

$$\liminf_{t \rightarrow \infty} \frac{\text{Sys}(\mathcal{F}_t X)}{\sqrt{\psi(t)}} \leq 1.$$

Obviously we can also suppose that elements of \mathcal{J} have neither horizontal connection nor vertical connections. Fix any X in \mathcal{J} . There exists a sequence of instants $t_1 < \dots < t_n < \dots$ with $t_n \rightarrow +\infty$ and such that $\text{Sys}(\mathcal{F}_{t_n} X) \leq \sqrt{\psi(t_n)}$. Let γ_n be a sequence of saddle connections for X such that for every $n \in \mathbb{N}$ we have $\text{Sys}(\mathcal{F}_{t_n} X) = |\gamma_n|_{t_n}$. For any such γ_n we obviously have $|\gamma_n|_{t_n}^2 \leq \psi(t_n)$. Since ψ has convergent tail, in particular it goes to zero at infinity and it follows that $|\gamma_n|_{t_n} \rightarrow 0$ as $n \rightarrow \infty$. We observe that $|\gamma_n|_t \rightarrow \infty$ as $t \rightarrow +\infty$ for any fixed n . Since $t_n \rightarrow +\infty$, then any γ_n occurs in the sequence $(\gamma_n)_{n \in \mathbb{N}}$ finitely many times. Moreover on any translation surface X there are just finitely many saddle connections with length smaller than some fixed bound, thus it follows that $|\gamma_n|_0 \rightarrow \infty$ as $n \rightarrow \infty$ (i.e. the length at $t = 0$). We also observe that this implies that $\cot(\gamma_n, 0) \rightarrow 0$ as $n \rightarrow \infty$. Indeed if we have $|\cot(\gamma_{n(k)}, 0)| > c$ for some positive constant c depending on X and for a subsequence $(n(k))_{k \in \mathbb{N}}$, then $\text{Area}(\gamma_{n(k)}) \rightarrow +\infty$ as $k \rightarrow \infty$, according to Equation (4.1). But Equation (4.3) implies that this last condition is in contradiction with our assumption on γ_n .

Let us fix any $\epsilon > 0$. Since $\cot(\gamma_n, 0) \rightarrow 0$ then for any n big enough we have

$$|\Re(\gamma_n, 0)| < (1 + \epsilon) \frac{\text{Area}(\gamma_n)}{|\gamma_n|_0}$$

and since by Equation (4.3) we have $\text{Area}(\gamma_n) \leq (1/2)|\gamma_n|_{t_n}^2$ then it follows that

$$|\Re(\text{Hol}(\gamma_n))| = |\Re(\gamma_n, 0)| < \frac{1 + \epsilon}{2} \frac{|\gamma_n|_{t_n}^2}{|\gamma_n|_0} < \frac{1 + \epsilon}{2} \frac{\psi(t_n)}{|\gamma_n|_0}.$$

Finally, recalling Lemma 4.1, we have $t_n > \log(|\gamma_n|_0)$ and it follows that

$$|\Re(\text{Hol}(\gamma_n))| < \frac{1 + \epsilon}{2} \frac{\psi(\log(|\gamma_n|_0))}{|\gamma_n|_0}.$$

We set $\varphi(t) := \psi(\log t)/t$ and according to Lemma 4.3 we have $\int_1^{+\infty} \varphi(t) dt < +\infty$. On the other hand, for a set \mathcal{J} of positive measure in $\mathcal{H}(k_1, \dots, k_r)$ and for all X in \mathcal{J} we can find infinitely many saddle connection γ_n for X such that $|\Re(\text{Hol}(\gamma_n))| < \varphi(|\text{Hol}(\gamma_n)|)$, which is absurd, since it is in contradiction with the convergent part of Theorem 1.1. Equation (4.4) is proved and the convergent part of Theorem 1.3 too.

4.2.2. *Divergent case.* — Now we consider the case $\int_0^\infty \psi(t)dt = +\infty$. We recall that for any translation surface with frame \widehat{X} , the Teichmüller flow \mathcal{F}_t on $\widehat{\mathcal{H}}^{(1)}(k_1, \dots, k_r)$ preserves the bundles of saddle connections $\mathcal{C}^{(p_j, p_i, m, l)}(\widehat{X})$ (Lemma 3.2). Let us fix any pair of points $p_j, p_i \in \Sigma$ and any pair of indexes m, l with $m \in \{1, \dots, k_j\}$ and $l \in \{1, \dots, k_i\}$. We prove that for almost any $\widehat{X} \in \widehat{\mathcal{H}}^{(1)}(k_1, \dots, k_r)$ we have

$$\liminf_{t \rightarrow \infty} \frac{\text{Sys}^{(p_j, p_i, m, l)}(\mathcal{F}_t \widehat{X})}{\sqrt{\psi(t)}} = 0.$$

Similarly to the convergent case, our strategy is to prove that for generic \widehat{X} we have

$$(4.5) \quad \liminf_{t \rightarrow \infty} \frac{\text{Sys}^{(p_j, p_i, m, l)}(\mathcal{F}_t \widehat{X})}{\sqrt{\psi(t)}} < 1.$$

Once (4.5) is proved, we observe that for any positive constant $\epsilon > 0$ the function $\epsilon^2 \psi$ still has divergent integral. Therefore we can change ψ with $\epsilon^2 \psi$ and (4.5) still holds for $\epsilon^2 \psi$, which is equivalent to

$$\liminf_{t \rightarrow \infty} \frac{\text{Sys}^{(p_j, p_i, m, l)}(\mathcal{F}_t \widehat{X})}{\sqrt{\psi(t)}} < \epsilon.$$

Since the argument works for all ϵ we get $\liminf_{t \rightarrow \infty} \text{Sys}^{(p_j, p_i, m, l)}(\mathcal{F}_t \widehat{X}) / \sqrt{\psi(t)} = 0$.

We pass to the proof of (4.5). Let us fix any $\epsilon > 0$ and let us consider the function

$$\varphi(s) := \frac{\psi((1 + \epsilon) \log(s))}{s}$$

Lemma 4.3 (modulo a trivial change of variable) implies $\int_1^{+\infty} \varphi(t)dt = +\infty$, therefore the divergent part of Theorem 1.1 applies. For almost any $\widehat{X} \in \widehat{\mathcal{H}}^{(1)}(k_1, \dots, k_r)$ the bundle $\mathcal{C}^{(p_j, p_i, m, l)}(\widehat{X})$ contains infinitely many saddle connections γ_n such that $|\Re(\text{Hol}(\gamma_n))| < \varphi(|\text{Hol}(\gamma_n)|)$. It follows that we have

$$\text{Area}(\gamma_n) < |\Im(\gamma_n, 0)| \varphi(|\gamma_n|) < |\gamma_n| \varphi(|\gamma_n|),$$

where $|\gamma_n| = |\text{Hol}(\gamma_n)|$ is the initial length $|\gamma_n|_0$. For any such γ_n let us consider the instant $\tau(n)$ such that $|\gamma_n|_{\tau(n)} = \min\{|\gamma_n|_t; t \in \mathbb{R}\}$. According to Equation (4.3), the length $|\gamma_n|_{\tau(n)}$ of the saddle connection γ_n with respect to $\mathcal{F}_{\tau(n)} \widehat{X}$ satisfies

$$|\gamma_n|_{\tau(n)}^2 < (1/2) |\gamma_n| \varphi(|\gamma_n|).$$

Since we are considering a generic \widehat{X} we may assume that the underlying translation surface X belongs to the full measure subset of those elements for which

Lemma 4.2 applies. For the length of γ_n at $t = 0$ we have $|\gamma_n| \geq \exp\left(\frac{\tau(n)}{1+\epsilon}\right)$ and since the function $t\varphi(t)$ is decreasing monotone we get

$$|\gamma_n|_{\tau(n)}^2 < \frac{1}{2} \exp\left(\frac{\tau(n)}{1+\epsilon}\right) \varphi\left(\exp\left(\frac{\tau(n)}{1+\epsilon}\right)\right) = \psi(\tau(n)).$$

Since the action of \mathcal{F}_t on $\widehat{\mathcal{H}}^{(1)}(k_1, \dots, k_r)$ preserves bundles of saddle connections, then for any $\tau(n)$ the saddle connection γ_n belongs to the bundle $\mathcal{C}^{(p_j, p_i, m, l)}(\mathcal{F}_{\tau(n)}\widehat{X})$, therefore

$$\text{Sys}^{(p_j, p_i, m, l)}(\mathcal{F}_{\tau(n)}\widehat{X}) \leq |\gamma_n|_{\tau(n)} < \sqrt{\psi(\tau(n))}.$$

Finally we observe that $|\Re(\gamma_n, 0)| \rightarrow 0$ and $|\gamma_n|_0 \rightarrow \infty$ as $n \rightarrow \infty$, therefore $\cot(\gamma_n, 0) \rightarrow 0$ and Equation (4.2) implies that $\tau(n) \rightarrow \infty$. Equation (4.5) therefore follows and the divergent part of Theorem 1.3 is proved.

5. Punctured tori

This section is devoted to the proof of Theorem 1.6. Let us consider a function $\varphi : (0, +\infty) \rightarrow (0, \infty)$ such that $t\varphi(t)$ is decreasing monotone and $\int_1^{+\infty} \varphi(t)dt = +\infty$. Let X be any flat torus with r punctures p_1, \dots, p_r . We want to prove that for almost any $\theta \in [0, 2\pi)$ there exist at least r pairs of points (p_j, p_i) such that $\text{Hol}^{(p_j, p_i)}(X_\theta)$ contains infinitely many solutions of Equation (1.1).

5.1. Veech’s construction for punctured flat tori

5.1.1. *First return to an horizontal section.* — Let X be a flat torus with r punctures and let $\Sigma = \{p_1, \dots, p_r\}$ be the set of its punctures. We suppose that there are not vertical saddle connections of X connecting points of Σ . Let I be an horizontal segment in X whose endpoints lie on two trajectories of the vertical field ∂_y , respectively ending and starting at the same point of Σ . It is a well known fact that the first return T of the vertical flow to I is an irrational rotation. Let us consider the length coordinate x on I , which gives an identification of I with the interval $(0, |I|) \subset \mathbb{R}$. There exists some $\delta \in (0, |I|)$ such that T acts by

$$T(x) = x + \delta \text{ if } 0 < x < |I| - \delta$$

$$T(x) = x - (|I| - \delta) \text{ if } |I| - \delta < x < |I|.$$

In particular T is an i.e.t. and $(0, |I| - \delta)$ and $(|I| - \delta, |I|)$ are the two maximal intervals where T acts as a translation. Anyway, since we want to keep track of all points in Σ , we consider a partition of I into $d = r + 1$ intervals I_1^t, \dots, I_d^t , any I_i^t being delimited by two endpoints whose positive trajectory under the

vertical flow ends in some point of Σ . Therefore T is described by a pair of data (π, λ) , where $\lambda_i = |I_i^t|$ for $i \in \{1, \dots, d\}$ and $\pi \in S_d$ is a *rotational combinatorial datum*, that is it satisfies

$$\pi(i) - i = \text{const} \pmod d.$$

With this notation T has a real discontinuity at the point on I whose coordinate is $u_{\pi^{-1}(1)}^t$ and is continuous at all u_i^t with $i \neq \pi^{-1}(1)$. Similarly T^{-1} is continuous at all points u_j^b with $j \neq \pi^1(1)$ and has a real discontinuity just at $u_{\pi(1)}^b$. We define the normalized length datum $\widehat{\lambda} := \frac{\lambda}{\lambda_*}$, where $\lambda_* = \sum_{i=1}^d \lambda_i$, and we consider the i.e.t. \widehat{T} defined by the data $(\pi, \widehat{\lambda})$. According to the discussion above, \widehat{T} is a rotation and it has two maximal intervals where it acts as a translation, anyway we look at all points \widehat{u}_p^t and \widehat{u}_q^b with $1 < q \leq d$ and $1 < p \leq d$. In terms of the data (π, λ) they are given by

$$\widehat{u}_p^t = \sum_{j < p} \widehat{\lambda}_j \text{ and } \widehat{u}_q^b = \sum_{\pi(j) < \pi(q)} \widehat{\lambda}_j.$$

In particular the *rotation number* of \widehat{T} is $\alpha = \alpha(\lambda) := \widehat{u}_{\pi(1)}^b$ and \widehat{T} acts by

$$\begin{aligned} \widehat{T}(x) &= x + \alpha \text{ if } 0 < x < 1 - \alpha \\ \widehat{T}(x) &= x - (1 - \alpha) \text{ if } 1 - \alpha < x < 1. \end{aligned}$$

We can reconstruct the original torus X from T with the Veech construction, that is there exists some $\tau \in \Delta_\pi$ such that $X = X(\pi, \lambda, \tau)$. In general, any translation surface obtained with the Veech construction from a rotational combinatorial datum $\pi \in S_d$ is a flat torus with $r = d - 1$ marked points.

5.1.2. *Section for the rotated vertical flow.* — Let $\pi \in S_{r+1}$ be a rotational combinatorial datum and for $(\lambda, \tau) \in \Delta_\pi \times \Theta_\pi$ let $X = X(\pi, \lambda, \tau)$ be the associated flat torus with r marked points. For θ in \mathbb{R} we define the vectors in \mathbb{R}^d

$$\lambda(\theta) := \cos \theta \lambda - \sin \theta \tau \text{ and } \tau(\theta) := \sin \theta \lambda + \cos \theta \tau.$$

There exists an open interval $J = J(\pi, \lambda, \tau) \subset \mathbb{R}$ containing 0 and depending on π, λ, τ such that $\lambda(\theta) \in \Delta_\pi$ and $\tau(\theta) \in \Theta_\pi$ for $\theta \in J$. For θ in the same interval the rotated torus X_θ can be therefore represented with the Veech construction as $X_\theta = X(\pi, \lambda(\theta), \tau(\theta))$. For $\theta \in \mathbb{R}$ we define $\lambda_*(\theta) = \sum_{i=1}^d \lambda_i(\theta)$ and $\widehat{\lambda}(\theta) := \frac{\lambda(\theta)}{\lambda_*(\theta)}$ and we consider the i.e.t. \widehat{T}_θ whose data are $(\pi, \widehat{\lambda}(\theta))$. It is easy to check that the map $\theta \mapsto \widehat{\lambda}(\theta)$ is the parametrization of a segment in the standard simplex $\Delta^{(1)} := \{\lambda \in \mathbb{R}_+^d; \lambda_* = 1\}$, that is we have

$$\widehat{\lambda}(\theta) = \widehat{\lambda} + s(\theta)v,$$

where $s : J \rightarrow \mathbb{R}$ is a smooth change of variable and v is a vector in \mathbb{R}^d . Their explicit expression is

$$s(\theta) = \frac{\tan \theta}{\lambda_* - \tau_* \tan \theta} \text{ and } v = \frac{\tau_*}{\lambda_*} \lambda - \tau,$$

where $\tau_* = \sum_{i=1}^d \tau_i$. It follows that for $\theta \in J(\pi, \lambda, \tau)$ and for $1 < q \leq d$ and $1 < p \leq d$ the marked points $\widehat{u}_p^t(\theta)$ and $\widehat{u}_q^b(\theta)$ in $(0, 1)$ are given by

$$\widehat{u}_p^t(\theta) = \widehat{u}_p^t(0) + s(\theta) \sum_{k < p} v_k \text{ and } \widehat{u}_q^b(\theta) = \widehat{u}_q^b(0) + s(\theta) \sum_{\pi(k) < \pi(q)} v_k.$$

According to the discussion above the only real singularities for \widehat{T}_θ and \widehat{T}_θ^{-1} are respectively $\widehat{u}_{\pi^{-1}(1)}^t(\theta)$ and $\widehat{u}_{\pi(1)}^b(\theta)$. In particular, observing that $\lambda_* \sum_{\pi(j) < \pi(1)} v_j = \text{Area}(X)$, the rotation number $\alpha(\theta)$ of \widehat{T}_θ is given by

$$\alpha(\theta) = \widehat{u}_{\pi(1)}^b(\theta) = \alpha(0) + s(\theta) \frac{\text{Area}(X)}{\lambda_*}.$$

Let us fix any triple of data (π, λ, τ) . We resume the discussion of these two paragraphs in the following Lemma

LEMMA 5.1. — *The map $\theta \mapsto \alpha(\theta)$, which assigns to any $\theta \in J(\pi, \lambda, \tau)$ the rotation number $\alpha(\theta)$ of \widehat{T}_θ , is a smooth change of variable. Moreover, for any $1 < i \leq d$ there exists two pairs of real numbers (A_i^t, B_i^t) and (A_i^b, B_i^b) , depending only from (π, λ, τ) , such that*

$$\widehat{u}_i^t(\theta) = A_i^t + \alpha(\theta)B_i^t \text{ and } \widehat{u}_i^b(\theta) = A_i^b + \alpha(\theta)B_i^b.$$

5.2. Proof of Theorem 1.6. — Let us fix a rotational $\pi \in S_d$ and a pair of length and suspension data $\lambda \in \Delta_\pi$ and $\tau \in \Theta_\pi$. Let $J = J(\pi, \lambda, \tau)$ be the interval such that $(\lambda_\theta, \tau_\theta) \in \Delta_\pi \times \Theta_\pi$ for all $\theta \in J$. For any such θ let T_θ be the i.e.t. whose data are (π, λ_θ) and for a pair (q, p) with $1 < q \leq d$ and $1 < p \leq d$ let $u_q^b(\theta)$ and $u_p^t(\theta)$ be the corresponding singularities. We look at triples (q, p, n) satisfying

$$(5.1) \quad |T_\theta^n u_q^b(\theta) - u_p^t(\theta)| < \varphi(n).$$

We call \mathcal{P}_θ the subset of $\{2, \dots, d\}^2$ of those pairs (q, p) as above such that there exist infinitely many triples (q, p, n) which are reduced for T_θ and satisfy (5.1).

LEMMA 5.2. — *For almost any $\theta \in J$ and for any pair (q, p) with $1 < q \leq d$ and $1 < p \leq d$ there exist infinitely many triples (q, p, n) which satisfy Equation (5.1), anyway these triples are not necessarily reduced for T_θ .*

Proof. — Let us fix any pair (q, p) with $1 < q \leq d$ and $1 < p \leq d$. Since \widehat{T}_θ is a rotation on $(0, 1)$ with rotation number $\alpha(\theta)$, for any $x \in (0, 1)$ and any $n \in \mathbb{N}$ we have $\widehat{T}_\theta^n(x) = \{x + n\alpha(\theta)\}$, where $\{\cdot\}$ denotes the fractionary part. Then we have

$$|T_\theta^n u_q^b(\theta) - u_p^t(\theta)| = \lambda_*(\theta) |\widehat{T}_\theta^n \widehat{u}_q^b(\theta) - \widehat{u}_p^t(\theta)| = \lambda_*(\theta) |\{\widehat{u}_q^b(\theta) + n\alpha(\theta)\} - \widehat{u}_p^t(\theta)|.$$

Since $\varphi(n) \rightarrow 0$ as $n \rightarrow \infty$, for any $\theta \in J$, in order to have infinite solutions $n \in \mathbb{N}$ of Equation (5.1), it is enough to find infinitely many solutions of

$$\{\widehat{u}_q^b(\theta) - \widehat{u}_p^t(\theta) + n\alpha(\theta)\} < \frac{\varphi(n)}{\lambda_*(\theta)}.$$

Let us consider the decomposition $J = \bigsqcup_{k \in \mathbb{Z}} J_k$, where $2^k \leq \lambda_*(\theta) < 2^{k+1}$ for $\theta \in J_k$. Let us fix $k \in \mathbb{Z}$ and let us consider the function $\varphi_k := 2^{-(k+1)}\varphi$. Using the formula in Lemma 5.1 for $\widehat{u}_q^b(\theta)$ and $\widehat{u}_p^t(\theta)$ it is evident that it is sufficient to have, for any $\theta \in J_k$, infinitely many solutions of

$$\{(B_q^b - B_p^t + n)\alpha(\theta) - (A_q^b - A_p^t)\} < \varphi_k(n).$$

Since $\int_1^\infty \varphi_k(t) dt = \infty$, Theorem A.1 in appendix A implies that there exist infinitely many $n \in \mathbb{N}$ satisfying the inequality above for almost any α , moreover according to Lemma 5.1, the change of variable $\alpha = \alpha(\theta)$ is smooth and in particular it preserves sets of measure zero. It follows that we have infinitely many solutions for almost any $\theta \in J_k$ too. Since the result holds for any k and the union is countable, the required condition holds for almost any $\theta \in J$. The lemma is proved. □

We need the following Lemma, which holds in general for any i.e.t. and not just for rotational ones.

LEMMA 5.3. — *For any i.e.t. T without connections there exists $\epsilon > 0$ such that the following is true. If the triple (q, p, N) is not reduced for T and satisfies*

$$|T^N u_q^b - u_p^t| < \epsilon$$

then there exist q' and p' with $1 < q' \leq d$ and $1 < p' \leq d$ and positive integers n and m with $n, m < N$ such that the triples (q, p', n) and (q', p, m) are reduced for T and satisfy

$$|T^n u_q^b - u_{p'}^t| < |T^N u_q^b - u_p^t| \text{ and } |T^m u_{q'}^b - u_p^t| < |T^N u_q^b - u_p^t|.$$

Proof. — We call $u(1) < \dots < u(2d - 2)$ the elements in the set $\{u_i^t; 1 < i \leq d\} \sqcup \{u_j^b; 1 < j \leq d\}$, displayed in increasing order. Then we put $2\epsilon := \min_{i=2, \dots, 2d-2} |u(i) - u(i - 1)|$, which is positive since T has not connections. In particular our choice implies that all the intervals I_i^t and I_j^b have length at least 2ϵ . Let (q, p, N) be a triple as in the statement and let $I(q, p, N)$ be the interval whose endpoints are $T^N(u_q^b)$ and u_p^t . Since (q, p, N) is not reduced for

T there exists some $k \in \{0, \dots, n\}$ and $l \in \{2, \dots, d\}$ such that $T^{-k}(I(q, p, N))$ contains in its interior either u_l^t or u_l^b , moreover it follows by our choice of ϵ that in fact $k \geq 1$. We consider the smallest k such that the last condition holds. By minimality we have that T^{-i} is a translation on $I(q, p, N)$ for any $i = 0, \dots, k$, that is $|T^{N-k}u_q^b - T^{-k}u_p^t| = |T^N u_q^b - u_p^t| < \epsilon$. Without any loss in generality we suppose that we have $u_l^t \in T^{-k}(I(q, p, n))$. We can also suppose that we have

$$T^{N-k}u_q^b < u_l^t < T^{-k}u_p^t.$$

We first look at the inequality on the right. By our choice of ϵ , no other singularity of T or T^{-1} is contained in the interval $(u_l^t, T^{-k}u_p^t)$ and by minimality of k this also holds for the iterates $T^i(u_l^t, T^{-k}u_p^t)$ for $i = 0, \dots, k$. The interval $(u_l^t, T^{-k}u_p^t)$ is mapped by T onto $(u_l^b, T^{-(k-1)}u_p^t)$. Then we apply T again $k - 1$ times and we get that the triple $(l, p, k - 1)$ is reduced and satisfies $|T^{k-1}u_l^b - u_p^t| < |T^N u_q^b - u_p^t|$.

Now we look at the inequality on the left in the condition above. It implies that the triple $(q, l, N - k)$ satisfies $|T^{N-k}u_q^b - u_l^t| < |T^N u_q^b - u_p^t|$. If $(q, l, N - k)$ is not reduced for T , we call $j(q, p, N)$ the number of singularities, both for T and T^{-1} , which are contained in the orbit

$$I(q, p, N), T^{-1}(I(q, p, N)), \dots, T^{-N}(I(q, p, N)).$$

We observe that $j(q, l, N - k) < j(q, p, N)$, therefore we can start a descending induction procedure until we get a reduced triple. The lemma is proved. \square

COROLLARY 5.4. — *For almost any $\theta \in J$ the set \mathcal{P}_θ contains at least r elements.*

Proof. — Let us fix any θ in the full measure subset of J given by Lemma 5.2. Let (q, p) be any pair in $\{2, \dots, d\}^2$ and according to Lemma 5.2 consider a family of infinitely many triples (q, p, N_k) satisfying (5.1). According to Lemma 5.3, either there exists a subsequence N_i of N_k such that the corresponding triples (q, p, N_i) are reduced for T_θ , that is $(q, p) \in \mathcal{P}_\theta$, or there are two indexes q', p' with $1 < q' \leq d$ and $1 < p' \leq d$ with $(q, p') \in \mathcal{P}_\theta$ and $(q', p) \in \mathcal{P}_\theta$. We display the elements of $\{2, \dots, d\}$ in a $r \times r$ table. The argument above implies that any row and any column contain at least an element of \mathcal{P}_θ , that is the latter has at least r elements. \square

5.2.1. End of the proof. — Let X_0 be any flat torus with r marked points and let X be an element in $\text{SO}(2, \mathbb{R})X_0$ without horizontal and vertical saddle connections.

According to the discussion in §5.1.1, let us chose a rotational $\pi \in S_d$ and length-suspension data $(\lambda, \tau) \in \Delta_\pi \times \Theta_\pi$ in order to have $X = X(\pi, \lambda, \tau)$. Then consider the open interval $J = J(\pi, \lambda, \tau)$ and for θ in the full measure subset of

$J(\pi, \lambda, \tau)$ given by Corollary 5.4 consider the rotated torus $X_\theta = X(\pi, \lambda_\theta, \tau_\theta)$ and the associated i.e.t. T_θ whose combinatorial and length data are (π, λ_θ) . Observe that we obviously have $\text{Area}(X_\theta) = \text{Area}(X)$. Let us fix any $\epsilon > 0$ and consider the function

$$\varphi_\epsilon(t) := \varphi\left(\frac{1 + \epsilon}{1 - \epsilon} \text{Area}(X)t\right).$$

Since φ_ϵ differs from φ just for a linear change of variable, then it satisfies the same properties, that is $t\varphi_\epsilon(t)$ is decreasing monotone and $\int_1^{+\infty} \varphi_\epsilon(t)dt = +\infty$.

For any pair (q, p) in \mathcal{D}_θ there are infinitely many $n \in \mathbb{N}$ such that the corresponding triples (q, p, n) are reduced for T_θ and satisfy Equation (5.1). Lemma 3.8 in §3.4.2 implies that the set $\mathcal{V}^{(q,p)}(\pi, \lambda_\theta, \tau_\theta)$ contains infinitely many saddle connections γ for X_θ such that

$$|\Re(\text{Hol}(\gamma))| < \varphi_\epsilon\left(\frac{1 - \epsilon}{1 + \epsilon} \frac{|\text{Hol}(\gamma)|}{\text{Area}(X)}\right) = \varphi(|\text{Hol}(\gamma)|).$$

According to Corollary 5.4 the set \mathcal{D}_θ contains at least r pairs (q, p) . Lemma 3.4 in §3.2.5 implies that to any such (q, p) it corresponds an unique pair $(p_j, p_i) \in \Sigma^2$ such that the set $\mathcal{V}^{(q,p)}(\pi, \lambda_\theta, \tau_\theta)$ coincides with $\mathcal{C}^{(p_j, p_i)}(X_\theta)$. The set $\text{Hol}^{(p_j, p_i)}(X_\theta)$ therefore contains infinitely many solutions of Equation (1.1) and we have at least r different sets like $\text{Hol}^{(p_j, p_i)}(X_\theta)$.

We recall that the set of X without saddle connections has full measure in $\text{SO}(2, \mathbb{R})X_0$. Since for any such X exists an interval $J_X \subset \mathbb{R}$ containing zero such that X_θ satisfies the condition in Theorem 1.6 for almost any $\theta \in J_X$, then the condition holds almost everywhere on $\text{SO}(2, \mathbb{R})X_0$. Theorem 1.6 is proved.

Appendix A

An arithmetic result

Consider a function $\varphi : (0, +\infty) \rightarrow (0, \infty)$ and any pair (x, y) of real numbers. By restriction of φ to \mathbb{N}^* we can consider it as a sequence. For any $\alpha \in (0, 1)$ we look at the solutions $n \in \mathbb{N}$ of the following condition

$$(A.1) \quad \{(n + x)\alpha - y\} < \varphi(n).$$

THEOREM A.1. — *Let φ be a function as above.*

1. *If $\sum_{n=1}^\infty \varphi(n) < +\infty$, then for almost every $\alpha \in \mathbb{R}_+$ there exist finitely many solutions $n \in \mathbb{N}$ of Equation (A.1).*
2. *If $n\varphi(n)$ is decreasing monotone and $\sum_{n=1}^\infty \varphi(n) = \infty$ then for almost any α Equation (A.1) has infinitely many solutions $n \in \mathbb{N}$.*

The convergent part of the Theorem is a straightforward application of the easy half of the *Borel-Cantelli Lemma*, anyway we give here a short proof. We fix any pair of positive integers m and n and we introduce the function $f_{n,m}$ from $[m, m+1)$ to $[0, 1)$, defined by $f_{n,m}(\alpha) := \{(n+x)\alpha - y\}$, which is piecewise linear and has the same slope on all its branches. The branches of $f_{n,m}$ are defined on sub-intervals of $[m, m+1)$, if we exclude the leftmost and the rightmost of these intervals all the others have the same length (equal to $(n+x)^{-1}$) and restricted to them $f_{n,m}$ is surjective onto $[0, 1)$. It follows that for any $\epsilon > 0$ and any n big enough, for any subinterval I of $[0, 1)$ we have $\text{Leb}(f_{n,m}^{-1}(I)) \leq (1 + \epsilon)\text{Leb}(I)$. In particular for any m in \mathbb{N} and for any n big enough we have

$$\text{Leb}(f_{n,m}^{-1}(0, \varphi(n))) \leq (1 + \epsilon)\varphi(n).$$

Since φ has convergent series it follows that $\sum_{n=1}^{\infty} \text{Leb}(f_{n,m}^{-1}(0, \varphi(n))) < +\infty$ and accordingly to the Borel-Cantelli Lemma Equation (A.1) has just finitely many solutions for almost any α in $[m, m+1)$. The same argument works on any other interval $[m, m+1)$ and the proof of the convergent part is complete. The proof of the divergent part of Theorem A.1 is more complicated and its proof takes the remaining of this section.

Notation. — For any positive real number α we introduce the symbol $\hat{\alpha}$ to denote the vector $(1, \alpha) \in \mathbb{R}_+^2$, that is the vector with unitary horizontal component and slope equal to α . For any pair $v, w \in \mathbb{R}_+^2$ of linear independent vectors in the first quadrant we denote $[v, w]$ the matrix in $\text{GL}(2, \mathbb{R})$ whose first and second columns are respectively v and w . We define a total ordering on $\mathbb{P}\mathbb{R}_+^2$ writing for any such pair v, w

$$v \prec w \text{ if } \det[v, w] > 0 \text{ and } v \preceq w \text{ if } \det[v, w] \geq 0.$$

When there is no ambiguity with absolute value of a real number, for any borel subset I of \mathbb{R} , we write $|I|$ instead of $\text{Leb}(I)$.

A.1. Classical continued fraction. — In this paragraph we recall some classical facts about the *continued fraction algorithm*, which gives good rational approximations p/q of a real number α . We follow a geometrical interpretation. We are interested to real numbers admitting an infinite sequence of approximations, therefore we just consider irrational α . In this case, for any pair p, q of positive integers, denoting $r = (q, p) \in \mathbb{N}^2$, we have

$$\det[r, \hat{\alpha}] \neq 0.$$

Let us set $r_{-2} := (1, 0) \in \mathbb{N}^2$ and $r_{-1} := (0, 1) \in \mathbb{N}^2$. For any $\alpha \in \mathbb{R}_+ \setminus \mathbb{Q}$ we define a vector $r_0 = r_0(\alpha) \in \mathbb{N}^2$ by

$$r_0(\alpha) := a_0(\alpha)r_{-1} + r_{-2}$$

where $a_0 = a_0(\alpha)$ is the only positive integer such that $a_0 r_{-1} + r_{-2} \prec \widehat{\alpha} \prec (a_0 + 1)r_{-1} + r_{-2}$. Let us write the vector $r_0(\alpha)$ as $r_0(\alpha) = (q_0(\alpha), p_0(\alpha))$, with $q_0(\alpha), p_0(\alpha) \in \mathbb{N}$ (in particular we have $q_0 = 1$ and $p_0 = a_0$). Letting α vary we introduce the family of integer vectors $Q_0 := \{r_0(\alpha); \alpha \in \mathbb{R}_+ \setminus \mathbb{Q}\}$ and for any $r_0 \in Q_0$ we call $I(r_0)$ the interval where the function $\alpha \mapsto r_0(\alpha)$ is constant with value r_0 . We call \mathcal{Q}_0 the sigma-algebra generated by the intervals $I(r_0)$ with constant value for the function $\alpha \mapsto r_0(\alpha)$.

Now let us suppose that for any $\alpha \in \mathbb{R}_+ \setminus \mathbb{Q}$ and for all $i < n$ we have defined the vectors $r_i(\alpha) = (q_i(\alpha), p_i(\alpha))$, with $q_i(\alpha), p_i(\alpha) \in \mathbb{N}$, the associated family of integer vectors $Q_i := \{r_i(\alpha); \alpha \in \mathbb{R}_+ \setminus \mathbb{Q}\}$ and the sigma-algebra \mathcal{Q}_i generated by the intervals $I(r_i)$ with constant value for the function $\alpha \mapsto r_i(\alpha)$. Then we define by induction

$$r_n(\alpha) := a_n(\alpha)r_{n-1}(\alpha) + r_{n-2}(\alpha)$$

where $a_n(\alpha)$ is the positive integer defined according to the following condition

$$a_n r_{n-1} + r_{n-2} \prec \widehat{\alpha} \prec (a_n + 1)r_{n-1} + r_{n-2} \text{ if } n \text{ is even}$$

$$(a_n + 1)r_{n-1} + r_{n-2} \prec \widehat{\alpha} \prec a_n r_{n-1} + r_{n-2} \text{ if } n \text{ is odd .}$$

We write $r_n(\alpha) = (q_n(\alpha), p_n(\alpha))$ with $q_n(\alpha), p_n(\alpha) \in \mathbb{N}$. Letting α vary we define the family of integer vectors $Q_n := \{r_n(\alpha); \alpha \in \mathbb{R}_+ \setminus \mathbb{Q}\}$ and for any $r_n \in Q_n$ we call $I(r_n)$ the interval where the function $\alpha \mapsto r_n(\alpha)$ is constant with value r_n . We call \mathcal{Q}_n the sigma-algebra generated by the intervals $I(r_n)$ with constant value for the function $\alpha \mapsto r_n(\alpha)$. For any fixed $\alpha \in \mathbb{R}_+ \setminus \mathbb{Q}$ the vectors $r_n(\alpha)$ are a sequence of approximations of $\widehat{\alpha}$ in $\mathbb{P}\mathbb{R}_+^2$. The corresponding sequence p_n/q_n of rational numbers is the sequence of approximations of α with respect to the continued fraction algorithm. We recall some well known properties for these approximations, proofs can be found in [5].

1. For any α and any n we have $r_{2n} \prec \widehat{\alpha} \prec r_{2n+1}$. The even approximations r_{2n} give an increasing monotone sequence and the odd ones a decreasing monotone sequence. Moreover, if n is even and if r_n is any element of Q_n , there exist a decreasing sequence $(r_{n+1}(a))_{a \in \mathbb{N}}$ of elements of Q_{n+1} and an increasing sequence $(r_{n+2}(a))_{a \in \mathbb{N}}$ of elements of Q_{n+2} such that respectively $r_{n+1}(a) \rightarrow r_n$ and $r_{n+2}(a) \rightarrow r_n$ in $\mathbb{P}\mathbb{R}_+^2$ as $a \rightarrow \infty$. The analogous property holds for n odd.
2. For any $n \in \mathbb{N}$ let us introduce the vector $r'_n = r'_n(\alpha) := r_n(\alpha) + r_{n-1}(\alpha)$ and let us write it as $r'_n = (q'_n, p'_n)$ with $q'_n, p'_n \in \mathbb{N}$. We have $|\det[r_n, r'_n]| = 1$, that is $\{r_n, r'_n\}$ is a basis of \mathbb{Z}^2 .
3. For any n and any r_n we have $|I(r_n)| = (q_n q'_n)^{-1}$ and $q_{n+2}^2 > 2q_n^2$. We also have the uniform estimate $|I(r'_n)| < |I(r_n)| < 3|I(r'_n)|$ for two consecutive atoms $I(r_n)$ and $I(r'_n)$ of \mathcal{Q}_n .

4. There exists a constant $\gamma > 1$ such that for almost any α and any n big enough the denominators $q_n = q_n(\alpha)$ of the approximations of α satisfy $q_n < e^{\gamma n}$.

The family of sigma-algebras $\{Q_n\}_{n \in \mathbb{N}}$ satisfies $Q_n \subset Q_{n+1}$ for any $n \in \mathbb{N}$, that is it defines a *monotone increasing filtration*, moreover $\sup_n Q_n$ is the Borel sigma-algebra. It follows that any interval J contained in \mathbb{R}_+ , with its endpoints included or not, admits a decomposition $J = \bigsqcup_{n=0}^\infty J_n$, where J_0 is the maximal subset of J measurable with respect to Q_0 and J_n is defined iteratively for $n \geq 1$ as the maximal subset of $J \setminus \bigsqcup_{i=0}^{n-1} J_i$ which is measurable with respect to Q_n . To any interval J we associate an integer index $i(J)$ defined as the minimum of those $n \in \mathbb{N}$ such that $J_n \neq \emptyset$. For any $n > i(J)$ we introduce the sets $G_n = \bigsqcup_{l=i(J)}^n J_l$ and $\mathcal{G}_n := J \setminus G_n$. Applying the properties 1 and 3 of the continued fraction algorithm listed above it is easy to get the following Lemma, whose proof is left to the reader.

LEMMA A.2. — *There exist a positive constant λ with $0 < \lambda < 1$ such that for any interval $J \subset \mathbb{R}_+$ and for any $n > i(J)$ the decomposition $J = G_n \sqcup \mathcal{G}_n$ satisfies the following properties*

1. G_n is measurable with respect to Q_n and $|\mathcal{G}_n| \leq \lambda^{n-i(J)}|J|$.
2. If $I(r_n)$ is any atom of Q_n contained in G_n , we have $|I(r_n)| < \lambda^{n-i(J)}|J|$.
3. If J' is any non-empty connected component of \mathcal{G}_n , we have $i(J') \leq n+2$.

A.2. Twisted continued fraction. — In this paragraph we develop some useful machinery to solve Equation A.1, our exposure is inspired by a construction given in [15] at pages 105,106.

We fix a pair of real numbers (x, y) and we look for approximations of α of the form $\frac{j+y}{k+x}$ with $j, k \in \mathbb{Z}$. We use the same vectorial notation of paragraph A.1, thus we introduce the vector $v = (x, y) \in \mathbb{R}^2$ and for any $r = (k, j) \in \mathbb{Z}^2$ we consider the vector $s = r + v = (k + x, j + y)$, which is an element of the coset $\mathbb{Z}^2 + v$ of \mathbb{Z}^2 in \mathbb{R}^2 . We assume that α is irrational and moreover that for any $s \in \mathbb{Z}^2 + v$ we have

$$\det[\widehat{\alpha}, s] \neq 0.$$

We denote \mathcal{A} the set of those α in $\mathbb{R}_+ \setminus \mathbb{Q}$ such that the condition above is satisfied. It is evidently a full measure subset of \mathbb{R}_+ . Let us fix $\alpha \in \mathcal{A}$ and let $(r_k(\alpha))_{k \in \mathbb{N}}$ be the sequence of its classical approximations introduced in paragraph A.1. For any $k \in \mathbb{N}$ let us define

$$\Lambda(r_k) := \{ar_k + br'_k; a, b \in (0, 1]\},$$

that is the parallelogram in \mathbb{R}^2 spanned by the pair of vectors (r_k, r'_k) , which is a fundamental domain for the action of $SL(2, \mathbb{Z})$ on \mathbb{R}^2 . Then we define $v(r_k)$ as the only element in $(\mathbb{Z}^2 + v) \cap \Lambda(r_k)$.

We observe that for k even the k -th approximation $r_k(\alpha) = (q_k(\alpha), p_k(\alpha))$ satisfies $r_k \prec \widehat{\alpha} \prec r'_k$. In order to define the sequence of the *twisted approximations* $s_n(\alpha)$ of α we introduce a parameter $N \in \mathbb{N}^*$ and for $n \in \mathbb{N}$ we consider the $2N(n-1)$ -th approximation $r_{2N(n-1)} = r_{2N(n-1)}(\alpha)$ of α with respect to the continued fraction algorithm of paragraph A.1. Observe that $r_{2N(n-1)} \prec v(r_{2N(n-1)}) \prec r'_{2N(n-1)}$. We define s_n according to the following two cases:

- If $r_{2N(n-1)} \prec \widehat{\alpha} \prec v(r_{2N(n-1)})$ we define $\nu_n = \nu_n(\alpha)$ as the minimum of those ν in \mathbb{N} such that $v(r_{2N(n-1)}) + \nu r_{2N(n-1)} \prec \widehat{\alpha}$. In this case we always have $\nu_n(\alpha) \geq 1$. Then we define the n -th twisted approximation as

$$(A.2) \quad s_n := v(r_{2N(n-1)}) + \nu_n r_{2N(n-1)}.$$

In this case we also define the vector $s'_n = s'_n(\alpha)$ by $s'_n := s_n - r_{2N(n-1)}$.

- If $v(r_{2N(n-1)}) \prec \widehat{\alpha} \prec r'_{2N(n-1)}$ we define $\nu_n = \nu_n(\alpha)$ as the maximum of those ν in \mathbb{N} such that $v(r_{2N(n-1)}) + \nu r'_{2N(n-1)} \prec \widehat{\alpha}$. In this case we can also have $\nu_n(\alpha) = 0$. Then we define the n -th twisted approximation as

$$(A.3) \quad s_n := v(r_{2N(n-1)}) + \nu_n r'_{2N(n-1)}.$$

In this case we also define the vector $s'_n = s'_n(\alpha)$ by $s'_n := s_n + r'_{2N(n-1)}$.

Observe that by definition we have $s_n(\alpha) \prec \widehat{\alpha} \prec s'_n(\alpha)$ for any $n \in \mathbb{N}$. We write $s_n(\alpha) = (k_n(\alpha) + x, j_n(\alpha) + y)$ with $k_n(\alpha), j_n(\alpha) \in \mathbb{Z}$ and similarly $s'_n = (k'_n + x, j'_n + y)$, with $k'_n, j'_n \in \mathbb{Z}$. It is easy to see that $k_n, j_n, k'_n, j'_n \in \mathbb{N}^*$ definitively for any α .

If $s_n(\alpha)$ is defined by (A.2) we have $\det[s_n, s'_n] = \det[r_{2N(n-1)}, v(r_{2N(n-1)})]$. On the other hand, if $s_n(\alpha)$ is defined by (A.3) we have $\det[s_n, s'_n] = \det[v(r_{2N(n-1)}), r'_{2N(n-1)}]$. In both cases, recalling that $\det[r_{2N(n-1)}, r'_{2N(n-1)}] = 1$ and that $v(r_{2N(n-1)})$ belongs to the fundamental domain spanned by these two vectors, we have

$$(A.4) \quad 0 < \det[s_n, s'_n] \leq 1.$$

Letting α vary we define the family of vectors $P_n := \{s_n(\alpha); \alpha \in \mathcal{A}\}$, which is a subset of $(\mathbb{Z}^2 + v) \cap \mathbb{R}^2_+$, then for any $s_n \in \mathcal{P}_n$ we call $I(s_n)$ the interval where the function $\alpha \mapsto s_n(\alpha)$ is constant with value s_n . We call \mathcal{P}_n the sigma-algebra generated by the intervals $I(s_n)$ with constant value for the function $\alpha \mapsto s_n(\alpha)$. We observe that for any $n \in \mathbb{N}$ the sigma-algebra \mathcal{P}_n is a refinement of $\mathcal{Q}_{2N(n-1)}$, anyway the family $\{\mathcal{P}_n\}_{n \in \mathbb{N}^*}$ is not a filtration.

LEMMA A.3. — *For any $n \in \mathbb{N}$ and for any atom $I(s_n)$ of \mathcal{P}_n we have $i(I(s_n)) \leq 2N(n-1) + 3$.*

Proof. — Recall that s_n is defined either by Equation (A.2) or by Equation (A.3). We treat separately the two cases. If $r_{2N(n-1)} \prec s_n \prec v(r_{2N(n-1)})$ we observe that there exists $a \in \mathbb{N}^*$ such that

$$s_n \prec r'_{2N(n-1)} + ar_{2N(n-1)} \prec s'_n.$$

For the real numbers $\alpha \in I(r_{2N(n-1)})$ with $a_{2N(n-1)+1}(\alpha) = a + 1$ we have $r'_{2N(n-1)} + ar_{2N(n-1)} = r_{2N(n-1)-1} + (a + 1)r_{2N(n-1)} = r_{2N(n-1)+1}(\alpha)$, that is $I(s_n)$ contains an element of $\mathcal{Q}_{2N(n-1)+1}$. It follows from the properties of the continued fraction that there exist a sequence $r_{2N(n-1)+2}$ of elements of $\mathcal{Q}_{2N(n-1)+2}$ which cumulates to $r_{2N(n-1)+1}$. The corresponding intervals $I(r_{2N(n-1)+2})$ are therefore eventually contained in $I(s_n)$ and thus $i(I(s_n)) = 2N(n - 1) + 2$.

If $v(r_{2N(n-1)}) \preceq s_n \prec r'_{2N(n-1)}$ we observe that there exists $a \in \mathbb{N}^*$ such that

$$s_n \prec r_{2N(n-1)} + ar'_{2N(n-1)} \prec s'_n.$$

For the real numbers $\alpha \in I(r_{2N(n-1)})$ with $a_{2N(n-1)+1}(\alpha) = 1$ and $a_{2N(n-1)+2}(\alpha) = a$ we have $r'_{2N(n-1)} = r_{2N(n-1)+1}(\alpha)$ and $r_{2N(n-1)} + ar'_{2N(n-1)} = r_{2N(n-1)+2}(\alpha)$, that is $I(s_n)$ contains an element of $\mathcal{Q}_{2N(n-1)+2}$. Arguing as in the previous case we get $i(I(s_n)) = 2N(n - 1) + 3$. \square

A.3. A sufficient condition. — For any $n \in \mathbb{N}^*$ and any $s_n \in \mathcal{P}_n$ we call $\Upsilon[s_n]$ the bijective affine function from $I(s_n)$ to $[0, 1)$. In formulae:

$$\Upsilon[s_n](\alpha) := \frac{k'_n + x}{\det[s_n, s'_n]} \det[s_n, \hat{\alpha}].$$

For any $\alpha \in I(s_n)$ we have by definition $s_n \prec \hat{\alpha} \prec s'_n$, thus Equation (A.4) implies $0 < \det[s_n, \hat{\alpha}] < \det[s_n, s'_n] \leq 1$. Since $\det[s_n, \hat{\alpha}] = (k_n(\alpha) + x)\alpha - (j_n(\alpha) + y)$ we get $\det[s_n, \hat{\alpha}] = \{(k_n + x)\alpha - y\}$. It follows that in order to have infinitely many solution of Equation (A.1) it is enough to have infinitely many solutions n of

$$\Upsilon[s_n(\alpha)](\alpha) < \frac{k'_n + x}{\det[s_n, s'_n]} \varphi(k_n).$$

Lemma A.3 implies $k_n + x = \Re(s_n) < \Re(r_{2N(n-1)+3}) = q_{2N(n-1)+3}$ and similarly for k'_n . Since there exists a constant $\gamma > 1$ such that for almost any α and any n big enough we have $q_n < e^{\gamma n}$, then we can find a constant $\gamma' > 1$ such that for almost any α and any n big enough we have

$$k_n + x < e^{\gamma' n} \text{ and } k'_n + x < e^{\gamma' n}.$$

We define a positive sequence setting $\psi_n := e^{\gamma' n} \varphi(e^{\gamma' n})$. Since $t\varphi(t)$ is decreasing monotone, then ψ_n is decreasing monotone and for almost any α and any n big enough we have $\psi_n < (k'_n(\alpha) + x)\varphi(k_n(\alpha))$. Moreover according to Remark 3.0.4 condition $\int_1^{+\infty} \varphi(t)dt = +\infty$ is equivalent to $\sum_{n=1}^{+\infty} \psi_n = +\infty$. Finally,

according to Equation (A.4) we have $0 < \det[s_n, s'_n] < 1$, therefore in order to have infinitely many solution of Equation (A.1) for almost any α it is enough to have infinitely many solutions n of

$$(A.5) \quad \Upsilon[s_n(\alpha)](\alpha) < \psi_n.$$

It follows that in order to prove the divergent part of Theorem A.1 it is enough to prove the following Proposition

PROPOSITION A.4. — *If $\psi : \mathbb{N} \rightarrow \mathbb{R}_+$ is a decreasing monotone sequence such that $\sum_{n \in \mathbb{N}} \psi_n = +\infty$ then for almost every $\alpha \in \mathbb{R}_+$ there are infinitely many solutions $n \in \mathbb{N}$ of (A.5).*

A.4. End of the proof. — In this paragraph we prove Proposition A.4. Since any function $\Upsilon[s_n]$ has image in $(0, 1)$ we can assume that there exists some positive $\epsilon \in (0, 1)$ such that we definitively have $\psi_n < 1 - \epsilon$, otherwise the statement is trivially true. Then we make a choice of the parameter N in the definition of the twisted approximations requiring that $N \geq 3$ and $\lambda^{5-2N} < \epsilon$, where λ is the constant in Lemma A.2.

For any $n \in \mathbb{N}^*$ and any $s_n \in \mathcal{P}_n$ we define $J(s_n)$ as the subinterval of $I(s_n)$ of those α with $\Upsilon[s_n](\alpha) \geq \psi_n$. We evidently have $|J(s_n)| = (1 - \psi_n)|I(s_n)|$.

LEMMA A.5. — *For any $n \in \mathbb{N}^*$ and any $s_n \in \mathcal{P}_n$ we have $i(J(s_n)) < 2Nn$.*

Proof. — If for some s_n in some \mathcal{P}_n we have $i(J(s_n)) \geq 2Nn$, then $J(s_n)$ cannot contain elements r_{2Nn-2} of Q_{2Nn-2} , otherwise a sequence of intervals $I(r_{2Nn-1})$ would be contained in $J(s_n)$. It follows that there exists some atom of Q_{2Nn-2} which contains $J(s_n)$ as subinterval. On the other hand $i(I(s_n)) \leq 2N(n-1)+3$, according to Lemma A.3, so that $2Nn-2-i(I(s_n)) \geq 2N-5 \geq 1$. Then we can apply the decomposition in Lemma A.2 and we write $I(s_n) = G_{2Nn-2} \sqcup \mathcal{G}_{2Nn-2}$. Let $I(r_{2Nn-2})$ be the atom of Q_{2Nn-2} which contains $J(s_n)$. Either $I(r_{2Nn-2})$ is contained in G_{2Nn-2} or $J(s_n)$ is a subset of \mathcal{G}_{2Nn-2} . In both cases Lemma A.2 implies that $|J(s_n)| < \lambda^{2N-5}|I(s_n)|$. Finally we recall that have $|J(s_n)| > \epsilon|I(s_n)|$, because $\psi_n < 1 - \epsilon$, thus we get to an absurd since $\lambda^{2N-5} < \epsilon$. The lemma is proved. □

We set $\mathcal{E}_n := \bigcup_{s_n \in \mathcal{P}_n} J(s_n)$ and $\mathcal{E} := \bigcup_{M \in \mathbb{N}} \bigcap_{n \geq M} \mathcal{E}_n$. The statement in Proposition A.4 corresponds to $\text{Leb}(\mathcal{E}) = 0$ and in order to show it it is sufficient to prove that for any $M \in \mathbb{N}$ we have $\text{Leb}(\bigcap_{n \geq M} \mathcal{E}_n) = 0$. Thus we fix any $M \in \mathbb{N}$ and we re-define $\mathcal{E} := \bigcap_{m \geq M} \mathcal{E}_m$. To show that \mathcal{E} has null measure we define a family of sets $\widehat{\mathcal{E}}_n$ with $\widehat{\mathcal{E}}_n \supset \widehat{\mathcal{E}}_{n+1}$ and such that $\mathcal{E} \subset \widehat{\mathcal{E}}_n$ for any $n \in \mathbb{N}$ and $\text{Leb}(\widehat{\mathcal{E}}_n) \rightarrow 0$ as $n \rightarrow \infty$. The definition is given with the following iterative procedure.

- The first element of the family corresponds to $n = M$ and we set $\widehat{\mathcal{C}}_M := \mathcal{C}_M$. Any connected components of $\widehat{\mathcal{C}}_M$ is a sub-interval $J(s_M)$ of some atom $I(s_M)$ of \mathcal{P}_M and according to Lemma A.5 we have $i(J(s_M)) < 2NM$. Therefore Lemma A.2 implies that we can decompose it as

$$J(s_M) = G_{2NM} \sqcup \mathcal{G}_{2NM},$$

where the subset G_{2NM} is measurable with respect to the sigma-algebra \mathcal{Q}_{2NM} , and therefore with respect to \mathcal{P}_{M+1} , and where the remaining part satisfies $|\mathcal{G}_{2NM}| < \lambda|J(s_M)|$ (λ is the constant appearing in Lemma A.2). Moreover, if J' is any non-empty connected component of \mathcal{G}_{2NM} we have $i(J') \leq 2NM + 2$.

- We fix $n > M$ and suppose that the sets $\widehat{\mathcal{C}}_M, \dots, \widehat{\mathcal{C}}_{n-1}$ are defined. We also assume that for any $k \in \{M, \dots, n-1\}$ any connected component J of $\widehat{\mathcal{C}}_k$ satisfies $i(J) < 2Nk$ and we observe that the assumption is satisfied for $k = M$.

Let J be any connected component of $\widehat{\mathcal{C}}_{n-1}$. Since $i(J) < 2N(n-1)$, using Lemma A.2 we decompose it as

$$J = G_{2N(n-1)} \sqcup \mathcal{G}_{2N(n-1)},$$

where the subset $G_{2N(n-1)}$ is measurable with respect to the sigma-algebra $\mathcal{Q}_{2N(n-1)}$, and therefore with respect to \mathcal{P}_n , and where the remaining part satisfies $|\mathcal{G}_{2N(n-1)}| < \lambda|J(s_n)|$ and $i(J') \leq 2N(n-1) + 2$ for any non-empty connected component J' of $\mathcal{G}_{2N(n-1)}$. We define

$$G_{2N(n-1)} \cap \widehat{\mathcal{C}}_n := G_{2N(n-1)} \cap \mathcal{C}_n,$$

that is we consider the union of the subintervals $J(s_n)$ of those $I(s_n)$ contained in $G_{2N(n-1)}$. According to Lemma A.5 any new connected component $J(s_n)$ of $\widehat{\mathcal{C}}_n$ arising in this way satisfies $i(J(s_n)) < 2Nn$. Then we complete the definition setting

$$\mathcal{G}_{2N(n-1)} \cap \widehat{\mathcal{C}}_n := \mathcal{G}_{2N(n-1)},$$

that is the rest $\mathcal{G}_{2N(n-1)}$ passes unchanged to $\widehat{\mathcal{C}}_n$. According to Lemma A.2, for these connected component of $\widehat{\mathcal{C}}_n$, i.e. for the non empty connected components J' of $\mathcal{G}_{2N(n-1)}$ we have $i(J') \leq 2N(n-1) + 2 < 2Nn$.

For any $n \geq M$ the set $\widehat{\mathcal{C}}_n$ is therefore defined and if J is any of its connected component we have $i(J) < 2Nn$.

LEMMA A.6. — *For any $n \geq M$ and any connected component J of $\widehat{\mathcal{C}}_n$ we have*

$$|J \cap \widehat{\mathcal{C}}_{n+1}| < (1 - \delta\psi_{n+1})|J|,$$

where $\delta = 1 - \lambda$ and $\lambda \in (0, 1)$ is the constant in Lemma A.2.

Proof. — Let J be a connected component of $\widehat{\mathcal{C}}_n$. Since $i(J) < 2Nn$, we apply Lemma A.2 and we decompose it as $J = G_{2Nn} \sqcup \mathcal{G}_{2Nn}$. According to the inductive definition of $\widehat{\mathcal{C}}_{n+1}$ we have

$$J \cap \widehat{\mathcal{C}}_{n+1} = (G_{2Nn} \cap \mathcal{C}_{n+1}) \sqcup \mathcal{G}_{2Nn}.$$

Since G_{2Nn} is union of atoms $I(s_{n+1})$ of \mathcal{P}_{n+1} and for any such atom we have $|I(s_{n+1}) \cap \mathcal{C}_{n+1}| = |J(s_{n+1})| = (1 - \psi_{n+1})|I(s_{n+1})|$, we get $|G_{2Nn} \cap \mathcal{C}_{n+1}| = (1 - \psi_{n+1})|G_{2Nn}|$. On the other hand, Lemma A.2 also implies $|\mathcal{G}_{2Nn}| < \lambda|J|$, therefore summing the two contributions we have

$$\begin{aligned} |J \cap \widehat{\mathcal{C}}_{n+1}| &= (1 - \psi_{n+1})|G_{2Nn}| + |\mathcal{G}_{2Nn}| = \left((1 - \psi_{n+1}) \frac{|G_{2Nn}|}{|J|} + \frac{|\mathcal{G}_{2Nn}|}{|J|} \right) |J| = \\ &= \left(1 - (1 - \frac{|\mathcal{G}_{2Nn}|}{|J|})\psi_{n+1} \right) |J| < (1 - (1 - \lambda)\psi_{n+1})|J|. \end{aligned}$$

□

Now let us consider any connected component J of $\widehat{\mathcal{C}}_M$. Lemma A.6 implies that for any $n > M$ we have

$$|J \cap \widehat{\mathcal{C}}_n| < \prod_{k=M+1}^n (1 - \delta\psi_k)|J|.$$

For any $M \geq 1$ condition $\sum_{n \in \mathbb{N}} \psi_n = +\infty$ on the sequence ψ_n is equivalent to $\lim_{n \rightarrow \infty} \prod_{k=M+1}^n (1 - \delta\psi_k) = 0$. It follows that $|J \cap \widehat{\mathcal{C}}_n| \rightarrow 0$ as $n \rightarrow \infty$ for any connected component J of $\widehat{\mathcal{C}}_M$. Therefore $|J \cap \mathcal{C}| = 0$ for any J . Since $\widehat{\mathcal{C}}_M$ is countable union of its connected component we get $|\mathcal{C}| = 0$. Proposition A.4 is proved and therefore the divergent part of Theorem A.1 too.

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