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Chapter 1. The Basic Notions

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Chapter 1. The basic notions

1. Generalities on complex and projective manifolds

I recall the basic objects and maps one works with in (complex) algebraic geometry: complex manifolds and holomorphic maps between them, projective and affine varieties and rational and regular maps between them.

First, some NOTATION.

Points in $\mathbb{C}^n$ are denoted by $z = (z_1, \ldots, z_n)$ where $z_j = x_j + iy_j$ is the standard decomposition of z_j into real and imaginary parts. Introduce

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\Big(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\Big), \quad \frac{\partial}{\partial \bar{z}_j} = \frac{1}{2}\Big(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\Big)$$

and either consider these as a differential operators acting on complex valued functions or as elements in the complex tangent space to any point in $\mathbb{C}^n$. They give a real basis for this complex tangent space. For the dual space, the cotangent space, the dual basis is given by

$$dz_j = dx_j + idy_j, \quad d\bar{z}_j = dx_j - idy_j.$$

With this notation one has

$$df = \underbrace{\sum_j \frac{\partial f}{\partial z_j} dz_j}_{\partial f} + \underbrace{\sum_j \frac{\partial f}{\partial \bar{z}_j} d\bar{z}_j}_{\bar{\partial} f}.$$

Definition 1. A C^∞ function $f = u + iv$ on an open set $U \in \mathbb{C}^n$ is called *holomorphic* if one of the following equivalent conditions hold:

1 The Cauchy-Riemann equations hold on U:

$$\frac{\partial u}{\partial x_j} = \frac{\partial v}{\partial y_j}, \quad \frac{\partial u}{\partial y_j} = -\frac{\partial v}{\partial x_j}.$$

2 $\bar{\partial} f = 0$ on U.

3 f admits an absolutely convergent powerseries expansion around every point of U.

For the equivalence of these definitions, see e.g. [G-H], p.2.

Remark 2. A continuous function is called analytic if it admits a convergent powerseries around each point. By Osgood's lemma [Gu-Ro, p2.] such a function is holomorphic in each variable separately and conversely. Hence a continuous function which is analytic automatically satisfies the properties 1) and 2).

Definition 3. A Hausdorff topological space M with countable basis for the topology is an *n-dimensional complex manifold* if it has a covering U_i, $i \in I$ by open sets which admit homeomorphisms $\varphi_i : U_i \to V_i \subset \mathbb{C}^n$ with V_i open and such that for all $i \in I$ and $j \in I$ the map $\varphi_i \circ \varphi_j^{-1}$ is a holomorphic map on the open set $\varphi_j(U_i \cap U_j) \subset \mathbb{C}^n$ where it is defined.

A function f on an open set $U \subset M$ is called *holomorphic*, if for all $i \in I$ the function $f \circ \varphi_i^{-1}$ is holomorphic on the open set $\varphi_i(U \cap U_i) \subset \mathbb{C}^n$. Also, a collection of functions $z = (z_1, \ldots, z_n)$ on an open subset U of M is called a *holomorphic coordinate system* if $z \circ \varphi_i^{-1}$ is a holomorphic bijection from $\varphi_i(U \cap U_i)$ to $z(U \cap U_i)$ with holomorphic inverse. The open set on which a coordinate system can be given is then called a *chart*. Finally, a map $f : M \to N$ between complex manifolds is called *holomorphic* if it is given in terms of local holomorphic coordinates on N by holomorphic functions.

Let me give some examples. The first three generalize the examples in the introduction. The fourth example is a very important basic example: complex projective space.

Examples 1. Any open subset in $\mathbb{C}^n$ is a complex manifold. More generally any open subset of a complex manifold is a complex manifold.

2. Let Γ be a discrete lattice in $\mathbb{C}^n$, i.e. the set of points $\mathbb{Z}\gamma_1 + \mathbb{Z}\gamma_2 + \ldots \mathbb{Z}\gamma_m$ where $\gamma_1, \ldots, \gamma_m$ are m independent points (over the reals). Then the quotient $\mathbb{C}^n/\Gamma$ is a complex manifold. If $m = 2n$, i.e. if the points $\gamma_1, \ldots, \gamma_m$ form a real basis, the manifold $\mathbb{C}^n/\Gamma$ is compact and is called a *complex torus*.

3. The *Hopf manifolds* are defined as the quotient of $\mathbb{C}^n \setminus \{0\}$ by the infinite cyclic group generated by the homothety $z \mapsto 2z$. As an exercise one may show that any Hopf manifold is homeomorphic to $S^1 \times S^{2n-1}$. If $n = 2$ this is the *Hopf surface*.

4. The set of complex lines through the origin in $\mathbb{C}^{n+1}$ forms *complex projective space* $\mathbb{P}^n$ and is a compact n-dimensional complex manifold in a natural way with $Z_0, \ldots, Z_n$ as homogeneous coordinates. A natural collection of coordinate charts is obtained by taking $U_j = \{\ (Z_0, \ldots, Z_n) \in \mathbb{P}^n \mid Z_j \neq 0\ \}$ with coordinates $z^{(j)} = (Z_0/Z_j, \ldots, Z_{j-1}/Z_j, Z_{j+1}/Z_j, \ldots, Z_n/Z_j)$. These are called *affine coordinates* in U_j.

As with differentiable manifolds an important tool to produce new manifolds is the implicit function theorem, which is stated now together with the inverse function theorem. But first I recall the notion of the jacobian matrix $J(f)$ of a holomorphic map $f = (f_1, \ldots, f_m)$ defined on some open set $U \in \mathbb{C}^n$:

$$J(f) = \begin{pmatrix} \dfrac{\partial f_1}{\partial z_1} & \dfrac{\partial f_1}{\partial z_2} & \cdots & \dfrac{\partial f_1}{\partial z_n} \\ \dfrac{\partial f_2}{\partial z_1} & \dfrac{\partial f_2}{\partial z_2} & \cdots & \dfrac{\partial f_2}{\partial z_n} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial f_m}{\partial z_1} & \dfrac{\partial f_m}{\partial z_2} & \cdots & \dfrac{\partial f_m}{\partial z_n} \end{pmatrix}.$$

The jacobian matrix $J(f)$ is non-singular at $a \in U$ if $m = n$ and the matrix $J(f)(a)$ is invertible.

Theorem 4. (Inverse Function Theorem) *Let U and V be open sets in $\mathbb{C}^n$ with $0 \in U$ and let $f : U \to V$ be a holomorphic map whose jacobian is non-singular at the origin. Then f is one-to-one in a neighbourhood of the origin and the inverse is holomorphic near $f(0)$.*

Theorem 5. (Implicit Function Theorem) *Given an open neighbourhood $U \subset \mathbb{C}^n$ of the origin and $f : U \to \mathbb{C}^m$ holomorphic and vanishing at the origin. Assume that the $m \times m$-matrix*

$$\begin{pmatrix} \dfrac{\partial f_1}{\partial z_1} & \dfrac{\partial f_1}{\partial z_2} & \cdots & \dfrac{\partial f_1}{\partial z_m} \\ \dfrac{\partial f_2}{\partial z_1} & \dfrac{\partial f_2}{\partial z_2} & \cdots & \dfrac{\partial f_2}{\partial z_m} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial f_m}{\partial z_1} & \dfrac{\partial f_m}{\partial z_2} & \cdots & \dfrac{\partial f_m}{\partial z_m} \end{pmatrix}$$

is non-singular at the origin. Then there exist open neighbourhoods of V of $0 \in \mathbb{C}^m$ and W of $0 \in \mathbb{C}^{n-m}$ with $V \times W \subset U$, and a holomorphic map $g : W \to \mathbb{C}^m$ such that $f(z_1, \ldots, z_m, z_{m+1}, \ldots, z_n) = 0$ if and only if $(z_1, \ldots, z_m) = g(z_{m+1}, \ldots, z_n)$ for $z = (z_1, \ldots, z_n) \in V \times W$.

For a proof of these theorems see Problem 2.

Note that the Inverse Function Theorem shows that the map $(g, \mathbb{1}) : W \to V \times W \cap V(f)$ has a holomorphic inverse in a neighbourhood of 0 and hence gives a local chart on

$$V(f) := f^{-1}(0).$$

If the rank of the jacobian $J(f)$ is m everywhere on points of $V(f)$, one can always reorder the coordinates and shift the origin in such a way that one can apply the implicit function theorem at any point of $J(f)$ and produce a coordinate patch at that point. Also, in the overlap the transition functions are clearly holomorphic so that $V(f)$ is a complex manifold of dimension $n - m$ in its own right.

More generally, if M is a complex manifold and a closed subset N of M is locally in coordinate patches given by a function f which always has the same rank m on $V(f)$, the set N inherits the structure of a complex manifold of dimension $n - m$ which by definition is a *complex submanifold of M*. If one drops the condition about the jacobian one has an *analytic subset* of M. It is called *irreducible* if it is not the union of non-empty smaller analytic subsets. An irreducible analytic subset is also called an *analytic subvariety* and the terms *smooth subvariety* and *non-singular subvariety* mean the same as "submanifold".

Each analytic subset is the finite irredundant union of analytic subvarieties. This is by no means trivial but it won't be made use of in these notes. The interested reader can find a proof in [Gu-Ro, Chapter IIE]. The essential ingredients are the Weierstrass Preparation Theorem and Weierstrass Division Theorem.

In the algebraic setting there is the concept of (affine or projective) algebraic variety, to be introduced now. If in the preceding set-up $U = \mathbb{C}^n$ and $f = (f_1, \ldots, f_m)$ is a polynomial mapping defined on $\mathbb{C}^n$, the zero set $V(f)$ is called an affine algebraic set. This set actually only depends on the ideal $\mathfrak{I} = (f_1, \ldots, f_m)$ in $\mathbb{C}[z_1, \ldots, z_n]$ generated by the f_j and therefore usually is denoted by $V(\mathfrak{I})$. If $V(\mathfrak{I})$ is *irreducible*, i.e it is not the union of non-empty smaller affine sets it is called an *affine variety.* This is for instance the case if $\mathfrak{I}$ is a prime ideal.

It is well known that each affine algebraic set is the finite irredundant union of affine varieties in a unique way. This fact won't be made use of, but for the interested reader, I remark that this follows from the fact that the ring $\mathbb{C}[Z_0, \ldots, Z_n]$ is Noetherian; see [Reid, section 3].

Now, instead of holomorphic maps between affine varieties $V \subset \mathbb{C}^n$ and $W \subset \mathbb{C}^m$ one may consider *rational maps* i.e maps $f = (f_1, \ldots, f_m)$ whose coordinates f_j are rational functions in the affine coordinates of the source space: $f_j = \dfrac{P_j}{Q_j}$, $j = 1, \ldots, m$ with P_j, Q_j polynomials such that Q_j does not vanish identically on V. The rational map is not defined on the locus where some coordinate function f_j has a pole. If this is not the case, i.e. if all the f_j are polynomials one has a *regular* map.

A *Zariski-open subset* $U \subset \mathbb{C}^n$ by definition is the complement of an affine algebraic set. The Zariski-open sets form the *Zariski-topology* on $\mathbb{C}^n$. The induced topology on any affine variety V is called the Zariski-topology on V. One says that a rational function is regular on a Zariski-open subset U of an affine variety if it has no poles on U. For example, if f is any irreducible polynomial there is the basic Zariski-open set

$$U_f := \mathbb{C}^n \setminus V(f), \ f \in \mathbb{C}[X_1, \ldots, X_n]$$

and any regular function on U_f is of the form $\dfrac{P}{f^k}$ with P some polynomial and $k \geq 0$.

The regular functions on U form a ring, denoted $\mathcal{O}(U)$. For instance $\mathcal{O}(U_f)$ is the localisation of the ring $\mathbb{C}[z_1, \ldots, z_n]$ in the multiplicative system f^n, $n \geq 0$. See Appendix A1 for this notion.

The rational functions give the same function on $V = V(\mathfrak{I})$ if their difference is of the form $\dfrac{P}{Q}$ with $P \in \mathfrak{I}$. An equivalence class of such functions is called a *rational function on* V. The set of rational functions on V form the *function field* $\mathbb{C}(V)$ of V. It is the field of fractions of $\mathbb{C}[z_1, \ldots, z_n]/\mathfrak{I}$ and in fact of any of the rings $\mathcal{O}(U)$, U Zariski-open in V.

Next, if there is given a homogeneous polynomial F in the variables $(Z_0, \ldots, Z_n)$ its zero-set in a natural way defines a subset of $\mathbb{P}^n$ denoted $V(F)$. The zero locus of a set of homogeneous polynomials $F_1, \ldots, F_N$ only depends on the ideal $\mathfrak{I}$ they generate and is denoted by $V(\mathfrak{I})$. These loci are called *projective algebraic sets.*

If the ideal $\mathfrak{I}$ is a prime ideal, $V(\mathfrak{I})$ is a *projective algebraic variety.* This is for instance the case, if F is irreducible.

In the projective case, rational functions on V are functions $f = \dfrac{P}{Q}$ where P and Q are homogeneous polynomials of the same degree (otherwise f is not well defined) with Q not identically vanishing on V. These form the function field $\mathbb{C}(V)$ of V. A *rational map* $f : V \dashrightarrow \mathbb{P}^n$ is defined by demanding that the homogeneous coordinates of f be rational functions. If the map f can be given by polynomials, it is a *morphism* or *regular map* and these are examples of holomorphic maps.

Of course, on projective varieties one can introduce the Zariski-topology as well and as before one can speak of the ring of regular functions on any Zariski-open subset of a projective variety. Its field of fractions again coincides with the function field of the variety.

Also, each projective algebraic set is the finite irredundant union of projective varieties in a unique way. This follows from the corresponding assertion for affine varieties. See [Mu, section 2A] for details.

A projective variety is a complex subvariety of $\mathbb{P}^n$ but in general not a submanifold because of the jacobian condition. If it is, it is a *projective manifold.* So by definition an algebraic surface is a projective manifold of dimension two.

Example 6. A hypersurface $V(F)$ where F is a homogeneous polynomial of degree d. Consider the open set U_0 and for simplicity set $z^{(0)} = z$. The inhomogeneous polynomial

$$f(z_1, \ldots, z_n) = F(1, \frac{Z_1}{Z_0}, \ldots, \frac{Z_n}{Z_0}) = \left(\frac{1}{Z_0}\right)^d F(Z_0, \ldots, Z_n)$$

vanishes in U_0 precisely where F vanishes and if at a point of $V(f)$ some partial, say $\dfrac{\partial f}{\partial z_1}$ is non-zero, the implicit function theorem implies that $z_2, \ldots, z_n$ can be taken as local coordinates on $V(F)$ and hence that $V(F)$ is a manifold locally at that point. The locus where all the partials $\dfrac{\partial f}{\partial z_j}$, $j = 1, \ldots, n$ vanish on $V(f)$ is the set of non-manifold points, the so-called *singular set* $S(V(f))$.

To treat all coordinate patches simultaneously, recall Euler's formula

$$d \cdot F = Z_0 \frac{\partial F}{\partial Z_0} + \ldots + Z_n \frac{\partial F}{\partial Z_n}.$$

It follows that the singular set $S(V(F))$ of $V(F)$ is nothing but $V(\dfrac{\partial F}{\partial Z_0}, \ldots, \dfrac{\partial F}{\partial Z_n})$ and is a proper algebraic subset of $V(F)$. The Zariski-open complement $V(F) \setminus S(V(F))$ is a manifold of dimension $n - 1$.

For the general case see Problem 3.

Observe that there is no reason why a compact complex manifold should be projective or why a submanifold of $\mathbb{P}^n$ or more generally an irreducible subvariety would be projective, i.e. can be given as the zero locus of finitely many polynomials. For dimension one one has the basic

FACT *Any compact Riemann surface is projective.*

The proof uses Hodge Theory in some form. See Appendix 3.3.

In higher dimensions this is not true. The easiest example perhaps is the Hopf surface. Again see Appendix 3 for details, more particularly, see Example A3.6.

As to subvarieties of projective space, astonishingly enough, they are always projective:

Theorem 7. (Chow's Theorem) *Any subvariety of* $\mathbb{P}^n$ *is a projective variety.*

A rather self contained proof of Chow's Theorem can be found in [Mu]. For a considerably shorter proof see p.167 in [G-H]. This proof however uses the so called Proper Mapping Theorem, a partial proof of which is supplied in [G-H, p.395-400].

In the same vein one can show that holomorphic maps between complex projective manifolds are in fact morphisms, i.e given by rational functions. See problem 5.

Problems.

1.1. Let U be an open subset of $\mathbb{C}^n$ and let $f = (f_1, \dots, f_m)$ be a holomorphic map defined on U. Write $f_j = u_j + iv_j$ with u_j the real part and v_j the imaginary part of f_j. Recall that $z = (z_1, \dots, z_n) \in \mathbb{C}^n$ with $z_j = x_j + iy_j$. The differentiable map $f_{\mathbb{R}} = (u, v) : U \to \mathbb{R}^{2m}$ has a jacobian $J(f_{\mathbb{R}})$ of size $2n \times 2m$. If $n = m$ show that $\det J(f_{\mathbb{R}}) = |\det J(f)|^2$ and hence is positive if and only f is invertible. Deduce that any complex manifold is oriented in a natural way.

1.2. Prove the inverse and the the implicit function theorem.
Hint: Use the previous problem to see that one can use the ordinary inverse function theorem to find a differentiable inverse g for f and then prove that this map is in fact holomorphic by differentiating the relation $g(f(z)) = z$. See [G-H], p.18. The argument for the implicit function theorem is similar. Loc. cit. p.19.

1.3. Let F_j, $j = 1, \dots, N$ be homogeneous polynomials in $(Z_0, \dots, Z_n)$ defining the algebraic set $V := V(F_1, \dots, F_N)$ in $\mathbb{P}^n$. Consider the jacobian matrix $J(F_1, \dots, F_N)$. Prove:
(i) The locus where the rank of the Jacobian is k or less is an algebraic set. It is denoted by $J_k(V)$.
(ii) There is a minimal number m such that $J_m(V) \cap V = V$. If $J_{m-1}(V) \cap V = \emptyset$, the variety V is a manifold of dimension $n - m$. (In general, the $m \times m$ subdeterminants vanish in a proper subset of V, the singularity set of V and the complement is a manifold.)

1.4. Prove that the product of two projective varieties is projective.
Hint: use the Segre embedding $\mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^{nm+m+n}$. One may consult [Mu, section 2B] for details.

1.5. Prove that holomorphic maps between projective manifolds are morphisms (Consider the graph of the holomorphic map and apply the previous problem).

2. Vector bundles

Vector bundles live on manifolds, varieties etc. I recall their basic properties, discuss the principal examples such as the canonical bundle, line bundles related to divisors and the notion of an ample line bundle. Important results are the canonical bundle formula and the Bertini theorem on hyperplane sections. The first tells you how to compute the canonical bundle of a subvariety in terms of the canonical bundle of the variety and the normal bundle of the subvariety and will be used a lot to say something about the genus of curves on surfaces. Bertini's theorem will be used to construct smooth subvarieties of a given projective manifold.

Let M be a differentiable manifold. Let me recall the notion of a *differentiable vector bundle* on M. It consists of a collection of vector spaces $E_m, m \in M$ parametrized by M such that their union E, the *total space*, is a manifold and such that

1. The natural projection $p : E \to M$ which maps E_m to m is differentiable,
2. Every point $m \in M$ has an open neighbourhood U and a diffeomorphism

$$\varphi_U : p^{-1}U \to U \times T,$$

where T is some fixed vector space and where φ_U maps E_m linearly and isomorphically onto $m \times T$. If T is a complex vector space of dimension d the manifold E is called a *complex vector bundle* of rank d. For $d = 1$ it is called *line bundle.*

The vector space E_m is called the *fibre over* m and the maps φ_U are called *trivializations* and over non-empty intersections $U \cap V$ they can be compared:

$$\varphi_V^{-1}(m,t) = \varphi_U^{-1}(m, (\varphi_{UV}(m))(t)),$$

where $\varphi_{UV} : U \cap V \to GL(T)$ is differentiable and is called the *transition function.* These transition functions satisfy a certain compatibility rule

$$\varphi_{UV} \circ \varphi_{VW} \circ \varphi_{WU} = \mathbb{1}, \quad \text{(Cocycle relation).}$$

Conversely, given some covering of M by open sets U_i, $i \in I$ and a collection of transition functions φ_{ij} for subsets U_i and U_j having a non-empty intersection, define a set E by taking the disjoint union of the $U \times T$ and identify (m,t) and $(m, (\varphi_{ij}(m))t)$ whenever $m \in U_i \cap U_j$. This yields a vector bundle precisely if the above compatibility rule is valid as one can easily verify.

A *vector bundle homomorphism* between two vector bundles $p : E \to M$ and $p' : F \to M$ consists of a differentiable map $f : E \to F$ such that

1. $p = p' \circ f$ so that fibres go to fibres,
2. $f|E_m$ is linear.

If f is an diffeomorphism you have a *vector bundle isomorphism.*

For any vector bundle homomorphism $f : E \to F$ you can form the kernel $\ker(f)$, which consists of the union of the kernels of $f|E_m$. One can easily see that the kernel forms a vector bundle. Similarly one can form $\operatorname{im}(f) = \bigcup_{m \in M} \operatorname{im}(f|E_m)$, the *image bundle.* Often *exact sequences* of vector bundles arise. A sequence of vector bundle homomorphisms

$$E' \xrightarrow{f} E \xrightarrow{g} E''$$

is called *exact* at E if $\ker(g) = \operatorname{im}(f)$. A sequence of vector bundles

$$\ldots E_{i-1} \xrightarrow{f_{i-1}} E_i \xrightarrow{f_i} E_{i+1} \ldots$$

of arbitrary length, it is called *exact* if it is exact at all E_i. Especially, a sequence

$$0 \to E' \xrightarrow{f} E \xrightarrow{g} E'' \to 0$$

is exact if and only if f is injective, g is surjective and $\ker(g) = \operatorname{im}(f)$.

A *section* s of a vector bundle $p : E \to M$ is a differentiable map $s : M \to E$ such that $p \circ s = \mathrm{id}_M$. Sections of a vector bundle E form a vector space denoted by $\Gamma(E)$ or $H^0(M, E)$.

Examples

1. The trivial bundle $M \times T$.

2. If E is a bundle, a *subbundle* consists of a subset $F \subset E$ such that the projection and trivialisation of E gives F the structure of a bundle. For a subbundle $F \subset E$, the fibres F_m are subspaces of E_m and one can form $\bigcup_{m \in M} E_m/F_m$ which inherits the structure of a bundle E/F, the *quotient bundle.* If

$$0 \to E' \xrightarrow{f} E \xrightarrow{g} E'' \to 0,$$

is an exact sequence, f identifies E' with a subbundle of E and g induces an isomorphism of E/E' with E''.

3. The tangent bundle $T(M)$. Sections are *vector fields.*

4. If E is a vector bundle, any linear algebra construction done with the fibres yields a vector bundle. You already saw the examples of subbundles and quotient bundles. One can also form $E^\vee$, the dual bundle by taking $\bigcup_{m \in M} E_m^\vee$ or the exterior powers $\bigwedge^k E$ by forming $\bigcup_{m \in M} \bigwedge^k E_m$. The highest wedge with $k = \dim T$ is also called the *determinant line bundle*

$$\det(E) = \bigwedge^{\operatorname{rank} E} E.$$

Combining these operations and applying them to the previous example you get the cotangent bundle or bundle of one-forms and its k-fold exterior power, the bundle of k-forms:

$$\mathcal{E}^k(M) = \bigwedge^k T(M)^\vee.$$

Sections in the bundle of k-forms are precisely the k-forms.

5. Likewise, if E and F are two bundles, one can form their direct sum $E \oplus F$ and their tensor product $E \otimes F$ by taking it fibre wise. The collection of line bundles on a fixed manifold form a group under the operation of tensor product provided you identify isomorphic bundles. This group plays an important role for complex manifolds and holomorphic bundles. See below.

6. The tangent bundle $T(N)$ of a submanifold N of a manifold M is a subbundle of the restriction $T(M)|N$ of the tangent bundle of M to N. The quotient $(T(M)|N)/T(N)$ is called the *normal bundle* and denoted by $N(N/M)$.

7. If $\varphi : M \to N$ is a differentiable map and $p' : F \to N$ a vector bundle, there is the *pull-back bundle* $\varphi^* F$. Its total space consists of the pairs $(m, f) \in M \times F$ with $\varphi(m) = p'(f)$. Projection comes from projection onto the first factor. One may verify that the trivialization of N induces one on $\varphi^* F$.

8. Consider the subbundle of the trivial bundle with fibre $\mathbb{C}^{n+1}$ on projective space $\mathbb{P}^n$ consisting of pairs $([w], z) \in \mathbb{P}^n \times \mathbb{C}^{n+1}$ with z belonging to the line defined by $[w]$. This is a line bundle, the *tautological line bundle* and denoted by $\mathcal{O}(-1)$.

9. Given an exact sequence

$$0 \to E' \xrightarrow{f} E \xrightarrow{g} E'' \to 0,$$

there is an isomorphism

$$\det E' \otimes \det E'' \xrightarrow{\simeq} \det E$$

(see Problem 1).

Over a complex manifold you have *holomorphic vector bundles*. In the preceding definition of a complex vector bundle one demands that E be a complex manifold and that the differentiable maps involved are actually holomorphic. All of the constructions of the previous examples do produce holomorphic bundles out of holomorphic bundles. In particular, since for any complex manifold M the tangent spaces admit a natural complex structure so does the tangent bundle. Let me denote this complex bundle by $T_{\mathbb{C}}(M) = \bigcup_{m \in M} T_m(M)$. It is in fact a holomorphic bundle. This is likewise true for the cotangent bundle and exterior wedges which now are denoted as follows:

$$\Omega^k(M) = \bigwedge^k T_{\mathbb{C}}^{\vee}(M).$$

The line bundle $\det \Omega^1(M)$ is called the *canonical line bundle* and is sometimes denoted by K_M. If N is a submanifold of M the complex normal bundle $N(N/M)$ of N in M is the quotient of $T_{\mathbb{C}}(M)|N$ by $T_{\mathbb{C}}(N)$. Applying the remark about determinant bundles from Example 9 to the exact sequence defining the normal bundle, you arrive at an important formula:

$$K_N \cong K_M \mid N \otimes \det N(N/M) \qquad \text{(Canonical Bundle Formula)}.$$

As already said before, the collection of holomorphic line bundles on a complex manifold M modulo isomorphism form a group under the tensor product. It is called the *Picard group* and denoted by $\operatorname{Pic} M$.

An important line bundle related to a codimension one subvariety D of a manifold M is the bundle $\mathcal{O}(D)$ on M defined by means of transition functions as follows. Choose a coordinate covering U_i, $i \in I$ of M in which D is given by the equation $f_i = 0$. In $U_j \cap U_j$ the relations $f_i =$ (a non-zero function $\varphi_{ij}) \cdot f_j$ enables one to form the line bundle given by the transition functions $\varphi_{ij} = f_i/f_j$. (Note that the functions φ_{ij} obviously satisfy the co-cycle relation.) Observe that the bundle $\mathcal{O}(D)$ always has a section s_D canonically defined by D. Indeed, over U_i the bundle is trivial and the function f_i defines a section over it. These patch to a section s_D of $\mathcal{O}(D)$ because $f_i = (f_i/f_j)f_j$ in $U_i \cap U_j$. Restricting the bundle $\mathcal{O}(D)$ to D itself in case D is a submanifold, you get back the normal bundle $N(D/M)$. See Problem 2. The Canonical Bundle Formula in this case reads therefore

$$K_D \cong (K_M \otimes \mathcal{O}(D))|D.$$

By definition a *divisor* is a formal linear combination $\sum_{i=1}^m n_i D_i$ with $n_i \in \mathbb{Z}$ and D_i a codimension one subvariety. If the numbers n_i are non-negative the divisor is called *effective*. Divisors on M form an abelian group $\operatorname{Div} M$. The line bundle $\mathcal{O}(D)$ is defined by setting $\mathcal{O}(D) = \mathcal{O}(D_1)^{\otimes n_1} \otimes \ldots \otimes \mathcal{O}(D_m)^{\otimes n_m}$ so that it yields a homomorphism $\operatorname{Div} M \to \operatorname{Pic} M$.

Let me next describe how divisors behave under surjective holomorphic maps $f : M' \to M$. Let g be a local defining equation for D. If the image of f avoids the support of D, the function $g \circ f$ is nowhere zero, but if f is surjective it defines a divisor on M' which is independent of the choice of the local defining equation for D. It is called *the pull-back* f^*D. It is related to the pull-back of the line bundle $\mathcal{O}_M(D)$ by means of the relation $\mathcal{O}_{M'}(f^*D) = f^*(\mathcal{O}(D))$.

In the framework of holomorphic bundles $E \to M$, the group of holomorphic sections is now denoted by $\Gamma(E)$ or $H^0(M, E)$. It is true, but by no means trivial, that for compact complex manifolds the space of sections is finite dimensional. See Appendix A3 for a treatment using Hodge theory. For projective manifolds it is easier. See Theorem 4.13

Now let me turn to projective manifolds. Note that one could have defined algebraic vector bundles using morphisms instead of holomorphic maps. Algebraic vector bundles are holomorphic. The converse is true over a projective manifold. This GAGA-principle (named after the first letters of the words in the title of the article [Se]) is considerably harder to prove than Chow's theorem and uses a lot of sheaf theory and the Kodaira embedding theorem. Let me refer to [G-H], Chapter 1 section 5 for a proof of this assertion. In a similar vein, *regular sections* of an algebraic bundle, i.e. sections which are morphisms, are holomorphic and over a projective manifold the converse is true. In Appendix A4 I collected the main results from [Se].

Since a projective manifold is compact, the space of sections of any algebraic bundle on it is finite dimensional as we have seen before. If L is a line bundle on a projective manifold M, and its space of sections is not zero, say $n+1$-dimensional with basis $x_0, \ldots, x_n$, one can define a rational map

$$\varphi_L : M \dashrightarrow \mathbb{P}^n$$

by associating to $m \in M$ the point in $\mathbb{P}^n$ with homogeneous coordinates $(x_0(m), \ldots, x_n(m))$. This map is not defined on the locus where all sections of L vanish. This locus is called the *base locus* and any point in it is called a *base point*. If φ_L is an embedding, the line bundle L is called *very ample*. If for some integer k the k-th tensor power $L^{\otimes k}$ is very ample, L is said to be *ample*.

Two numbers, generalizing the genus of a projective curve, play an important role in higher dimensions:

The dimension of the space of holomorphic m-forms is called the *geometric genus* of M and denoted by $p_g(M)$.

The dimension of the space of holomorphic 1-forms is called the *irregularity* $q(M)$ of M.

Finally, the definition of divisors and of the Picard group for projective manifolds can be modified in the obvious way by using projective codimension one subvarieties instead.

Again there is a suitable GAGA-principle.

Examples

1. The hyperplane in $\mathbb{P}^n$ defines an algebraic line bundle, the *hyperplane bundle* $\mathcal{O}(1)$. The tautological bundle is the dual of this bundle which explains the notation $\mathcal{O}(-1)$ for the tautological bundle. The line bundle $\mathcal{O}(d)$ is defined as $\mathcal{O}(1)^{\otimes d}$ for $d > 0$ and as $\mathcal{O}(-1)^{\otimes -d}$ if $d < 0$. The line bundle associated to a hypersurface of degree d is (isomorphic to) $\mathcal{O}(d)$, see Problem 5.

2. The canonical line bundle of projective space $\mathbb{P}^n$ is isomorphic to $\mathcal{O}(-n-1)$. See Problem 4. Using the Canonical Bundle Formula you find that the canonical bundle for a smooth degree d hypersurface D in $\mathbb{P}^n$ is the restriction to D of $\mathcal{O}(d-n-1)$.

Any polynomial P which does not vanish identically on V defines a divisor (P) on V by taking $V(P) \cap V$. Any rational function $f = \dfrac{P}{Q}$ on a projective manifold V defines the divisor $(f) = (P) - (Q)$. Since one can represent rational functions on V in different ways, it is not a priori clear that this definition make sense. To see this, one has to use the fact that the ring of holomorphic functions near the origin in $\mathbb{C}^{n-1}$ is a unique factorization domain. This is a corollary of the Weierstrass preparation theorem and I won't give a proof but refer to [G-H, p.10]. One now argues as follows.

Let $f \in \mathbb{C}(M)$ be a rational function on M and D be an irreducible hypersurface. Let $p \in M$ and let $f_D = 0$ be a local equation for D at p. Since the ring $\mathcal{O}_{M,p}$ of germs of holomorphic functions at p is a unique factorization domain one can write

$$f = f_D^m \cdot (u/v)$$

with u and v not identically zero along D. It is easily verified that m does not depend on f_D and the chosen point $p \in M$ so that one can now unambiguously define m to be the *order of vanishing of f along D* denoted $\operatorname{ord}_D(f)$ and one introduces the *divisor of the rational function f* by

$$(f) = \sum_{D \text{ an irreducible hypersurface}} \operatorname{ord}_D(f) D.$$

One checks easily that this definition is the same as the previous one.

Divisors of rational functions form the subgroup of *principal divisors* of $\operatorname{Div} M$. Two divisors D and D' are said to be *linearly equivalent*, notation $D \equiv D'$ if their difference is the divisor of a rational function. Equivalent divisors define isomorphic line bundles and hence there is a well defined map

$$\operatorname{Div} M/\text{principal divisors} \longrightarrow \operatorname{Pic} M.$$

This is in fact is an isomorphism. That it is injective is not so difficult. See Problem 3. The surjectivity is not entirely trivial. See Corollary 4.21

Rational functions f with the property that $(f) + D$ is effective form a vector space traditionally denoted by $\mathcal{L}(D)$. The resulting effective divisors $(f) + D$ linearly equivalent to D form a projective space $|D|$, which is nothing but $\mathbb{P}\mathcal{L}(D)$. Any projective subspace of $|D|$ is called a *linear system of divisors*, whereas $|D|$ itself is called a *complete linear system.* Let me come back to the rational map defined by the line bundle $\mathcal{O}(D)$ associated to D. This can be generalised by taking a basis $\{ s_0, \dots, s_n \}$ for any linear subspace W of $\Gamma(\mathcal{O}(D))$ and the rational map $p \mapsto (s_0(p), \dots, s_n(p))$ then is said to be given by the linear system $\mathbb{P}(W)$. A *fixed component* of the linear system $\mathbb{P}(W)$ is any divisor F which occurs as a component of all divisors in $\mathbb{P}(W)$. The map defined by taking away this fixed part then is the same. The resulting divisors form the *moving part* of $\mathbb{P}(W)$ and now there still can be *fixed points* which however form at most a codimension 2 subspace.

The notion of ampleness has been introduced in connection with line bundles. A divisor D is called ample if the corresponding line bundle $\mathcal{O}(D)$ is ample.

If D is a hypersurface, the line bundle $\mathcal{O}(D)$ has a section s vanishing along D and if $f \in \mathcal{L}(D)$ the product $f \cdot s$ is in a natural way a section of $\mathcal{O}(D)$ and every section can be obtained in this way (see Problem 8). So

$$\mathcal{L}(D) \xrightarrow{\otimes s} H^0(\mathcal{O}(D)) \quad \text{is an isomorphism.}$$

Let me finish this chapter with an important theorem.

Theorem 1. (Bertini) *A generic hyperplane section of a smooth projective variety is smooth.*

Proof: Let $X \subset \mathbb{P}^n$ be a smooth projective variety and let $(\mathbb{P}^n)^\vee$ be the dual projective space of hyperplanes of $\mathbb{P}^n$. Inside $X \times (\mathbb{P}^n)^\vee$ let me consider the set B consisting of pairs (x, H) such that the projectivized tangent space $T_x(X)$ to X at x and H are NOT transversal, i.e. such that $T_x(X) \subset H$. If $\dim X = k$ the possible hyperplanes with this "bad" behaviour form a projective space parametrized by the $\mathbb{P}^{n-k-1}$ disjoint from $T_x(X)$. So the projection $B \to X$ realises B as a projective bundle over X and hence is a variety of dimension $k + n - 1 - k = n - 1$. Consequently, the projection of B into $(\mathbb{P}^n)^\vee$ is not surjective. The complement of this variety parametrizes the "good" hyperplanes. ∎

Problems.

2.1. Let

$$0 \to E' \xrightarrow{f} E \xrightarrow{g} E'' \to 0$$

be an exact sequence of vector bundles over a manifold M. Introduce the subbundle F^r of $\bigwedge^k E$ whose fibre over $m \in M$ is the subspace generated by the wedges of the form $e_1 \wedge e_2 \dots \wedge e_k$ with r of the e_j in $f(E')_m$. Prove that F^{r+1} is a subbundle of F^r and that g induces an isomorphism

$$F^r / F^{r+1} \xrightarrow{\sim} \bigwedge^r E' \otimes \bigwedge^{k-r} E''.$$

In particular, one has an isomorphism

$$\det E' \otimes \det E'' \xrightarrow{\sim} \det E.$$

2.2. Prove that the normal bundle for a smooth hypersurface D of M is isomorphic to the restriction of $\mathcal{O}(D)$ to D.

2.3. Prove that two divisors D and D' on a projective manifold give isomorphic line bundles if and only if the divisors are linearly equivalent.

2.4. Prove that the canonical bundle of $\mathbb{P}^n$ is isomorphic to $\mathcal{O}(-n-1)$.

2.5. Prove that any hypersurface in $\mathbb{P}^n$ is linearly equivalent to dH, where d is the degree of the hypersurface and H is a hyperplane. Deduce that $\operatorname{Pic}\mathbb{P}^n \cong \mathbb{Z}$.

2.6. Prove that for a smooth hypersurface D of degree d in $\mathbb{P}^n$ the normal bundle is given by $N(D/\mathbb{P}^n) = \mathcal{O}(d)|D$.

2.7. Let $M = M_1 \times M_2$ and let $p_1 : M \to M_1$ and $p_2 : M \to M_2$ be the projections onto the factors.

i) Let $\mathcal{V}_1$ be a vector bundle on M_1 and $\mathcal{V}_2$ a vector bundle on M_2. There is a natural homomorphism

$$H^0(M_1, \mathcal{V}_1) \otimes H^0(M_2, \mathcal{V}_2) \to H^0(M_1 \times M_2, p_1^*\mathcal{V}_1 \otimes p_2^*\mathcal{V}_2).$$

Show that this is an isomorphism. Hint: restrict a section s of $p_1^*\mathcal{V}_1 \otimes p_2^*\mathcal{V}_2$ to the fibre $p_1^{-1}(x)$. This yields a section $s(x) \in H^0(M_2, \mathcal{V}_2) \otimes (\mathcal{V}_1)_x$ depending holomorphically on x.

ii) Prove that $\Omega^1(M) = p_1^*\Omega^1(M_1) \oplus p_2^*\Omega^1(M_2)$ and that $K_M = p_1^*K_{M_1} \otimes p_2^*K_{M_2}$.

iii) Prove that $q(M) = q(M_1) + q(M_2)$ and that $p_g(M) = p_g(M_1) \cdot p_g(M_2)$.

Specialize this to products of compact Riemann surfaces.

2.8. Let D be a projective hypersurface of the projective manifold M and let s_D be a regular section of $\mathcal{O}(D)$ vanishing along D. Let $f \in \mathcal{L}(D)$. Prove that $f \cdot s$ is a regular section of $\mathcal{O}(D)$ and that any regular section of $\mathcal{O}(D)$ is obtained in this way.