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Chapter 2. Cohomological Tools

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Chapter 2. Cohomological tools

3. Sheaves and their cohomology

A. Sheaves

As stated in the preface, sheaves form an indispensable tool for algebraic geometers. For cohomology theory fine and flasque sheaves turn out to be useful.

Let me fix a principal ideal domain R and a topological space M.

Definition 1. *A presheaf $\mathcal{F}$ of R-modules over M consists of a collection of R-modules $\mathcal{F}(U)$, one for every open set $U \subset M$, and a collection of R-module homomorphisms $\rho_V^U : \mathcal{F}(U) \to \mathcal{F}(V)$ for pairs of open sets U, V with $V \subset U$ (the restriction homomorphisms) such that:*

a. $\rho_U^U = \mathrm{Id}_{\mathcal{F}(U)}$ *for all open* $U \subset M$,

b. $\rho_W^V \circ \rho_V^U = \rho_W^U$ *for all* $W \subset V \subset U$.

If in addition the following property holds, $\mathcal{F}$ is called a *sheaf:*

c. *If U is a union of open sets* $U = \bigcup_{i \in I} U_i$ *then*

   1) if $f, g \in \mathcal{F}(U)$ *and* $\rho_{U_i}^U(f) = \rho_{U_i}^U(g)$ *for all* $i \in I$ *, then* $f = g$,

   2) if $f_i \in \mathcal{F}(U_i)$ *with* $\rho_{U_i \cap U_j}^{U_i}(f_i) = \rho_{U_i \cap U_j}^{U_j}(f_j)$ *for all* $i, j \in I$ *, then there exists a, because of 1) unique element* $f \in \mathcal{F}(U)$ *with* $\rho_{U_i}^U(f) = f_i$ *for all* $i \in I$.

An element of $\mathcal{F}(U)$ is called a *section* of $\mathcal{F}$ over U. The module $\mathcal{F}(M)$ of sections over M is also denoted by $\Gamma(M)$ or $H^0(M, \mathcal{F})$. The latter notation will be justified later.

Another useful concept is that of the *stalks.* To define it fix $m \in M$ and consider the collection of neighbourhoods of m. The stalk $\mathcal{F}_m$ at m is defined as the *direct limit*

$$\mathcal{F}_m := \operatorname*{dirlim}_{U \ni m} \mathcal{F}(U), \quad U \text{ a neighbourhood of } m,$$

which by definition is obtained by taking the disjoint union of the modules $\mathcal{F}(U)$, U a neighbourhood of m, and then identifying $m \in \mathcal{F}(U)$ with $m' \in \mathcal{F}(U')$ if there is some neighbourhood $U'' \subset U \cap U'$ of m such that $\rho_{U''}^U m = \rho_{U''}^{U'} m'$.

If $\mathcal{F}$ is a presheaf, but not a sheaf, one may enlarge it to a sheaf, the *sheaf associated to the presheaf* $\mathcal{F}$ (see [Wa, p.166]). This is sometimes useful since natural constructions which start on the level of the R-modules $\mathcal{F}(U)$ with $\mathcal{F}$ a sheaf do often give presheaves, but not always sheaves, as will be seen when homomorphisms between sheaves are treated. Although the explicit construction of the sheaf associated to a presheaf is not needed, let me give it for the sake of completeness.

So let $\mathcal{F}$ be a presheaf and define for each open set U the module $F(U)$ consisting of functions $U \ni x \mapsto s(x) \in \mathcal{F}_x$ with the property that for each point $x \in U$ there is a neigbourhood $V \subset U$ and a section $t \in \mathcal{F}(V)$ such that the germ of t at y is equal to $s(y)$ for all $y \in Y$. Then the $F(U)$ form a sheaf which by definition is the sheaf associated to $\mathcal{F}$.

It should be clear what is meant by a *sheaf homomorphism* $h : \mathcal{F} \to \mathcal{F}'$: one should have homomorphisms $\mathcal{F}(U) \to \mathcal{F}'(U)$ for each open set $U \subset M$ commuting with the restriction maps. In particular there are induced homomorphisms $h_m : \mathcal{F}_m \to \mathcal{F}'_m$ for the stalks. Observe that the modules of the kernels of $\mathcal{F}(U) \to \mathcal{F}'(U)$ do form a sheaf ker h, but the modules of the cokernels only form a presheaf. By definition coker h is the sheaf associated to this presheaf.

A sequence of sheaf homomorphisms on M:

$$\ldots \to \mathcal{F}_{i-1} \xrightarrow{f_{i-1}} \mathcal{F}_i \xrightarrow{f_i} \mathcal{F}_{i+1} \to \ldots$$

is *exact* if for all $m \in M$ the corresponding sequence of the stalks at m is exact. This does NOT mean that the corresponding sequence of the sections over all open $U \subset M$ is exact, which is the definition of an exact sequence of presheaves. The reason is that, as noticed before, the cokernels on presheaf level do not always form a sheaf.

Examples

1. Let M be any topological space. Let G be any R-module. For any open $U \subset M$ let $G(U) = \{$ locally constant functions $f : U \to G \}$. The restriction maps are the obvious ones. The properties a), b) and c) are immediate. This sheaf is called the *constant sheaf* G_M.

2. Let $f : M \to N$ be a continuous mapping between topological spaces and let $\mathcal{F}$ be a sheaf on M. The image sheaf $f_*\mathcal{F}$ is the sheaf defined by $f_*\mathcal{F}(U) = \mathcal{F}(f^{-1}U)$ (and the obvious restriction maps). It is easy to see that $(g \circ f)_*\mathcal{F} = g_*(f_*\mathcal{F})$ when $g : N \to P$ is a further continuous map between topological spaces. Note also that there is a canonical homomorphism $\hat{f} : (f_*\mathcal{F})_{f(m)} \to \mathcal{F}_m$ which associates to a germ $g_{f(m)} \in (f_*\mathcal{F})_{f(m)}$ represented by a section $g \in \mathcal{F}(f^{-1}U)$, U a neighbourhood of $f(m)$, the germ of g at m.

3. Let M be a differentiable manifold and for any open $U \subset M$ let $\mathcal{E}(U)$ be the ring of differentiable functions (it is a module over the real numbers) and take the usual restriction maps. Again, one verifies that this defines a sheaf, the sheaf $\mathcal{E}_M$ of differentiable functions on M. The elements of $\mathcal{E}_{M,m}$ are called *germs at $m \in M$ of differentiable functions.*

Similarly, if M is a complex manifold, there is the sheaf $\mathcal{O}_M^{\mathrm{hol}}$ of holomorphic functions and on a projective manifold there is the sheaf $\mathcal{O}_M^{\mathrm{alg}}$ of regular functions. If no confusion arises the same notation $\mathcal{O}_M$ for these sheaves will be used although the holomorphic sheaf is much bigger in general.

4. If M is a complex manifold, for any open $U \subset M$ one can form the ring of fractions $Q(\mathcal{O}(U))$ of the ring $\mathcal{O}(U)$ and the obvious restriction maps between them. These form only a presheaf, since Axiom C2) does not hold. To make it into a sheaf, let me define *meromorphic functions* over U so that this Axiom holds automatically. So, a meromorphic function over U should be given by a collection $\{ U_i, f_i \}$ with $\{ U_i \}$ an open cover of U, $f_i \in Q(\mathcal{O}(U_i))$ such that in $U_i \cap U_j$ one has $f_i = f_j$. Meromorphic functions on U form a complex vectorspace $\mathcal{M}(U)$ and in this way one does get a sheaf, the sheaf $\mathcal{M}$ of

germs of meromorphic functions on M. By definition, a *meromorphic function* on M is a global section of this sheaf. Denote by $\mathcal{M}^*_M$ the sheaf (of multiplicative groups) of non-zero elements in $\mathcal{M}_M$. The sheaf $\mathcal{O}^*_M$ of germs of nowhere zero holomorphic functions on M forms a subsheaf of $\mathcal{M}^*_M$. A *Cartier divisor* on M is a global section of the sheaf $\mathcal{M}^*_M/\mathcal{O}^*_M$. In concrete terms, a Cartier divisor consists of a collection of open sets $\{ U_i \}$ covering M and non-zero meromorphic functions f_i on U_i such that in the overlaps $f_i = g_{ij} \cdot f_j$ with g_{ij} a nowhere vanishing holomorphic function on $U_i \cap U_j$. Two sets $\{ U_i, f_i \}$ and $\{ U'_j, f'_j \}$ define the same Cartier divisor if in overlaps $U_i \cap U'_j$ one has $f_i/f'_j \in \mathcal{O}^*(U_i \cap U'_j)$.

For a projective manifold, working with $\mathcal{O}^{\mathrm{alg}}_M$ one obtains the algebraic Cartier divisors. On a variety any rational function is completely determined by knowing it on any non-empty Zariski open subset U. So the sheaf of germs of rational functions, i.e. the sheaf of quotients of $\mathcal{O}^{\mathrm{alg}}_M$ is just the constant sheaf $\mathbb{C}(M)_M$. There is no need to do patchwork for defining rational functions. In particular the algebraic analogue of a meromorphic function on projective manifolds just is a rational function. Any irreducible hypersurface D defines a Cartier divisor by taking the local defining equations. A different choice of local defining equations yield the same Cartier divisor, more or less by definition. It follows that any divisor defines a unique Cartier divisor. Conversely, any Cartier divisor $\{ U_i, f_i \}$ yields a divisor by taking $\sum_D \mathrm{ord}_D(f_i)D$. This indeed gives a well defined divisor since in $\mathcal{O}(U_i \cap U_j)$ the function f_i/f_j is nowhere vanishing and hence $\mathrm{ord}_D(f_i) = \mathrm{ord}_D(f_j)$. This shows that on a projective manifold one may identify divisors and Cartier divisors. For the case of general complex manifolds see Problem 3.

The GAGA-principle tells us that there is no difference between the group of algebraic Cartier divisors and the group of Cartier divisors.

5. A sheaf $\mathcal{F}$ of R-modules on M is a *fine sheaf*, if for every locally finite cover $\{U_i\}$ of M by open sets there are endomorphisms $h_i : \mathcal{F} \to \mathcal{F}$ with support in U_i such that $\sum_i h_i = \mathrm{Id}_{\mathcal{F}}$. Here the *support* of a homomorphism h is the closure of the points $m \in M$ where h_m is not zero. Examples include the sheaves $\mathcal{E}_M$ of differentiable functions on a differentiable manifold M, since there are partitions of unity subordinate to any locally finite open cover of M. See [Wa, p. 170].

6. A sheaf $\mathcal{F}$ is called a *flasque sheaf* if for any pair of open subsets $U \subset V$ the restriction map $\mathcal{F}(V) \to \mathcal{F}(U)$ is surjective. Any sheaf $\mathcal{F}$ embeds in a flasque sheaf $\mathcal{C}^0(\mathcal{F})$, its sheaf of discontinuous sections which is defined by letting $\mathcal{C}^0(\mathcal{F})(U)$ be the set of maps $U \ni x \mapsto s(x) \in \mathcal{F}_x$.

7. Let E be a vector bundle on a manifold M. For any $U \subset M$ take $\Gamma(U, E)$ and the obvious restriction maps. This gives the *sheaf of sections associated to E*. In the differentiable setting this sheaf is denoted by $\mathcal{E}(E)$, in the holomorphic (or algebraic) setting by $\mathcal{O}(E)$. Particular cases are the sheaves $\mathcal{E}^p_M$ of differentiable p-forms on a differentiable manifold M and the sheaf Ω^p_M of holomorphic p-forms on a complex manifold. The sheaves $\mathcal{E}^p_M$ are fine. This follows with partitions of unity.

8. If M is a complex manifold, an affine or a projective variety one often uses *sheaves of $\mathcal{O}_M$-modules*, which by definition are sheaves $\mathcal{F}$ of complex vector spaces such that for every open $U \subset M$ the vector space $\mathcal{F}(U)$ in addition is an $\mathcal{O}(U)$-module and if $V \subset U$ is open, the restriction $\mathcal{F}(U) \to \mathcal{F}(V)$ is compatible with the module structures via the ring homomorphism $\mathcal{O}(U) \to \mathcal{O}(V)$. It should be clear what is meant by a *morphism of*

$\mathcal{O}_M$-modules.

Special cases of $\mathcal{O}_M$-modules include $\oplus^n\mathcal{O}_M$, the ***trivial*** $\mathcal{O}_M$-module of rank n and the *locally trivial $\mathcal{O}_M$-modules* $\mathcal{F}$ of rank n, which by definition have the property that there is a cover of M by open sets U over which $\mathcal{F}$ is trivial of rank n. The sheaf $\mathcal{O}(E)$ of holomorphic sections of a vector bundle ise locally free and conversely. See Problem 1.

Let E be a holomorphic vector bundle on a complex manifold M and let $m \in M$. There is the following useful relation between the fibre of E and the stalk of $\mathcal{O}(E)$ at m.

$$\mathcal{O}(E)_m/\mathfrak{m}_m \cdot \mathcal{O}(E)_m \xrightarrow{\cong} E_m$$

where $\mathfrak{m}_m$ is the maximal ideal of the point m in $(\mathcal{O}_M)_m$ and where the isomorphism comes from evaluating germs of sections of E at m.

In the usual way, out of the two sheaves of $\mathcal{O}_M$-modules $\mathcal{F}$ and $\mathcal{G}$ one produces $\mathcal{F}\otimes_{\mathcal{O}_M}\mathcal{G}$ by forming the presheaf given over U by $\mathcal{F}(U)\otimes_{\mathcal{O}(U)}\mathcal{G}(U)$ which in fact is a sheaf. The sheaf $\mathcal{H}om_{\mathcal{O}_M}(\mathcal{F},\mathcal{G})$ is constructed in an essentially different way by taking the $\mathcal{O}(U)$-module $\mathrm{Hom}_U(\mathcal{F}|U,\mathcal{G}|U)$ of the $\mathcal{O}_M|U$-module homomorphisms $\mathcal{F}|U \to \mathcal{G}|U$ and the obvious restrictions. One cannot take $\mathrm{Hom}_{\mathcal{O}(U)}(\mathcal{F}(U),\mathcal{G}(U))$ since then there would be no apparent way to define the restrictions.

Finally, if $f : M \to N$ is a holomorphic map between complex manifolds (or a morphism between varieties) the image sheaf $f_*\mathcal{O}_M$ is a sheaf of $\mathcal{O}_N$-modules in a natural way (holomorphic (or regular) functions on $U \subset N$ pull back to holomorphic (or regular) functions on $f^{-1}U$).. Thus you can view the image of any sheaf $\mathcal{F}$ of $\mathcal{O}_M$-modules as a sheaf of $\mathcal{O}_N$-modules. This is the *(analytic or algebraic) direct image sheaf* which is still denoted by $f_*\mathcal{F}$.

8. The exponential sequence on a complex manifold M:

$$0 \to \mathbb{Z}_M \to \mathcal{O}_M \xrightarrow{exp} \mathcal{O}_M^* \to 0,$$

where "exp" means the map $f \mapsto \exp(2\pi i f)$. This sequence is a typical example of an exact sequence of sheaves which is not exact as sequence of presheaves. See Problem 2.

B. Cohomology

Cohomology for sheaves is introduced axiomatically. Not all proofs are presented here, but the reader can find them in the references given. De Rham's theorem, Dolbeault's theorem and Leray's theorem are explained in some more detail. An important application is given: the cohomological interpretation of the Picard group.

Let me briefly and informally recall the axiomatic set-up for a cohomology theory. You start with a fixed topological manifold and a class of sheaves on the manifold. Of course, in order to ensure that the axioms that follow make sense, the manifold should have some good properties and the same holds for the sheaves on it. Let me not be precise about this now. Let it be sufficient to say that one may take for example an arbitrary topological space and sheaves of abelian groups on it. Another possibility is that you take a Hausdorff space with countable basis for the topology and any sheaf of R-modules on it, where R is a fixed principal ideal domain. Lastly, there is the most widely used example of an algebraic variety X with sheaves of $\mathcal{O}_X$-modules.

For a cohomology theory you want groups $H^q(M,\mathcal{F})$, $q \in \mathbb{Z}$, for any allowable sheaf $\mathcal{F}$ and topological space M and for any allowable sheaf homomorphism $h : \mathcal{F} \to \mathcal{F}'$ there should be induced homomorphisms $H^q(h) : H^q(M,\mathcal{F}) \to H^q(M,\mathcal{F}')$. These groups and homomorphisms should satisfy the following axioms.

A. $H^q(M,\mathcal{F}) = 0$ for $q < 0$ and there are isomorphisms $H^0(\mathcal{F}) \cong \mathcal{F}(M)$ commuting with any induced homomorphism $H^0(f)$, where $f : \mathcal{F} \to \mathcal{F}'$ is a homomorphism of sheaves of R-modules.

B. $H^q(\mathcal{F}) = 0$ for all $q > 0$ if $\mathcal{F}$ is a fine sheaf or a flasque sheaf.

C. The correspondence which associates sheaves of R-modules and homomorphisms to their q-th cohomology groups and induced homomorphisms is *functorial*:

C1. $H^q(\mathrm{Id} : \mathcal{F} \to \mathcal{F}) = \mathrm{Id}_{H^q(\mathcal{F})}$ and

C2. $H^q(f \circ g) = H^q(f) \circ H^q(g)$.

D. For any short exact sequence

$$0 \to \mathcal{F}' \xrightarrow{i} \mathcal{F} \xrightarrow{j} \mathcal{F}'' \to 0$$

there exist coboundary homomorphisms $H^q(M,\mathcal{F}'') \to H^{q+1}(M,\mathcal{F}')$ so that the sequence

$$\ldots \to H^q(M,\mathcal{F}') \xrightarrow{H^q(i)} H^q(M,\mathcal{F}) \xrightarrow{H^q(j)} H^q(M,\mathcal{F}'') \longrightarrow H^{q+1}(M,\mathcal{F}') \to \ldots$$

is exact. Furthermore, any homomorphism between short exact sequences of sheaves yields a homomorphism between the corresponding long exact sequences in cohomology.

As to *existence* of cohomology theories let me only remark that on an arbitrary topological space M and any sheaf of R-modules $\mathcal{F}$ on M, one can define $H^q(M,\mathcal{F})$ as the q-th cohomology group of the complex $\Gamma(M,\mathcal{C}^\bullet(\mathcal{F}))$. Here $\mathcal{C}^0(\mathcal{F})$ is the sheaf of discontinuous sections of $\mathcal{F}$ as introduced in Example 6. Setting $\mathcal{Z}^1(\mathcal{F}) = \mathcal{C}^0(\mathcal{F})/\mathcal{F}$ one defines $\mathcal{C}^1(\mathcal{F})$ as the sheaf of discontinuous sections of the sheaf $\mathcal{Z}^1$. Next, one inductively introduces $\mathcal{Z}^q = \mathcal{C}^{q-1}(\mathcal{F})/\mathcal{Z}^{q-1}$ and $\mathcal{C}^q(\mathcal{F})$ as the sheaf of discontinuous sections of $\mathcal{Z}^q$. For the verification of the axioms see [Go]. More precisely, axiom A is clear, B is II, Théorème 4.4.3 (Fine sheaves on a Hausdorff space with countable basis for the topology satisfy (b) in this theorem by [Go II, 3.7]) , C and D are the content of [Go II, Théorème 4.4.2.]

From the preceding definition it is virtually impossible to compute cohomology groups. Now, a cohomology theory is essentially unique (I come back to this in a little while) and so one might try to find another theory which is more suitable for computations. Such a theory is Čech cohomology-theory with values in a sheaf $\mathcal{F}$ on a topological space M. Although it can be defined for any M, this does NOT yield a good cohomology theory unless M is a Hausdorff space with countable basis for the topology. The delicate point is the exactness of long exact cohomology sequences. For details of the following discussion see [Wa, p. 200-204]. Since I shall be using Čech cohomology also on algebraic varieties X with the Zariski-topology, one has to be careful with long exact sequences. I shall apply them only for sheaves of $\mathcal{O}_X$-modules and for these one can prove that there are no problems with long exact sequences. See Proposition 4.8.

To define Čech cohomology, you start with an open cover $\mathfrak{U} = \{ U_i \}$ of M. A collection $(U_0, \dots, U_q)$ of members of $\mathfrak{U}$ with non-empty intersection is called a q-simplex $\sigma = \{ 0, \dots, q \}$ and its support $|\sigma|$ is by definition $U_0 \cap \dots \cap U_q$. The i-th face of σ is the $q-1$-simplex $\sigma^i = \{ 0, \dots, i-1, i+1, \dots, q \}$. A *$q$-cochain* is a function f which assigns to any q-simplex σ an element $f(\sigma) \in \mathcal{F}(|\sigma|)$. This is the same as saying that f is an element of the *free product* of the R-modules $\mathcal{F}(|\sigma|)$ where σ runs over the q-simplices of $\mathfrak{U}$. This free product is again an R-module (with the obvious module-operations):

$$\prod_{\sigma \text{ a q-simplex of } \mathfrak{U}} \mathcal{F}(|\sigma|) = C^q(\mathfrak{U}, \mathcal{F}).$$

There is the coboundary homomorphism

$$d : C^q(\mathfrak{U}, \mathcal{F}) \to C^{q+1}(\mathfrak{U}, \mathcal{F})$$

defined by

$$df(\sigma) = \sum_{i=0}^{q+1} (-1)^i \rho_{|\sigma|}^{|\sigma^i|} f(\sigma^i)$$

which satisfies $d \circ d = 0$ and hence one obtains a cochain complex (see Appendix 2), the *Čech cochain complex* $C^\bullet(\mathfrak{U}, \mathcal{F})$and it has cohomology groups $H^q(\mathfrak{U}, \mathcal{F})$. By definition,

$$H^q(M, \mathcal{F}) := \operatorname*{dirlim}_{\mathfrak{U}} H^q(\mathfrak{U}, \mathcal{F}),$$

where the direct limit is taken over the set of coverings, partially ordered under the refinement relation. If $\mathfrak{U}'$ is a refinement of $\mathfrak{U}$, there are indeed natural homomorphisms $H^q(\mathfrak{U}, \mathcal{F}) \to H^q(\mathfrak{U}', \mathcal{F})$ which are to be used in forming the direct limit. See Appendix 1.

Clearly one has $H^q(M, \mathcal{F}) = 0$ for $q < 0$ and $H^0(M, \mathcal{F}) = \mathcal{F}(M)$. For fine sheaves, one even has $H^q(\mathcal{F}) = 0$ for all $q > 0$ (loc. cit.).

For any sheaf homomorphism $h : \mathcal{F} \to \mathcal{F}'$ there are induced R-module homomorphisms $H^q(h) : H^q(M, \mathcal{F}) \to H^q(M, \mathcal{F}')$. Moreover, if for a short exact sequence

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

one can define a coboundary operator $H^q(M, \mathcal{F}'') \to H^{q+1}(M, \mathcal{F}')$ which fits into a long cohomology sequence

$$\dots \to H^{q-1}(M, \mathcal{F}'') \to H^q(M, \mathcal{F}') \to H^q(M, \mathcal{F}) \to H^q(M, \mathcal{F}'') \to H^{q+1}(M, \mathcal{F}') \to \dots.$$

If M is Hausdorff and has a countable basis for the topology, this sequence is exact. (In taking a limit one might have problems with the exactness on more general spaces.)

The modules $H^q(M, \mathcal{F})$ for the various sheaves of R-modules and induced homomorphisms $H^q(f)$ taken together therefore constitute a cohomology theory.

There is essentially only one cohomology theory up to natural isomorphism (loc. cit.). From the axioms it follows then for example that one can calculate $H^q(M, \mathcal{F})$ using exact sequences of the form

$$0 \to \mathcal{F} \to \mathcal{F}^0 \to \mathcal{F}^1 \to \dots$$

with $H^q(\mathcal{F}^j) = 0$, $q > 0$, $j = 0, 1, \ldots$. These are called *cohomological resolutions.* Examples arise when $\mathcal{F}^j$ is fine, resp. flasque, since then $H^q(\mathcal{F}^j) = 0$. Such resolutions are called *fine, resp. flasque resolutions.* Observe that by functoriality the sections of $\mathcal{F}^j$ form a complex $\Gamma(\mathcal{F}^\bullet)$. The q-th cohomology group of this complex $H^q(\Gamma(\mathcal{F}^\bullet))$ is naturally isomorphic to $H^q(\mathcal{F})$. See [Wa, Theorem 5.25].

As an example, the De Rham complex

$$0 \to \mathbb{R}_M \to \mathcal{E}^0_M \to \mathcal{E}^1_M \to \ldots$$

is a fine resolution of the constant sheaf $\mathbb{R}_M$ (by the Poincaré lemma) and hence one has

Theorem 2. (De Rham) *The sheaf cohomology group $H^q(M, \mathbb{R}_M)$ is canonically isomorphic to the q-th De Rham group $H^q_{\mathrm{DR}}(M)$.*

In a similar vein one has the Dolbeault complex

$$0 \to \Omega^p_M \to \mathcal{E}^{p,0} \xrightarrow{\bar\partial} \mathcal{E}^{p,1} \xrightarrow{\bar\partial} \mathcal{E}^{p,2} \xrightarrow{\bar\partial} \cdots,$$

which is a fine resolution of Ω^p_M (by the Dolbeault lemma) and so one obtains:

Theorem 3. (Dolbeault) *The sheaf cohomology group $H^q(M, \Omega^p_M)$ is canonically isomorphic to the q-th Dolbeault group*

$$\frac{\ker\left(\mathcal{E}^{p,q} \xrightarrow{\bar\partial} \mathcal{E}^{p,q+1}\right)}{\operatorname{im}\left(\mathcal{E}^{p,q-1} \xrightarrow{\bar\partial} \mathcal{E}^{p,q}\right)}.$$

IN THE REST OF THE SECTION ČECH COHOMOLOGY IS USED FOR A CLASS OF SHEAVES FOR WHICH ČECH COHOMOLOGY IS A GOOD COHOMOLOGY THEORY.

A useful tool for computing cohomology directly from a so-called *acyclic covering* is Leray's theorem. By definition, given a sheaf $\mathcal{F}$, a covering $\mathfrak{U}$ is $\mathcal{F}$- acyclic if for every simplex σ of the covering one has $H^q(|\sigma|, \mathcal{F}) = 0$ for $q > 0$.

Theorem 4. (Leray) *Let $\mathcal{F}$ be a sheaf of abelian groups on a topological space M and $\mathfrak{U}$ an $\mathcal{F}$-acyclic covering. Assume that either M is Hausdorff with countable basis for the topology or that M is Noetherian. The natural homomorphism $H^q(\mathfrak{U}, \mathcal{F}) \to H^q(M, \mathcal{F})$ is an isomorphism.*

Remark A topological space is said to be *Noetherian* if any descending chain of closed subsets becomes stationary, which is the case for instance for the Zariski-topology. The conditions in the theorem are used to ensure that one can interchange limits and cohomology groups. For a simple proof of this property see [Go, II,4.12]. There are other proofs valid for any topological space and a sheaf of abelian groups on it, but these make use of spectral sequences (loc. cit. II, Théorème 5.4.1, Corollaire.)

Proof: Consider the sheaf $\mathcal{C}^p(\mathfrak{U},\mathcal{F})$ defined by $U \mapsto C^p(U \cap \mathfrak{U},\mathcal{F})$. The reader may verify that this is indeed a sheaf. There is a natural map $j : \mathcal{F} \to \mathcal{C}^0(\mathfrak{U},\mathcal{F})$ defined by sending $f \in \mathcal{F}(U)$ to the 0-cochain which associates to $U_i \in \mathfrak{U}$ the restriction of f to $U \cap U_i$. This is an embedding by the sheaf axiom C1. So you get an embedding into a complex of sheaves

$$0 \to \mathcal{F} \to \mathcal{C}^\bullet(\mathfrak{U},\mathcal{F})$$

and I claim that this gives a cohomological resolution of $\mathcal{F}$. So I have to show that the complex is exact and that $H^q(\mathcal{C}^p(\mathfrak{U},\mathcal{F})) = 0$ for $q > 0$.

a. Exactness at $\mathcal{C}^0(\mathfrak{U},\mathcal{F})$ is the sheaf axiom C2. For $p > 0$ let me consider the germ at $x \in M$ of a p-cocycle. I may assume that for some open neighbourhood U of x there is a representing cocycle $\alpha \in C^p(U \cap \mathfrak{U},\mathcal{F})$. Moreover I may assume that $U \subset U_i$ for some index i. Then, if σ is a $p-1$-simplex in $\mathfrak{U}$ one has $|i\sigma| \cap U = |\sigma| \cap U$. So one may define a $(p-1)$-cochain $\beta \in C^{p-1}(U \cap \mathfrak{U},\mathcal{F})$ by setting

$$\beta(\sigma) = \alpha(i\sigma).$$

So then one computes for a p-simplex $\tau = \{\, j_0 \cdots j_p \,\}$ of the covering

$$d(\beta)(j_0 \cdots j_p) = \sum_{0 \le k \le p} (-1)^k \alpha(ij_0 \cdots \widehat{j_k} \cdots j_p).$$

Since $d(\alpha) = 0$ one has in $U \cap |i\tau| = U \cap |\tau|$ that

$$\alpha(j_0 \cdots j_p) - \sum_{0 \le k \le p} (-1)^k \alpha(ij_0 \cdots \widehat{j_k} \cdots j_p) = 0$$

and hence $d(\beta) = \alpha$.

b. Next I have to show that $H^q(\mathcal{C}^p(\mathfrak{U},\mathcal{F})) = 0$. For the moment, for any covering $\mathfrak{U}$ let $N_p(\mathfrak{U})$ be the collection of its p-simplices so that

$$C^p(\mathfrak{U},\mathcal{G}) = \prod_{\sigma \in N_p(\mathfrak{U})} \mathcal{G}(|\sigma|)$$

and so for any other covering $\mathfrak{U}'$ one has

$$C^q(\mathfrak{U}',\mathcal{C}^p(\mathfrak{U},\mathcal{F})) = \prod_{\sigma \in N_p(\mathfrak{U})} \prod_{\sigma' \in N_q(\mathfrak{U}')} \mathcal{F}(|\sigma| \cap |\sigma'|).$$

and so

$$H^q(\mathfrak{U}',\mathcal{C}^p(\mathfrak{U},\mathcal{F})) = H^q\Big(\prod_{\sigma \in N_p(\mathfrak{U})} C^\bullet(\mathfrak{U}' \cap |\sigma|,\mathcal{F})\Big) = \prod_{\sigma \in N_p(\mathfrak{U})} H^q(C^\bullet(\mathfrak{U}' \cap |\sigma|,\mathcal{F})).$$

For any p-simplex σ of $\mathfrak{U}$ one has $H^q(|\sigma|,\mathcal{F}) = 0$ if $q > 0$. So the direct limit of the groups $H^q(C^\bullet(\mathfrak{U}') \cap \sigma,\mathcal{F})$ vanishes. But then also the free product

$$\prod_{\sigma \in N_p(\mathfrak{U})} \operatorname*{dirlim}_{\mathfrak{U}'} H^q(C^\bullet(\mathfrak{U}' \cap |\sigma|,\mathcal{F}))$$

vanishes. Interchanging product and limit, which is allowed thanks to the assumptions (see the remark preceding the proof), you find

$$0 = \operatorname*{dirlim}_{\mathfrak{U}'} \prod_{\sigma \in N_p(\mathfrak{U})} H^q(C^\bullet(\mathfrak{U}' \cap |\sigma|, \mathcal{F})) = H^q(M, C^p(\mathfrak{U}, \mathcal{F})).$$

To complete the proof I have to verify that the isomorphism $H^q(\mathfrak{U}, \mathcal{F}) \to H^q(M, \mathcal{F})$ obtained in this way from the cohomological resolution is exactly the canonical map. This I leave to the reader.

■

As an example of the use of sheaf theory, let me come back to the group Pic M of isomorphism classes of line bundles on a projective manifold M. Let me recall that a line bundle $\mathcal{L}$ can be given by a trivialising open cover $\mathfrak{U} = \{ U_i \}$ and nowhere zero transition functions $f_{ij} \in \mathcal{O}(U_i \cap U_j)$. The collection of transition functions defines a cochain $f \in C^1(\mathfrak{U}, \mathcal{O}_M^*)$ satisfying the cocycle relation $f_{ij} \cdot f_{jk} \cdot f_{li} = 1$, (written multiplicatively) i.e $df = 0$ and hence f defines an element in $H^1(\mathfrak{U}, \mathcal{O}_M^*)$. If one chooses a different trivialisation over the same cover, the new transition functions are seen to give a cocycle differing by a coboundary from f. So the class $[f] \in H^1(\mathfrak{U}, \mathcal{O}_M^*)$ is well defined. If you look at trivialisations on a different open cover their union is a common refinement $\mathfrak{U}'$. The two cohomology classes associated to the two trivialisations coincide in $H^1(\mathfrak{U}', \mathcal{O}_M^*)$. So the isomorphism class of the line bundle $\mathcal{L}$ gives a well defined element in $H^1(M, \mathcal{O}_M^*)$. Conversely, any element in $H^1(M, \mathcal{O}_M^*)$ gives a line bundle up to isomorphism. So one has

$$\operatorname{Pic} M \cong H^1(M, \mathcal{O}_M^*).$$

Let me now come back to the exponential sequence

$$0 \to \mathbb{Z}_M \to \mathcal{O}_M \xrightarrow{exp} \mathcal{O}_M^* \to 0,$$

and look at its induced cohomology sequence

$$\ldots \to H^1(M, \mathbb{Z}_M) \to H^1(M, \mathcal{O}_M) \to H^1(M, \mathcal{O}_M^*) \xrightarrow{c_1} H^2(M, \mathbb{Z}_M) \to \ldots.$$

The coboundary map is called the *First Chern Class* map. Its kernel is a subgroup $\operatorname{Pic}^0 M \subset \operatorname{Pic} M$. The exact sequence shows that there is a natural isomorphism

$$\operatorname{Pic}^0 M \cong \frac{H^1(M, \mathcal{O}_M)}{\operatorname{im} H^1(M, \mathbb{Z})}.$$

The latter quotient group in fact is a torus, the Picard variety. See Corollary 11.2. For curves you get the *jacobian* of C in this way.

The remaining part of the Picard group, the image under c_1, by definition is the *Néron-Severi group* $\mathrm{NS}\, M$ of M which is the group of isomorphism classes of divisors modulo homological equivalence: two divisors are said to be homologically equivalent if they have the same first Chern class. There is an exact sequence which summarises this situation

$$0 \to \mathrm{Pic}^0 M \to \mathrm{Pic}\, M \xrightarrow{c_1} \mathrm{NS}\, M \to 0.$$

Since for a compact manifold the cohomology groups are finite-dimensional (see Appendix 3) the Néron-Severi-group must be a finitely generated group, a fact which will be used several times later on.

Problems.

3.1. Prove that for any holomorphic vector bundle E of rank d on a complex manifold M the sheaf $\mathcal{O}(E)$ is locally free of rank d and that conversely any locally free sheaf of $\mathcal{O}_M$-modules of rank d is of the form $\mathcal{O}(E)$ with E a holomorphic vector bundle of rank d.

3.2. Give an example of an open set U in $\mathbb{C}$ such that the sequence

$$0 \to \mathbb{Z}(U) \to \mathcal{O}(U) \xrightarrow{exp} \mathcal{O}^*(U) \to 0$$

is not exact.

3.3. For a meromorphic function f and an irreducible hypersurface $D \subset M$ one can define the order of vanishing of f along D $\mathrm{ord}_D(f)$ in the same way as for rational functions on a projective manifold and hence one can speak of divisors of meromorphic functions. Generalise the concepts $\mathrm{Div}\, M$ and $\mathrm{Pic}\, M$. See [G-H, Chapter 1.1].

3.4. Show that for a connected compact complex manifold M the sequence

$$0 \to \mathbb{Z}(M) \to \mathcal{O}(M) \to \mathcal{O}^*(M) \to 0$$

is exact. Deduce that $H^1(M, \mathbb{Z}_M)$ embeds naturally into $H^1(M, \mathcal{O}_M)$.

3.5. Show that $\mathrm{Pic}^0(\mathbb{P}^n) = 0$.

3.6. Show that for the quadric surface $Q \subset \mathbb{P}^3$, the Picard group is the free abelian group generated by the divisor classes corresponding to the two rulings.

3.7. Let $Q \subset \mathbb{C}^3$ be the singular quadric defined by $xy - z^2$. The x-axis gives a subvariety L of Q of codimension one, which one may consider as a divisor. Prove that locally near the vertex, L cannot be given by an equation, while $2L$ is cut out by $z = 0$.

4. Serre's Finiteness and Vanishing Theorems

In this section $\mathbb{C}^n$ and projective space equipped with the *Zariski-topology* and *algebraic* sheaves on them will be studied.

A. Coherent sheaves

Coherent and quasi-coherent sheaves form global objects which are defined algebraically over affine sets thus permitting to translate their geometric properties into algebra.

Projective varieties $M \subset \mathbb{P}^n$ are to be looked at first. Introduce the homogeneous coordinates $X_0, \ldots, X_n$ on $\mathbb{P}^n$. Let me recall that $U_j = \{\ X_j \neq 0\ \} \cong \mathbb{C}^n$ are the basic affine open sets.

By definition a sheaf $\mathcal{F}$ of $\mathcal{O}_M$-modules on a projective variety M is coherent if every point of M has a Zariski neighbourhood U over which there is an exact sequence of the form

$$\mathcal{O}_U^{\oplus m} \to \mathcal{O}_U^{\oplus n} \to \mathcal{F}|U \to 0.$$

In other words: there is a Zariski-open cover over which the sheaf is a quotient of a finitely generated locally free module by a finitely generated submodule. In particular, any locally free sheaf is coherent. More examples can be found upon proving:

Proposition 1. *Let U be Zariski-open in a projective variety. A sheaf $\mathcal{F}$ of $\mathcal{O}_U$-modules is coherent if and only if it is, locally in the Zariski-topology, the quotient of a free $\mathcal{O}_U$-module of finite rank.*

Proof: Any Zariski-open subset of $\mathbb{C}^n$ is the union of the basic open sets

$$U_f := \mathbb{C}^n \setminus V(f),\ f \in \mathbb{C}[X_1, \ldots, X_n]$$

and so any Zariski-open set in $\mathbb{P}^n$ can also built up from such basic open sets by restricting to any of the affine open sets $U_j = \{ X_j \neq 0 \}$. Let me for the moment fix such an affine open set and identify it with $\mathbb{C}^n$. Consider its intersection with M. This intersection is the zero-locus $V(\mathfrak{p})$ of some prime-ideal $\mathfrak{p} \subset \mathbb{C}[x_1, \ldots, x_n]$. The ring of regular functions on $V(\mathfrak{p})$ is just the quotient ring $\mathbb{C}[X_1, \ldots, X_n]/\mathfrak{p}$ and hence Noetherian (see Appendix A1). Hence also the ring R of regular functions on $V(\mathfrak{p}) \cap U_f$ is Noetherian since it is a localisation of a Noetherian ring. It follows that any submodule of an R-module of finite rank again is of finite rank. ∎

Example 2.

The sheaf of ideals $\mathcal{I}_M$ of any projective variety $M \subset \mathbb{P}^n$ is coherent. The sheaf of ideals $\mathcal{I}_M$ is defined in the usual way by letting $\mathcal{I}_M(U)$ be the ideal of $\mathcal{O}(U)$ generated by the equations of M. Since finitely many suffice (Hilbert's Basis Theorem) you get a surjection $\mathcal{O}(U)^{\oplus n} \to \mathcal{I}_M(U)$. This even gives a surjective sheaf homomorphism, since the same equations for M are used over every open set.

Remark 3. Of course, one can likewise introduce the concept of coherent sheaf on any complex manifold. But the validity of the preceding proposition is much less trivial. This result is known as *Oka's lemma* a proof of which can be found in [Gu-Ro, Chapter IV B,C]. Note that the definition of coherent sheaf given there differs from ours and the results proved there essentially say that the definitions agree for sheaves of $\mathcal{O}_U$-modules where U is some open subset of $\mathbb{C}^n$ (in the ordinary topology).

B. Coherent sheaves on $\mathbb{C}^n$

The central result here is the vanishing of higher cohomology groups for coherent sheaves on affine varieties. This is needed in the next section

In this subsection put

$$R := \mathbb{C}[X_1, \ldots, X_n].$$

Note that the structure sheaf on $\mathbb{C}^n$ is completely determined by the modules

$$\mathcal{O}(U_f) := R_f,$$

the *localisations* of the ring R in f.

Let me now study in detail the coherent sheaves on $\mathbb{C}^n$.

Given any R-module M define a sheaf $M^{\sim}$ on $\mathbb{C}^n$ by

$$M^{\sim}(U_f) = M_f = \{\ m/f^n \mid n \in \mathbb{Z}_{\geq 0}, m \in M\ \}.$$

Such a sheaf by definition is called *quasi-coherent.* This is motivated by the remark that $M^{\sim}$ is coherent if M is a finitely-generated R-module. Indeed, $M(U_f)$ is finitely generated and so a quotient of $R_f^{\oplus n}$ by the submodule of the relations. So $M(U_f)$ is of the desired shape and over U_f the sheaf $M^{\sim}$ itself then is a quotient of the free sheaf $\mathcal{O}_{U_f}^{\oplus n}$ by the subsheaf coming from the relations.

The need for quasi-coherent sheaves originates from the following example.

Example 4. Let $f : X \to Y$ be a morphism between affine varieties and let $\mathcal{F}$ be quasi-coherent. Then $f_*\mathcal{F}$ is quasi-coherent. Indeed, one has $\mathcal{F} = M^{\sim}$ for some $R = \mathcal{O}(X)$-module M. You can also consider M as an $S = \mathcal{O}(Y)$-module using the natural pull-back of functions. The associated sheaf is just $f_*\mathcal{F}$. Note that even if $\mathcal{F}$ is coherent $f_*\mathcal{F}$ need not be coherent.

Now the following proposition says that a coherent sheaf on $\mathbb{C}^n$ is a quasi-coherent sheaf associated to a finitely generated $\mathcal{O}(\mathbb{C}^n)$ module.

Proposition 5. *If $\mathcal{F}$ is a coherent sheaf of $\mathcal{O}$-modules on $\mathbb{C}^n$, the module of its global sections $\Gamma(\mathcal{F})$ is a finitely generated R-module. The associated sheaf $\Gamma(\mathcal{F})^{\sim}$ is naturally isomorphic to $\mathcal{F}$.*

Proof: Cover $\mathbb{C}^n$ by finitely many Zariski-open sets U_{f_i}, $i = 1, \ldots, N$ over which $\mathcal{F}|U_{f_i} := \mathcal{G}_i$ is the quotient of $\mathcal{O}_{U_{f_i}}^{\oplus n}$ by a free $\mathcal{O}_{U_{f_i}}$-submodule

$$\mathcal{O}_{U_{f_i}}^{\oplus m} \xrightarrow{\alpha_i} \mathcal{O}_{U_{f_i}}^{\oplus n} \to \mathcal{G}_i \to 0.$$

Observe that now for all $g \in R$ the module $\mathcal{G}_i(U_{f_i} \cap U_g)$ is the cokernel of the restriction of α_i to $U_{f_i} \cap U_g = U_{f_i g}$. Look at the diagram

$$\begin{array}{ccccccc} R_{f_i}^{\oplus m} & \xrightarrow{\alpha_i} & R_{f_i}^{\oplus n} & \to & \mathcal{G}_i(U_{f_i}) = G_i & \to & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ R_{f_i g}^{\oplus m} & \to & R_{f_i g}^{\oplus n} & \to & \mathcal{G}_i(U_{f_i g}) & \to & 0 \end{array}$$

It follows that $\mathcal{G}_i(U_{f_i g}) = (G_i)_g$.

The natural restrictions $M = \mathcal{F}(\mathbb{C}^n) \xrightarrow{\rho} \mathcal{F}(U_g)$ induce a sheaf-homomorphism $M^{\sim} \to \mathcal{F}$ and I claim that it is an isomorphism. I must show that restriction

$$M^{\sim}(U_g) = M_g \xrightarrow{\rho_g} \mathcal{F}(U_g)$$

gives isomorphisms over the basis for the topology $\{ U_g \mid g \in R \}$.

1. ρ_g is injective. Suppose that for some $s \in M_g$ one has $\rho_g(s) = 0$. Let $s_i \in \mathcal{F}(U_{f_i}) = G_i$ be the restriction of s to U_{f_i}. Since s_i restricts to zero in $\mathcal{F}(U_g \cap U_{f_i}) = (G_i)_g$ one gets $s_i \cdot g^n = 0$. Since there are only finitely many U_{f_i} one can find an n which works simultaneously for all U_{f_i}. So $s \cdot g^n = 0$ and hence $s = 0$ since it is an element of the localisation M_g.

2. ρ_g is surjective. This can be shown in a similar fashion. One considers $t \in \mathcal{F}(U_g)$ and its restriction t_i to $U_{f_i} \cap U_g$. Since $t|U_{f_i} \in \mathcal{F}(U_g \cap U_{f_i}) = (G_i)_g$ one can write $t|U_{f_i} = s_i/g^{n_i}$ with $s_i \in G_i$. Now take $n = n_i$ independent of i. Now the sections t_i and t_j agree on $U_{f_i} \cap U_{f_j} \cap U_g$ (there they are equal to $s \cdot g^n$). So on $U_{f_i} \cap U_{f_j}$ itself you must have $g^m(t_i - t_j) = 0$. Again you can assume that m is independent of i and j. So the sections $g^{m+n}t_i$ glue to a section, say s of $\mathcal{F}(\mathbb{C}^n) = M$ with the property that $s/g^{m+n} = t|U_g$.

Finally, to complete the proof one has to see that M is a finitely generated R-module. The localisations M_{f_i} are known to be finitely generated R_{f_i} modules for a covering U_{f_i} of $\mathbb{C}^n$. The fact that this is a covering means that the f_i generate the ring R.

FACT: Let $N \subset M$ be a submodule and let $\mathrm{loc}_i : M \to M_{f_i}$ be the localisation map. Then
$$N = \bigcap_i \mathrm{loc}_i^{-1}(\mathrm{loc}_i(N) \cdot M_{f_i}).$$

Assume this fact. To show that M is finitely generated one only has to show that an increasing sequence of submodules $N_1 \subset N_2 \ldots$ becomes stationary. The submodules $\mathrm{loc}_i(N_1) \cdot M_{f_i}$, $\mathrm{loc}_i(N_2) \cdot M_{f_i}, \ldots$ become stationary in M_{f_i} since these are finitely generated. But this is true for any of the finitely many i. So there is some index independent of i beyond which the sequences become stationary. But then the fact can be applied to see that $N_1 \subset N_2 \ldots$ becomes stationary.

So it remains to establish the fact. Only the inclusion $\supset$ is non trivial. So suppose that $m \in M$ with $\mathrm{loc}_i(m) = n_i/f_i^{k_i}$ with $n_i \in N$. You may assume that $k = k_i$ independent of i and hence $f_i^k m - n_i = 0$ in M_{f_i}. So $f_i^{l_i}(f_i^k m - n_i) = 0$ in M. Again one may assume that $l_i = l$ independent of i and so
$$f_i^{k+l} m = f_i^l n_i \in N.$$
Now the f_i generate R, and hence this is true for the powers f_i^{l+k}, i.e. for some combination of R-coefficients one has $1 = \sum_i c_i f_i^{k+l}$ and it follows that
$$m = \sum_i c_i f_i^{k+l} m \in N$$
as required. ∎

In the next subsection one needs that on affine varieties a short exact sequence of sheaves sometimes gives a short exact sequence for the sections.

Lemma 6. *Let $X \subset \mathbb{C}^n$ be an affine variety and let*

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

be an exact sequence of $\mathcal{O}_X$-modules. Assume that $\mathcal{F}'$ is quasi-coherent. Then there is an induced exact sequence for the global sections

$$0 \to \Gamma(X, \mathcal{F}') \to \Gamma(X, \mathcal{F}) \to \Gamma(X, \mathcal{F}'') \to 0.$$

Proof: The only non-trivial point here is the fact that $\Gamma(X, \mathcal{F}) \to \Gamma(X, \mathcal{F}'')$ is a surjection. So let $s \in \Gamma(X, \mathcal{F}'')$ be a global section. Lift s locally, say over an open subset U_f to a section t of $\mathcal{F}$. I CLAIM that first of all for suitable natural number N the section $f^N s$ lifts to a global section of $\mathcal{F}$. Indeed, cover X by finitely many sets $U_i = U_{f_i}$ such that s lifts over U_i to $t_i \in \Gamma(U_i, \mathcal{F})$. Over $U_f \cap U_i$ the sections t and t_i both lift s and so their difference is a section of $\mathcal{F}'$. Since $\mathcal{F}'$ is quasi-coherent, Problem 4 shows that for suitable $n \in \mathbb{N}$ the section $f^n(t - t_i)$ extends to a section $u_i \in \Gamma(U_i, \mathcal{F}')$. One can take the same n for all U_i. Then $v_i := f^n t_i + u_i \in \Gamma(U_i, \mathcal{F})$ is a lifting of $f^n s$ which coincides with $f^n t$ on $U_i \cap U_f$. On $U_i \cap U_j$ the two sections v_i and v_j both lift $f^n s$ and so $v_i - v_j \in \Gamma(U_i \cap U_j, \mathcal{F}')$. Since v_i and v_j coincide over $U_i \cap U_j \cap U_f$, again by Problem 4 for some $m \in \mathbb{N}$, which can be taken independent of i and j, you have $f^m(v_i - v_j) = 0$. Now the sections $f^m v_i \in \Gamma(U_i, \mathcal{F})$ agree on overlaps so define a global section lifting $f^{n+m}s$.

Now cover X by a finite number of open sets $U_i' = U_{g_i}$ over which $g_i^n s$ lifts to section t_i of $\mathcal{F}$. Since the sets U_i' cover X, the ideal generated by the n-th powers of g_i generate the unit ideal in the coordinate ring $\mathcal{O}(X)$ of X and one can write

$$1 = \sum_i r_i g_i^n, \quad r_i \in \mathcal{O}(X).$$

The section $t := \sum_i r_i t_i \in \Gamma(X, \mathcal{F})$ has image $\sum_i r_i g_i^n s = s$ in $\Gamma(X, \mathcal{F}'')$. ■

Let me now prove a fundamental result which is seemingly stronger (remember, I am working with Čech cohomology for which the exactness for the cohomology sequence has not been established; In 5C it will be shown for coherent sheaves on projective varieties):

Proposition 7. *Let $X \subset \mathbb{C}^n$ be an affine variety and $\mathcal{F}$ a quasi-coherent sheaf on X. Then for the Čech groups one has $H^q(X, \mathcal{F}) = 0$ for $q \geq 1$.*

Proof: Extend $\mathcal{F}$ by zero outside of X. The resulting sheaf now is quasi-coherent as a sheaf of $\mathcal{O}_{\mathbb{C}^n}$-modules (see Problem 2.) So I may assume that $X = \mathbb{C}^n$. From the previous proposition it follows that $\mathcal{F} = M^{\sim}$ for some R-module M of finite rank. I show that for any finite affine covering $\mathfrak{U}$ of $\mathbb{C}^n$ given by open sets of the form U_f the groups $H^q(\mathfrak{U}, \mathcal{F})$ vanish for $q > 0$. Suppose $\mathfrak{U} = \{ U_{f_i} \}$, $i = 1, \ldots, N$. Let $c \in C^q(\mathfrak{U}, \mathcal{F})$ and let $\sigma = \{ i_0, \ldots, i_q \}$ be a q-simplex of $\mathfrak{U}$. Then

$$c(\sigma) = \frac{m_{i_0 \cdots i_q}}{f_{i_0}^{n_{i_0}} \cdots f_{i_q}^{n_{i_q}}}, \quad m_{i_0 \cdots i_q} \in M.$$

There are polynomials P_j such that

$$\sum_j P_j f_j^{n_j} = 1.$$

This is the case because $\mathfrak{U}$ is a covering so that the f_j and hence also the $f_j^{n_j}$ generate R. Define $g \in C^{q-1}(\mathfrak{U}, \mathcal{F})$ by setting for any $(q-1)$-simplex $\tau = \{ i_0 \dots i_{q-1} \}$

$$g(\tau) = (-1)^q \sum_k P_k \frac{m_{i_0 \cdots i_k \cdots i_{q-1} k}}{f_{i_0}^{n_{i_0}} \cdots f_{i_{q-1}}^{n_{i_{q-1}}}}.$$

Suppose that $d(c) = 0$. This implies

$$\sum_{k=0}^{q+1} (-1)^k m_{i_0 \cdots \widehat{i_k} \cdots i_{q+1}} f_{i_k}^{n_{i_k}} = 0.$$

Then I have

$$\begin{aligned} d(g)(\sigma) =& (-1)^q \sum_k \sum_{l=0}^{q} \frac{(-1)^l m_{i_0 \cdots \widehat{i_l} \cdots i_q k} P_k f_{i_l}^{n_{i_l}}}{f_{i_0}^{n_{i_0}} \cdots f_{i_q}^{n_{i_q}}} \\ =& \sum_k P_k \frac{m_{i_0 \cdots i_q} f_k^{n_k}}{f_{i_0}^{n_{i_0}} \cdots f_{i_q}^{n_{i_q}}} \\ =& \frac{m_{i_0 \cdots i_q}}{f_{i_0}^{n_{i_0}} \cdots f_{i_q}^{n_{i_q}}} \\ =& c(\sigma). \end{aligned}$$

So every cocycle is a coboundary. ∎

C. Coherent sheaves on $\mathbb{P}^n$

Here, the explicit description, due to Serre, of the cohomology groups of the basic coherent sheaves $\mathcal{O}(k)$ on projective space is given. Using this, Serre's finiteness and vanishing results are proven for coherent sheaves on arbitrary projective varieties. A suitable relative version allows one to deduce that the higher direct images of coherent sheaves remain coherent (under a morphism between projective varieties).

In this subsection, let me put

$$S := \mathbb{C}[X_0, \dots, X_n]$$

and consider it as a graded ring, where you grade by degree. The rings $S(d)$ are the same ring as S but you shift the grading up, i.e. the degree of a homogeneous polynomial of degree e is given degree $e - d$ in $S(d)$. It is a graded S-module.

Now follows a fundamental construction for coherent sheaves on $\mathbb{P}^n$. Let M be a graded S-module and define the associated sheaf $M^\sim$ by

$$M^\sim(U) = \{ m/f \mid m \in M, \, f \in S, \, \deg f = \deg m, \, f(x) \neq 0, \forall x \in U \}.$$

If M is finitely generated this sheaf is coherent.

Example $S(d)^\sim \cong \mathcal{O}(d)$. See Problem 3.

Before passing to the central result of this section let me briefly pause to look back at the Čech cohomology groups. These do not give a good cohomology theory: for a short exact sequence of sheaves, the induced sequence in cohomology need not be exact. Now I show that this is a good cohomology theory when restricted to quasi-coherent sheaves on (quasi-)projective varieties:

Proposition 8. *Let X be any (quasi-) projective variety and let*

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

be an exact sequence of quasi-coherent sheaves. The associated sequence in Čech cohomology is exact.

Proof: Use Lemma 4.6 to see that the induced sequence of groups of Čech cochains

$$0 \to C^q(\mathcal{F}') \to C^q(\mathcal{F}) \to C^q(\mathcal{F}'') \to 0$$

is exact. Now these are ordinary cochain complexes and thus there is a long exact sequence for the associated cohomology. Then one passes to the direct limit. ■

In view of the remarks made in §4 on the axiomatic aspect of cohomology theory, one can now unambiguously speak of THE cohomology groups for quasi-coherent sheaves on (quasi-)projective varieties. In particular one can apply Leray's theorem.

Proposition 9. *Let X be a projective variety, $\mathfrak{U}$ an open affine cover and $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. The natural map*

$$H^p(\mathfrak{U}, \mathcal{F}) \to H^p(X, \mathcal{F})$$

is an isomorphism.

Proof: The intersection of two affine open sets is again affine. By Proposition 4.7, the higher cohomology groups of a coherent sheaf vanish on any affine set. One can then apply Leray's result, Theorem 3.4. ■

Concerning the sheaves $\mathcal{O}(d)$ there is the following fundamental result due to Serre:

Theorem 10.

(i). The natural map $S \to \bigoplus_{n\in\mathbb{Z}} H^0(\mathcal{O}(n))$ is a graded isomorphism.

(ii). $H^i(\mathcal{O}(k)) = 0$ for $0 < i < n$.

(iii). $H^n(\mathcal{O}(-n-1)) \cong \mathbb{C}$ and $H^n(\mathcal{O}(k)) = 0$ for $k > -n-1$.

(iv). For $k \geq 0$ the natural map

$$H^0(\mathcal{O}(k)) \times H^n(\mathcal{O}(-k-n-1)) \longrightarrow H^n(\mathcal{O}(-n-1) \cong \mathbb{C}$$

is a perfect pairing.

Proof: Introduce the sheaf $\mathcal{F} = \bigoplus_{n\in\mathbb{Z}} \mathcal{O}(n)$ and let $\mathfrak{U}$ be the standard cover of $\mathbb{P}^n$ by affines $X_j \neq 0$, $j = 0, \ldots, n$. Although this sheaf is not coherent, it is the direct sum of coherent sheaves and by the previous theorem $H^q(|\sigma|, \mathcal{F}) = 0$ for $q > 0$ and any simplex σ of $\mathfrak{U}$. So Leray's theorem (see Theorem 3.4) then shows that the cohomology of $\mathcal{F}$ on $\mathbb{P}^n$ can be computed as the Čech -cohomology with respect to $\mathfrak{U}$. First consider

$$\mathcal{F}(|\sigma|) = \{ \frac{F}{G} \mid F, G \in S,\ G \neq 0 \text{ on } |\sigma| \}.$$

Now $G \neq 0$ on $|\sigma|$ with $\sigma = \{ i_0, \ldots, i_p \}$ means that G is a polynomial in $X_{i_0}, \ldots, X_{i_p}$ only. To compute H^0 one considers quotients $F_i/X_i^{d_i}$, $i = 0, \ldots, n$ so that $F_i/X_i^{d_i} = F_j/X_j^{d_j}$ in the overlaps. But then $F_i = FX_i^{d_i}$ for some $F \in S$. Hence (i). follows.

Next, note that

$$\mathcal{F}(U_0 \cap U_1 \cap \ldots \cap U_n) = \bigoplus_{d_i\in\mathbb{Z}} \mathbb{C}X_0^{d_0} \cdots X_n^{d_n},$$
$$\mathcal{F}(U_0 \ldots \cap \widehat{U_j} \cap \ldots \cap U_n) = \bigoplus_{d_i\in\mathbb{Z},\ d_j\geq 0} \mathbb{C}X_0^{d_0} \cdots X_n^{d_n}$$

and the Čech -coboundary $C^{n-1}(\mathfrak{U}, \mathcal{F}) \to C^n(\mathfrak{U}, \mathcal{F})$ is the natural inclusion

$$\bigoplus_j \bigoplus_{d_i\in\mathbb{Z},\ d_j\geq 0} \mathbb{C}X_0^{d_0} \cdots X_n^{d_n} \to \bigoplus_{d_i\in\mathbb{Z}} \mathbb{C}X_0^{d_0} \cdots X_n^{d_n}.$$

It follows that $H^n(\mathfrak{U}, \mathcal{F}) = \bigoplus_{d_i<0} \mathbb{C}X_0^{d_0} \cdots X_n^{d_n}$ and the part in degree $-n-1$ which computes $H^n(\mathfrak{U}, \mathcal{O}(-n-1))$ is one-dimensional with basis $\mathbb{C}X_0^{-1} \cdots X_n^{-1}$. This proves the first part of (iii). Furthermore, observe that there is nothing in the cokernel of degree $> -n-1$, proving the remaining assertion of (iii).

Now, continuing with the previous computation, the part in degree $-k-n-1$ is the $\mathbb{C}$-vector space with basis consisting of the 'monomials' of the form $X_0^{d_0} \cdots X_n^{d_n}$ with all degrees d_j negative and with total degree $-k-n-1$. Consider the multiplication

$$H^0(\mathcal{O}(k)) \times H^n(\mathcal{O}(-k-n-1)) \longrightarrow H^n(\mathcal{O}(-n-1))$$

which translates into the natural multiplication

$$\bigoplus_{\substack{d_i\geq 0\\ \sum d_i = k}} \mathbb{C}X_0^{d_0} \cdots X_n^{d_n} \times \bigoplus_{\substack{d_i'<0\\ \sum d_i' = -k-n-1}} \mathbb{C}X_0^{d_0'} \cdots X_n^{d_n'} \to \mathbb{C}X_0^{-1} \cdots X_n^{-1}.$$

The product of $X_0^{d_0} \cdots X_n^{d_n}$ with $X_0^{d_0'} \cdots X_n^{d_n'}$ is zero in $H^n(\mathfrak{U}, \mathcal{O}(-n-1))$ if any $d_i + d_i' \geq 0$. So one only gets a non-zero element if $d_i' = -d_i - 1$ for all $i = 0, \ldots n$. So the pairing is perfect since the basis dual to the basis $\{ X_0^{d_0} \cdots X_n^{d_n} \mid d_i \geq 0; \sum d_i = k \}$ is the basis $\{ X_0^{-d_0-1} \cdots X_n^{-d_n-1} \}$. This proves (iv).

I prove (ii) by induction on n. This is done in two steps:

Step 1. I show that multiplication by X_n induces a bijection on $H^k(\mathcal{F})$.

Consider the exact sequence

$$0 \to \mathcal{F}(-1) \xrightarrow{\cdot X_n} \to \mathcal{F} \to \mathcal{F}|\{X_n = 0\} \to 0.$$

Let me put $H = \{ X_n = 0 \}$. Part of the long exact sequence in cohomology reads as follows

$$H^{i-1}(\mathcal{F}|H) \to H^i(\mathcal{F}(-1)) \xrightarrow{\cdot X_n} H^i(\mathcal{F}) \to H^i(\mathcal{F}|H).$$

Note that $\mathcal{F}|H = \bigoplus_{k\in\mathbb{Z}} \mathcal{O}_H(k)$ and so induction shows that for $i = 2, \ldots, n-2$ multiplication by X_n gives an isomorphism. When $i = 1$ you have a surjection and when $i = n-1$ you have an injection. Applying (i) one obtains an exact sequence

$$0 \to H^0(\mathcal{F}(-1)) \to H^0(\mathcal{F}) \to H^0(\mathcal{F}|H) \to 0$$

which implies that the next map $H^1(\mathcal{F}(-1)) \xrightarrow{X_n} H^1(\mathcal{F})$ in the sequence is injective in addition to being surjective.

Similarly, applying (iii) one finds an exact sequence

$$0 \to H^{n-1}(\mathcal{F}|H) \to H^n(\mathcal{F}(-1)) \to H^n(\mathcal{F}) \to 0$$

and so multiplication by X_n is surjective on $H^{n-1}(\mathcal{F})$ in addition to being injective.

Step 2. I show that for a given $u \in H^k(\mathcal{F})$ one has $X_n^s u = 0$ for a suitable non-negative power of X_n.

Note that U_n is affine and so $0 = H^k(U_n, \mathcal{F}) = H^k(\mathfrak{U} \cap U_n, \mathcal{F}) = H^k(C^\bullet(\mathfrak{U} \cap U_n, \mathcal{F}))$. But the module $C^q(\mathfrak{U} \cap U_n, \mathcal{F})$ is nothing but the localisation $C^q(\mathfrak{U}, \mathcal{F})_{X_n}$ and so $H^k(C^\bullet(\mathfrak{U} \cap U_n, \mathcal{F})) = H^k(C^\bullet(\mathfrak{U}, \mathcal{F}))_{X_n}$. This localisation vanishes precisely when for all $u \in H^k(\mathcal{F})$ some power of X_n kills u. ■

Let me derive an important consequence of this computation. First you need to know that any coherent sheaf on $\mathbb{P}^n$ is the quotient of a direct sum of line bundles.

Proposition 11. *There is a short exact sequence*

$$\bigoplus_{j=1}^{k} \mathcal{O}(n_j) \to \mathcal{F} \to 0.$$

Proof: I'll show that in fact for large enough N there is a surjection of the trivial sheaf $\mathcal{O}^{\oplus m}$ onto $\mathcal{G} = \mathcal{F}(N)$. This means that $\mathcal{G}$ is generated by sections, i.e. there are sections s_j, $j = 1, \ldots m$ such that every stalk $\mathcal{G}_x$ is generated by the $s_j(x)$. Indeed the standard generators of $\mathcal{O}^{\oplus m}$ map to generators of $\mathcal{G}$.

To prove this, consider the standard affine cover U_i, $i = 0, \ldots, n$. Now $\mathcal{F}|U_i = M_i^{\sim}$ for some R_i-module M_i, where $R_i = \mathbb{C}[X_0/X_i, \ldots, X_n/X_i]$. I shall make use of the following lemma.

Lemma 12. *Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}^n$ and let f be a global section of $\mathcal{O}(d)$, i.e. a homogeneous polynomial of degree d. Let $U_f = \{ x \in \mathbb{P}^n \mid f(x) \neq 0 \}$ and suppose that one has $t \in \Gamma(U_f, \mathcal{F})$. Then for some $N \in \mathbb{N}$ the section $f^N t$ of $\mathcal{F}(Nd)$ over U_f extends over $\mathbb{P}^n$.*

For a proof I refer to Problem 4.

View X_i as a section of $\mathcal{O}(1)$. It follows that for any $s \in M_i$ for large enough N the section $X_i^N s \in M_i^{\sim}(N)$ extends as a section of $\mathcal{F}(N)$ over $\mathbb{P}^n$. Let me take N large enough so that I can use it for all $i = 0, \ldots, n$. Let me apply this simultaneously to the finitely many generators $\{ s_{ij} \}$ of the module M_i. Now multiplication by x_i^N induces an isomorphism $M_i^{\sim} \to \mathcal{F}(N)|U_i$ and so the sections $s_{ij} x_i^N$ generate the latter sheaf. But these sections extend to sections of $\mathcal{F}$, so together they generate $\mathcal{F}(N)$ everywhere.

■

Theorem 13. (Serre's Finiteness and Vanishing Theorem) *Let $X \subset \mathbb{P}^n$ be a projective variety and let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Then*

(a) for each $i \geq 0$ the vector space $H^i(X, \mathcal{F})$ is finite-dimensional.

(b) there is an integer n_0 depending only on $\mathcal{F}$ so that $H^i(\mathcal{F}(n)) = 0$ for all $i > 0$ and all $n \geq n_0$.

Proof: One can reduce to the case $X = \mathbb{P}^n$ since the sheaf obtained from $\mathcal{F}$ by extending it by zero on the complement of X is a coherent sheaf of $\mathcal{O}_{\mathbb{P}^n}$-modules (Problem 1.) It follows immediately from Theorem 4.10 that the theorem holds for any sheaf which is a direct sum of sheaves of the form $\mathcal{O}(n)$. Now by the previous lemma, $\mathcal{F}$ is a quotient of such a direct sum $\mathcal{E}$ and so there is an exact sequence

$$0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{F} \to 0$$

with $\mathcal{E}'$ again coherent. From the resulting exact sequence one gets

$$\ldots \to H^i(\mathcal{E}) \to H^i(\mathcal{F}) \to H^{i+1}(\mathcal{E}') \to \ldots.$$

The vector space on the left is finite-dimensional by the previous theorem.

To prove the theorem I now use descending induction on i. For $i > n$ one has $H^i(\mathbb{P}^n, \mathcal{F}) = 0$ since the standard affine covering of $\mathbb{P}^n$, which computes the cohomology, consists of $n+1$ elements. It follows that I may assume that $H^{i+1}(\mathcal{E}') = 0$ and so $H^i(\mathcal{F})$ is finite-dimensional. This proves (a).

To prove (b) let me twist the preceding sequence and consider the following piece of the resulting long exact sequence

$$\ldots \to H^i(\mathcal{E}(n)) \to H^i(\mathcal{F}(n)) \to H^{i+1}(\mathcal{E}'(n)) \to \ldots.$$

Now again, by the previous theorem the vector space on the left vanishes for all n larger than a certain number m_0 which works for all i. By the induction hypothesis $H^{i+1}(\mathcal{E}'(n)) = 0$ for $n \geq m_1$ independent of i. Now take $n_0 = \max(m_0, m_1)$. ■

Inspecting the proof of this theorem more closely one derives the following proposition.

Proposition 14. *Let X be a projective variety of dimension n and $\mathcal{F}$ a coherent sheaf on X. Then $H^q(X, \mathcal{F}) = 0$ if $q > n$.*

Proof: If $X \subset \mathbb{P}^N$ a general linear subspace of codimension $n+1$ is disjoint from X so that projecting from it yields a morphism of X onto $\mathbb{P}^n$. Now, the standard affine covering of $\mathbb{P}^n$ consists of $n+1$ elements. Since q is a projection, the inverse by q of an affine open set on $\mathbb{P}^n$ gives an affine open set on X and so you get an acyclic cover of X by $n+1$ open sets. Leray's theorem then implies that $H^q(X, \mathcal{F}) = 0$ for $q > n$. ■

Note that the preceding finiteness theorem can be formulated in relative form.

Proposition 15. *Let $U \subset \mathbb{C}^m$ be affine and let X be any irreducible Zariski-closed subset in $\mathbb{P}^n \times U$. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Then $H^i(X, \mathcal{F})$ is a finitely generated $\mathcal{O}(U)$-module.*

For the proof let me refer to Problem 5.

Corollary 16. *Let $f : X \to Y$ be a morphism between projective varieties and $\mathcal{F}$ any coherent sheaf of $\mathcal{O}_X$-modules. The direct image sheaf $f_*\mathcal{F}$ is coherent.*

Proof: Since the question is local one may assume that Y is affine. Take a covering of X by a finite number of affines U_i and let $V \subset Y$ be affine. Giving a section of $\mathcal{F}$ over $f^{-1}V$ is the same as giving sections over $U_i \cap f^{-1}V$ which patch over the intersections, i.e there is an exact sequence

$$0 \to f_*\mathcal{F} \to \bigoplus_i f_*(\mathcal{F}|U_i) \to \bigoplus_{i,j} f_*(\mathcal{F}|U_i \cap U_j),$$

and since the last two terms are quasi-coherent (Example 4.4) the first sheaf is quasi-coherent as well (Problem 1). So it is the sheaf associated to $f_*\mathcal{F}(Y) = \mathcal{F}(X) = H^0(X, \mathcal{F})$ which however is a finitely generated $\mathcal{O}(Y)$-module by the previous Proposition.

■

Next, let me introduce higher direct images.

Definition 17. *For any continuous map $f : X \to Y$ between topological spaces and any sheaf $\mathcal{F}$ of abelian groups on X let the q-the direct image sheaf $R^q f_*\mathcal{F}$ be the sheaf associated to the presheaf*

$$V \mapsto H^q(f^{-1}(V), \mathcal{F}|f^{-1}V).$$

To compute higher direct images in the case of morphisms between projective manifolds let me first consider the case where the target space is an affine variety.

Lemma 18. *let X be a (quasi-)projective variety, Y affine, $\varphi : X \to Y$ a morphism and $\mathcal{F}$ a quasi-coherent sheaf on X. The higher direct image $R^q\varphi_*\mathcal{F}$ is the sheaf on Y associated to the module $H^q(X, \mathcal{F})$.*

Proof: Choose an affine open cover $\mathfrak{U}$ of X. By proposition 4.7 this is an acyclic cover. Recall that in the proof of Leray's theorem, Theorem 3.4, the sheaf $\mathcal{C}^p(\mathfrak{U},\mathcal{F})$ given by $U \mapsto C^p(U \cap \mathfrak{U},\mathcal{F})$ has been introduced. The complex $\mathcal{C}^\bullet(\mathfrak{U},\mathcal{F})$ has been shown to be exact and to give a cohomological resolution for $\mathcal{F}$ since the covering is acyclic. So, if f is in the coordinate ring $R(Y)$ of Y defining the open set U_f, one has $H^q(\varphi^{-1}U_f,\mathcal{F}) = H^q(\Gamma(\mathcal{C}^\bullet(\mathfrak{U}\cap U_f,\mathcal{F})))$. Now $\mathcal{C}^q(\mathfrak{U},\mathcal{F})$ is the quasi-coherent sheaf associated to the $R(Y)$-module $C^q(\mathfrak{U},\mathcal{F}) = \prod_\sigma \mathcal{F}(|\sigma|)$ where you take the product over all q-simplices. So one can write

$$H^q(\varphi^{-1}U_f,\mathcal{F}) = H^q(C^\bullet(\mathfrak{U},\mathcal{F})\tilde{\ }(U_f)) = H^q(C^\bullet(\mathfrak{U},\mathcal{F})\tilde{\ }(U_f)),$$

and by Leray's theorem, this last module is isomorphic to $H^q(X,\mathcal{F})\tilde{\ }(U_f)$. ∎

Corollary 19. *Let $f : X \to Y$ be a morphism between projective varieties and $\mathcal{F}$ any coherent sheaf of $\mathcal{O}_X$-modules. The higher direct image sheaves $R^q f_*\mathcal{F}$ are coherent.*

Proof: The assertion is local on Y and so one can assume that Y is affine and then one can apply the previous result. Now, by Proposition 4.15, $H^q(X,\mathcal{F})$ is a module of finite rank over the affine coordinate ring of Y and so the sheaf $R^q f_*\mathcal{F}$ is not only quasi-coherent but even coherent. ∎

D. Applications to very ampleness.

I derive an ampleness criterion for line bundles which play a central role in the proof of Nakai's ampleness criterion for divisors on surfaces, to be treated later.

Next, let me study line bundles L on compact complex manifolds X. To prove that L is very ample one has to show:

1. The map φ_L must be everywhere defined. So, for every point $x \in X$ there is a section of L which is non-zero in x. Let $\mathfrak{m}_x$ be the maximal ideal in the ring $\mathcal{O}_x$, i.e. the set of germs of functions vanishing at x and let $\mathcal{L} = \mathcal{O}(L)$. The exact sequence

$$0 \to \mathfrak{m}_x \cdot \mathcal{L} \to \mathcal{L} \to L_x \to 0$$

shows that it is sufficient to prove that $H^1(\mathfrak{m}_x \cdot \mathcal{L}) = 0$.

2. The map φ_L must be injective, i.e. sections must separate pairs of points. So, for every two points $x, y \in X$ there must be sections s, s' of L with $s(x) = 0$, $s(y) \neq 0$ and $s'(x) \neq 0, s'(y) = 0$. The exact sequence

$$0 \to \mathfrak{m}_x \cdot \mathfrak{m}_y \cdot \mathcal{L} \to \mathcal{L} \to L_x \oplus L_y \to 0$$

shows that it is enough to show that $H^1(\mathfrak{m}_x \cdot \mathfrak{m}_y \cdot \mathcal{L}) = 0$.

3. Sections must separate tangent directions. This means the following. Locally sections of L are holomorphic functions on an open subset U of X and if $x \in U$ there must be enough sections vanishing at x so that their differentials span $T_x(X)$. So, for every cotangent direction $v^* \in T_x^\vee X$ there must be a section s of L vanishing at x and with

$ds(x) = v^*$. This can be formulated more intrinsically as follows. There is a well-defined sheaf map

$$d_x : \mathfrak{m}_x \cdot \mathcal{L} \to L_x \otimes T_x^\vee$$

and it should be surjective on the level of sections. The exact sequence

$$0 \to \mathfrak{m}_x^2 \cdot \mathcal{L} \to \mathfrak{m}_x \cdot \mathcal{L} \xrightarrow{d_x} L_x \otimes T_x^\vee \to 0$$

shows that it is enough to show the vanishing of $H^1(\mathfrak{m}_x^2 \cdot \mathcal{L})$.

Let me collect the results:

Proposition 20. *Let L be a line bundle on a compact complex manifold X. Let $\mathcal{L} = \mathcal{O}_X(L)$ be the corresponding locally free sheaf. The map φ_L is defined at x if $H^1(X, \mathfrak{m}_x \cdot \mathcal{L}) = 0$. It separates x from y if $H^1(X, \mathfrak{m}_x \cdot \mathfrak{m}_y \cdot \mathcal{L}) = 0$ and it separates tangents at x if $H^1(X, \mathfrak{m}_x^2 \cdot \mathcal{L}) = 0$.*

Now, if φ_L is defined at x this is true in a Zariski-open neigbourhood if x. Similarly, if φ_L separates x and y it will separate points in a neigbourhood of x from points in a neighbourhood of y and if φ_L is an immersion at x it will be so in a neighbourhood. By compactness, the previous remarks show that it sufficient to prove vanishing of $H^1(\mathfrak{m}_x \cdot \mathcal{L})$, $H^1(\mathfrak{m}_x \cdot \mathfrak{m}_y \cdot \mathcal{L})$ and $H^1(\mathfrak{m}_x^2 \cdot \mathcal{L})$ for a certain finite number of points x and y. Since this involves a finite number of coherent sheaves on X, by the previous theorem one can find some large integer N so that the desired groups vanish provided you replace $\mathcal{L}$ by $\mathcal{L}(N)$. In other words, $\mathcal{L}(N)$ will be very ample. So

Corollary 21. *Any line bundle L on a projective manifold is of the form $L' \otimes L''^{-1}$ with L' and L'' very ample. In particular, every line bundle on X is of the form $\mathcal{O}_X(D)$ for some divisor D.*

From this Corollary it follows that the map

$$\operatorname{Div}(X) \to \operatorname{Pic} X,$$

introduced in Chapter 3, is surjective.

Remark 22. Since a generic hyperplane section of a projective manifold is smooth (by Bertini, see 2.1) and connected by A2.21, it follows that one can assume $D = D_1 - D_2$ with D_1 and D_2 smooth and connected.

Let me finish by proving a very useful criterion for ampleness which is used when proving the Nakai Ampleness Criterion 11.14 for surfaces.

Proposition 23. (Criterion for Ampleness) *Let M be a projective manifold and L a line bundle on M. The following are equivalent.*

1. *L is ample,*
2. *$H^p(\mathcal{F} \otimes \mathcal{O}_M(L^{\otimes n})) = 0$, $p > 0$, for all coherent sheaves $\mathcal{F}$ on M and $n > n(\mathcal{F})$.*
3. *$\mathcal{F} \otimes \mathcal{O}_M(L^{\otimes n})$ is spanned by its sections for all coherent sheaves $\mathcal{F}$ on M and $n > m(\mathcal{F})$.*

Proof:
1. $\Longrightarrow$ 2. For very ample L this is Serre's Theorem. 4.13. Otherwise, if $L^{\otimes m}$ is very ample, one has by loc. cit. $H^p(\mathcal{F} \otimes \mathcal{O}_M(L^{\otimes r} \otimes L^{\otimes ms}) = 0$, $p > 0$ for $s > n_r$, $r = 0, \ldots, m-1$. Take $n(\mathcal{F}) = m \max n_r$. Then, writing $n = ms + r$ with $0 \leq r < m$ one has for $n > n(\mathcal{F})$ that $H^p(\mathcal{F} \otimes L^{\otimes n}) = 0$, $p > 0$.

2. $\Longrightarrow$ 3. To prove that $\mathcal{F} \otimes \mathcal{O}_M(L^{\otimes n})$ is spanned by sections at $x \in M$ it is sufficient to show that $H^1(\mathfrak{m}_x \cdot (\mathcal{F} \otimes \mathcal{O}_M(L^{\otimes n})) = 0$ which by assumption is the case for $n > n_x$. Spannedness then holds in a Zariski-open neighbourhood of x, say U_x. By compactness finitely many such sets cover M, say U_{x_i}, $i = 1, \ldots, N$. Now take $n(\mathcal{F}) = \max(n_{x_i})$.

3. $\Longrightarrow$ 1. Start with an affine neighbourhood U of x and let $N \subset M$ be the complement. Then for some $n > 0$ the sheaf $\mathcal{I}_N \otimes \mathcal{O}_M(L^{\otimes n})$ is generated by sections and in particular there is a section s of $L^{\otimes n}$ vanishing at x. By construction $U_s = \{ y \in M \mid s(y) \neq 0 \} \subset U$ and hence is an affine neighbourhood of x over which $L^{\otimes n}$ is trivial. Finitely many such sets $U_{s_1}, \ldots, U_{s_k}$ cover M, since M is compact. Let $X_1^j, \ldots, X_n^j$ be affine coordinates in U_{s_j}. By Problem 4 there is an integer m such that all of the functions $s_j^m X_k^j$ extend to global sections t_j^k of $L^{\otimes mn}$ and one replaces $L^{\otimes n}$ by $L^{\otimes mn}$ and s_j by s_j^m. Already the sections s_j^m generate the bundle $L^{\otimes mn}$ globally and so these sections together with the t_j^k define a morphism $X \to \mathbb{P}^N$ which is constructed in such a way that it gives an embedding when restricted to each U_{s_i}. But points on U_{s_i} separate from points in the complement, since s_i^m is not zero on U_{s_i} while this section vanishes on the complement. ■

Corollary 24. *Let $f : X \to Y$ a morphism between projective varieties with finite fibres and let L be ample on Y. Then f^*L is ample on X.*

Proof: The sheaf $f_*\mathcal{F}$ is coherent on Y by 4.16. Furthermore, one has $f_*(\mathcal{F} \otimes \mathcal{O}_X(f^*L^{\otimes n}))$ $= f_*\mathcal{F} \otimes \mathcal{O}_Y(L^{\otimes n})$ (see Problem 8). Now, since f has finite fibres, $H^p(f_*(\mathcal{O}_X(f_*\mathcal{F} \otimes \mathcal{O}_Y(L^{\otimes n})) = H^p(f_*\mathcal{F} \otimes \mathcal{O}_Y((L^{\otimes n}))$ by Problem 7. The result follows from the previous criterion. ■

Problems.

4.1. Let $\mathcal{F} \to \mathcal{G}$ be a homomorphism between (quasi-) coherent sheaves on a Zariski-open subset of a projective variety. Show that the kernel, the cokernel and the image are (quasi-) coherent. Next, let
$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$
be an exact sequence of sheaves. Show that if any two of the preceding sheaves is (quasi-) coherent then so is the third.

4.2. Let U be a Zariski-open subset of $\mathbb{P}^n$ and let $X \subset \mathbb{P}^n$ be a projective variety. If $\mathcal{F}$ is coherent on $U \cap X$ show that the sheaf $\mathcal{F}$ considered as a sheaf of $\mathcal{O}_U$-modules is also coherent.

4.3. Prove that that $S(d)^{\sim} \cong \mathcal{O}(d)$.

4.4. Let X be a projective variety and $\mathcal{F}$ a quasi-coherent sheaf on X. Let L a line bundle on X, s a section of L and set $U_s = \{ y \in X \mid s(y) \neq 0 \}$.

a. If t is a global section of $\mathcal{F}$ restricting to zero on U_s, there exists $n > 0$ such that $f^n s = 0$.

b. Suppose that t now is a section of $\mathcal{F}$ over U_s. Prove that there is an integer $n > 0$ such that $s^n t$ extends to a global section of $\mathcal{F} \otimes \mathcal{O}_X(L)$.

4.5. Prove Lemma 4.12. Hint: use the previous Problem.

4.6. Prove the relative version of Serre's finiteness theorem (Theorem 4.15).

4.7. Let $f : X \to Y$ be a continuous map between topological spaces and $\mathcal{F}$ a sheaf on X with the property that $R^p f_* \mathcal{F} = 0$, $p > 0$. Prove that $H^q(X, \mathcal{F}) \cong H^q(Y, p_* \mathcal{F})$, $q \geq 0$. Show that this can be applied to morphisms between projective varieties with finite fibres. (This is a special case of Leray's Spectral Sequence)

4.8. Let $f : X \to Y$ be a morphism between projective varieties. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module and E a locally free $\mathcal{O}_X$-module of finite rank. Prove the *projection formula*

$$R^p f_*(\mathcal{F} \otimes_{\mathcal{O}_X} f^* E) \cong R^p f_* \mathcal{F} \otimes_{\mathcal{O}_Y} E.$$