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Chapter 6. The Enriques Classification

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13. Statement of the Enriques Classification Theorem

I introduce the classes of surfaces comprising the Enriques classification and give examples of each of these classes before I stae the classification theorem.

So I first introduce a number of useful concepts related to the classification theorem and illustrate these by examples.

Definition 1. *Let S be a surface and C a smooth (projective) curve. A morphism $f : S \to C$ is called a fibration if f is surjective and has connected fibres. If the generic fibre (which is a smooth projective curve) has genus g, the fibration f is called a genus-g fibration. A genus-1 fibration is also called an elliptic fibration. Any surface admitting an elliptic fibration is called an elliptic surface.*

Before giving some examples let me summarise the basic properties of the Kodaira-dimension as treated in §10.

Definition-Proposition 2. *The Kodaira-dimension $\kappa(X)$ of a projective manifold X is the maximal dimension of a pluricanonical image. Equivalently $\kappa(X) = k$ if and only if there are two positive numbers A, B with $A < \frac{P_m}{m^k} < B$ for all $m \gg 0$ for which $P_m(X) \neq 0$. If all plurigenera of X vanish, one sets $\kappa(X) = -\infty$.*

Examples 3.

1. The easiest example is of course a product $C \times F$ of two smooth curves, which is a fibration in two ways. Let me calculate the invariants. It was shown before that $q(C \times F) = g(C) + g(F)$, the sum of the genera of the factors and that $p_g(C \times F) = g(C) \cdot g(F)$. In a similar way one finds $P_n(C \times F) = P_n(C) \cdot P_n(F)$. So this gives examples of Kodaira-dimensions $-\infty$ (one of the factors $\mathbb{P}^1$), 0 (both factors elliptic), 1 (one factor elliptic and one of genus ≥ 2) or 2 (both factors of genus ≥ 2).

2. Another type are the fibre bundles over a smooth curve C with fibre F and structure group $\operatorname{Aut} F$, the group of biholomorphic automorphisms of F. You construct them by covering C by Zariski-open sets U_j and glueing the union $U_j \times F$ by means of transition functions $U_i \cap U_j \to \operatorname{Aut} F$.

Concrete examples are the quotients of a product of two curves, $C' \times F'$ by a finite group G, where G is a group of automorphisms of C' and F'. Even more concretely, one may take $G = \mathbb{Z}/a\mathbb{Z} \oplus \mathbb{Z}/b\mathbb{Z}$ as a subgroup of translations of some elliptic curve F' and construct a ramified Galois-cover $C' \to \mathbb{P}^1$ ramified in three points with covering group G. See Problem 1 for the construction of such coverings. This yields $C' \times F'/G$, at the same time a fibration over $\mathbb{P}^1$ with fibre F' and a fibration over F'/G with fibres C'.

To compute the invariants of $S = C' \times F'/G$ note that

$$\begin{aligned} H^0(S,\Omega^1) =& H^0(C' \times F',\Omega^1)^G = H^0(C',\Omega^1)^G \oplus H^0(F',\Omega^1)^G \\ H^0(S,K_S^{\otimes m}) =& H^0(C' \times F',K_{C\times F'}^{\otimes m})^G = H^0(C',K_{C'}^{\otimes m})^G \otimes H^0(F',K_{F'}^{\otimes m})^G. \end{aligned}$$

Now one uses a special case of Hurwitz formula for mappings $f : C' \to C$ between curves. Recall that locally f is given by $z \mapsto w = z^e$ and e is the ramification index and it equal to 1 except for finitely many points, the ramification points R_j with corresponding e_j. The divisor $R = \sum_j (e_j - 1)R_j$ is called the ramification-divisor and Hurwitz formula states

$$K_{C'} = f^* K_C \otimes \mathcal{O}(R).$$

This is most easily seen by observing that $dw = ez^{e-1}dz$ so that the divisor of any meromorphic 1-form on C' is $R + f^*($ the divisor of a meromorphic 1-form on $C)$.

If C is the quotient of C' by a group G acting on C', the group-action forces the ramification to be the same on all points of a fibre of $C' \to C = C'/G$. So $R = \sum(e_k - 1)f^{-1}(Q_k)$, where $f^{-1}Q_k$ is a complete fibre above Q_j. Now $f^*Q_k = e_k f^{-1}Q_k$, and hence $R = \sum_k (1 - \frac{1}{e_k})Q_k$. It follows that

$$K_{C'}^{\otimes m} = f^*\Big(K_C^{\otimes m} \otimes \sum_k (1 - \frac{1}{e_k}) \cdot mQ_k\Big).$$

To compute $H^0(C', K_C^{\otimes m})^G$ note that any G-invariant m-canonical holomorphic form comes from an m-canonical meromorphic form on C and the preceding formula then shows that

$$H^0(C', K_{C'}^{\otimes m})^G = H^0(K_C^{\otimes m} \otimes \sum_k [(1 - \frac{1}{e_k}) \cdot m]Q_k),$$

where $[s]$ means the integral part of the number s. For simplicity, let us write

$$R_m(C', G) = \sum_k [(1 - \frac{1}{e_k}) \cdot m]Q_k).$$

Combining the formulas yields

$$\begin{aligned} q(S) =& g(C'/G) + g(F'/G) \\ p_g(S) =& g(C'/G) \cdot g(F'/G) \\ P_m(S) =& h^0(C'/G, K_{C'/G}^{\otimes m} \otimes \mathcal{O}(R_m(C',G))) \cdot h^0(F'/G, K_{F'/G}^{\otimes m} \otimes \mathcal{O}(R_m(F',G))). \end{aligned}$$

3. A special case of the previous case form the so-called bi-elliptic surfaces.

Definition 4. *A surface $S = E \times F/G$, where E and F are elliptic curves, G a group of translations of E acting on F in such a way that $p_g(S) = 0$ is called bi-elliptic.*

By the previous calculation $p_g(S) = 0$ if and only if F/G is a rational curve. It is relatively simple to classify the possibilities for G and F (any E will work). Since G is a translation subgroup of E it must be abelian and as a transformation group of F it then is the direct product $T \times A$ of its subgroup T of translations and the subgroup A consisting of automorphisms of F preserving the origin. Since the product is direct, the points of T must be invariant under A, which strongly restricts the possible T. Furthermore, since F/G is rational, G cannot consist of translations of F only, and so A must be cyclic of order 2, 3, 4 or 6. Since G is a group of translations of E it is either cyclic or a direct sum of two cyclic groups. From these remarks the following list of possibilities is almost immediate:

1a. $G = \mathbb{Z}/2\mathbb{Z}$ with generator acting as the canonical involution $x \mapsto -x$ on F.

1b. $G = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ with one generator acting as in 1 a., while the other generator acts as translation over a point of order 2.

2a. $F = \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}i)$ and $G = \mathbb{Z}/4\mathbb{Z}$, the generator acting as multiplication by i.

2b. $F = \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}i)$ and $G = \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, one generator acting as before, the other by translation over a point of order 2.

3a. $F = \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\rho)$, $\rho = e^{2\pi i/3}$ and $G = \mathbb{Z}/3\mathbb{Z}$, the generator acting as multiplication by ρ.

3b. $F = \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\rho)$, $\rho = e^{2\pi i/3}$ and $G = \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$, one generator acting as multiplication by ρ, the other by translation over $(1-\rho)/3$.

4. $F = \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\rho)$, $\rho = e^{2\pi i/3}$ and $G = \mathbb{Z}/6\mathbb{Z}$, with generator acting as multiplication by $-\rho$.

The formulas established in Example 2 can be used to show that first of all $p_g = 0$ and $q = 1$. Then one sees that $P_m \leq 1$ and that $K^{\otimes m} = \mathcal{O}$ where $m = 2$ (in case 1a,b), $m = 4$ (in case 2a,b), $m = 3$ (in case 3a,b), $m = 6$ (in case 4). So the Kodaira-dimension is 0 in all cases.

4. Take a fixed point free linear system $|D|$ on a curve C. Let p and q be the two projections of $C \times \mathbb{P}^2$ onto C and $\mathbb{P}^2$ and consider a generic member S of the the linear system $|p^*D \otimes q^*(3H)|$. By Bertini, S will be smooth. The projection p induces a fibration $S \to C$ with fibres plane cubic curves, i.e. this is an elliptic fibration. The canonical bundle formula shows that $K_S = p^*(K_C \otimes D)$ and hence the Kodaira dimension is 1 whenever $\deg D > -\deg K_C = 2 - 2g(C)$.

Definition 5.

1. *A surface with $q = 0$ and trivial canonical bundle is called a K3-surface.*

2. *A surface with $q = 0, p_g = 0$ and $K^{\otimes 2}$ trivial is called an Enriques surface,*

3. *A complex 2-torus which admits an embedding into a projective space is called an Abelian surface,*

4. *A surface is a surface of general type if its Kodaira-dimension is 2.*

Examples 6.

1. Let us consider complete intersections S of multidegrees $d_1, d_2, \ldots, d_n$ in $\mathbb{P}^{n+2}$. The canonical bundle formula shows that $K_S = \mathcal{O}_S(d_1+d_2+\ldots+d_n-n-3)$. It is easy to see that the only combinations giving trivial K_S are $d_1 = 4$, $(d_1, d_2) = (2,3)$, $(d_1, d_2, d_3) = (2,2,2)$. An application of Lefschetz theorem (A2.21) shows that S is simply connected so that $q(S) = 0$. So there are three types of K3-surfaces which are complete intersections.

2. To find an Enriques surface S one first observes that K_S gives an element of order exactly 2 in the Picard group (since $p_g = 0$, the canonical bundle cannot be trivial) and so there exists an unramified Galois cover $\tilde{S}$ of degree 2 with $K_{\tilde{S}}$ trivial. Indeed, one may take

$$\tilde{S} = \{ \, s \in \mathcal{K}_S \mid s^{\otimes 2} = 1 \, \},$$

where one considers the total space of K_S and 1 is the global section corresponding with the constant section 1 of the trivial bundle. Noether's Formula expresses $\chi(\mathcal{O}(S))$ as a linear combination of $e(S)$ and (K_S, K_S). By Corollary A2.8 the Euler number gets multiplied by the degree of the cover, while the selfintersection number of K_S of course also gets multiplied by the degree. So $\chi(\mathcal{O}_{\tilde{S}}) = 2\chi(\mathcal{O}_S) = 2$ and hence $q(\tilde{S}) = 0$. Conversely, any K3-surface $\tilde{S}$ with a fixed point-free involution i yields an Enriques surface. This one sees as follows. Let $q : \tilde{S} \to \tilde{S}/i = S$ be the natural degree 2 cover. Then for any divisor D on S one has $q_*q^*D = 2D$. This is clear for irreducible curves and it then follows by linearity. In particular $q_*q^*K_S = 2K_S$, but $q^*K_S = K_{\tilde{S}}$ as locally any holomorphic 2-form on $\tilde{S}$ is a lift of a holomorphic 2-form on S and so $2K_S$ is trivial.

Now you construct a fixed point free-involution on a suitable K3 which is an intersection of three quadrics in $\mathbb{P}^5$. Let $X_0, \ldots, X_5$ be homogeneous coordinates on $\mathbb{P}^5$ and consider the intersection $\tilde{S}$ of three quadrics $Q'_j(X_0, X_1, X_2) + Q''_j(X_3, X_4, X_5)$, $j = 1,2,3$. For generic choices of Q'_j and Q''_j this intersection $\tilde{S}$ will be a smooth surface. The involution ι given by $(X_0, X_1, X_2, X_3, X_4, X_5) \mapsto (X_0, X_1, X_2, -X_3, -X_4, -X_5)$ has two planes of fixed points: the planes $P_1 = \{ \, X_0 = X_1 = X_2 = 0 \, \}$ and $P_2 = \{ \, X_3 = X_4 = X_5 = 0 \, \}$. They miss $\tilde{S}$ precisely if the three quadrics Q'_1, Q'_2, Q'_3, resp. Q''_1, Q''_2, Q''_3 (considered as quadrics in P_2, resp. P_1) have no point in common. For generic choices of Q'_j and Q''_j this will be the case. So then $\tilde{S}/\iota$ will be an Enriques surface.

3. A classical construction of a K3-surface is the Kummer surface. One starts out with an Abelian surface A (for example the product of two elliptic curves), blows up the surface A in the sixteen points of order 2, obtaining $\sigma : \tilde{A} \to A$. The canonical involution ι (sending x to $-x$) extends to an involution $\tilde{\iota}$ on $\tilde{A}$ and the quotient $S = \tilde{A}/\tilde{\iota}$ is a K3-surface. To show this let me calculate the invariants.

Since $\tilde{\iota}$ has only fixed points along the exceptional curves of $\tilde{A}$, the possible singular points of the quotient are among these. Choose local coordinates (x, y) in a neighbourhood U of a point of order two such that ι is given by $(x, y) \mapsto (-x, -y)$. Locally in one of the two standard coordinate patches of $\sigma^{-1}U$ there are coordinates (u, v) such that $\sigma(u, v) = (uv, v)$ and $\tilde{\iota}(u, v) = (u, -v)$. In particular, the quotient by the involution is smooth, and (u, v^2) can be taken as local coordinates on the quotient. If ω is a non-zero holomorphic 2-form on A, in these local coordinates, after multiplying with a non-zero constant, you find

$$\sigma^*\omega = d(uv) \wedge dv = \frac{1}{2} du \wedge d(v^2).$$

Now let $q : \tilde{A} \to S := \tilde{A}/\tilde{\iota}$ be the natural projection. The preceding formula then shows that $\sigma^*\omega = q^*\alpha$ for some nowhere zero 2-form α on S. In particular K_S is trivial. Furthermore, any non-trivial holomorphic 1-form on S would lift to a non-trivial $\tilde{\iota}$-invariant holomorphic 1-form on $\tilde{A}$ and such a form corresponds to a non-trivial holomorphic one-form on A which is invariant under the natural involution. These don't exist and hence $q(S) = 0$.

After these preparations let me state the Enriques Classification theorem.

Theorem 7. (Enriques Classification) *Let S be a minimal algebraic surface. Then S belongs to one of the following non-overlapping classes:*

1. *($\kappa = -\infty$, $q = 0$) $S = \mathbb{P}^2$, $S = \mathbb{F}_n$, ($n = 0, 2, 3, \ldots$).*
2. *($\kappa = -\infty$, $q > 0$) S a geometrically ruled, surface over a curve of genus > 0.*
3. *($\kappa = 0$, $q = 2, p_g = 1$) S is an Abelian surface,*
4. *($\kappa = 0$, $q = 1, p_g = 0$) S is bi-elliptic,*
5. *($\kappa = 0$, $q = 0, p_g = 1$) S is K3,*
6. *($\kappa = 0$, $q = 0, p_g = 0$) S is Enriques,*
7. *($\kappa = 1$) S is minimal elliptic but NOT $\kappa = 0$ or $\kappa = -\infty$,*
8. *($\kappa = 2$) S is of general type.*

Problems.

13.1. Let C be a compact Riemann surface and let $T \subset C$ be a finite set of points. Let G be a finite quotient of the fundamental group of $C \setminus T$ and let $f' : D' \to C \setminus T$ be the finite unramified covering with covering group G. Show that there exists a compact Riemann surface $D \supset D'$ and a holomorphic map $f : D \to C$ extending f'. Apply this to the case $C = \mathbb{P}^1$, and T consisting of three points to construct a covering of $\mathbb{P}^1$ with group the direct product of two finite cyclic groups.

13.2. Let X and Y be compact complex manifolds of the same dimension and let $f : X \to Y$ be a finite surjective map of degree d. The ramification locus is defined as the locus $R_f \subset Y$ of points where df does not have maximal rank. Show that R_f is a divisor and that $K_Y = f^*K_X \otimes \mathcal{O}(R_f)$. Derive the Hurwitz formulas from it in case X and Y are curves.

14. The Enriques Classification: First reduction

An important part of the classification theorem rests on the following proposition which deals with Case 4. of Proposition 12.4. The proof of this proposition is very technical and will be dealt with in the following sections. In this section, the proof of the Kodaira classification will be reduced it.

Proposition 1. *Suppose S is a surface with K_S nef and $(K_S, K_S) = 0$, $q = 1$ and $p_g = 0$. Then $\kappa(S) = 0$ or 1 and $\kappa(S) = 0$ if and only if S is bi-elliptic.*

Let me give two immediate consequences of this proposition: the characterisation of $\kappa = -\infty$-surfaces and the characterisation of the minimal rational surfaces.

Corollary 2. *For a minimal surface K_S is nef if and only if $\kappa \geq 0$. In particular a surface is ruled (or rational) if and only if $\kappa = -\infty$.*

Proof: If K_S is nef Proposition 12.4 combined with the previous proposition shows that either $P_2 > 0$ so that $\kappa \geq 0$ or the surface is bielliptic and then also $\kappa \geq 0$. Conversely, if K_S is nef, the surface must be minimal by Reformulation 6.14.

This proves the first part of the corollary.

For the second part, one may assume that S is minimal and from the first part one may assume that K_S is not nef. But then S is geometrically ruled or $S = \mathbb{P}_2$. ∎

Corollary 3. *Let S be a minimal rational surface. Then $S \cong \mathbb{P}^2$ or $S \cong \mathbb{F}_n$, $n \neq 1$.*

Proof: Since S is rational, $\kappa(S) = -\infty$. So, by Proposition 12.2 S is the projective plane or S is geometrically ruled. In the latter case, since $q(S) = 0$, by Corollary 7.13 the surface S must be a Hirzebruch surface. ∎

Next, let me continue the proof of the classification theorem by considering the case of an elliptic fibration.

Theorem 4. *Suppose that S is a surface with K_S nef and $(K_S, K_S) = 0$. Then $\kappa(S) = 0$ or 1. In the last case S admits the structure of an elliptic fibration.*

Proof: By Proposition 14.2 one has $\kappa(S) \geq 0$. Assume that $\kappa(S) \geq 1$. Then for n large enough $|nK_S|$ is at least 1-dimensional. Let D_{f}, resp $|D|$ be the fixed part, resp. the variable part of this linear system.

Claim $(D, D) = (K_S, D) = 0$,

Proof: (of Claim) One has $0 = n(K_S, K_S) = (D_{\mathrm{f}}, K_S) + (D, K_S)$ and since each term is ≥ 0 by nefness of K_S these must vanish. Now $0 = n(D, K_S) = (D, D) + (D, D_{\mathrm{f}})$ and again, each term is non-negative, since D moves and so $(D, D) = 0 = (D, D_{\mathrm{f}})$. ∎

The claim implies that the rational map $f = \varphi_{nK_S}$ is a morphism and that $f : S \to C$ maps every divisor $D \in |D|$ to a point and so C is a curve. This is true for all n large enough so that $|nK_S|$ is at most 1-dimensional and hence $\kappa(S) = 1$ in this case. If D is a smooth fibre of f, the adjunction formula says that the connected components are elliptic curves and so, taking the Stein factorisation of f, one obtains an elliptic fibration. ∎

Finally, consider the case of Kodaira dimension 0.

Proposition 5. *Suppose that K_S is nef, that $(K_S, K_S) = 0$ and that $\kappa(S) = 0$. Then S is bi-elliptic, an abelian surface, a K3-surface or an Enriques surface.*

Proof: By Proposition 12.4 and Proposition 14.1 you only have to consider the cases $p_g(S) = 0 = q(S)$ and the case $p_g(S) > 0$ (and hence $p_g = 1$). Moreover, if $p_g = 0$ one must have $P_2 = 1$, again by 12.4. Let me first deal with this case. I claim that $P_3 = 0$. If not, then $P_3 = 1$. Let $D_2 \in |2K_S|$ and $D_3 \in |3K_S|$. So $3D_2$ and $2D_3$ are both divisors in $|6K_S|$. Since $P_6 \leq 1$ you must have $3D_2 = 2D_3$ and there must be an effective divisor D with $D_2 = 2D$ and $D_3 = 3D$. Necessarily $D = D_3 - D_2 \in |K_S|$, but $p_g = 0$. So indeed $P_3 = 0$.

Now apply the Riemann-Roch inequality to $3K_S$. One has

$$h^0(3K_S) + h^0(-2K_S) \geq 1$$

and hence $h^0(-2K_S) \geq 1$. Since $P_2 = h^0(2K_S) = 1$ this is only possible if $2K_S$ is trivial. It follows that S is an Enriques surface.

I next suppose that $p_g = 1$. Consider the Noether formula in this case. It reads as follows.

$$12(2 - q(S)) = e(S) = 2 - 4q(S) + b_2$$

and hence $b_2 = 22 - 8q(S)$. So $q(S) = 0,\ 1,\ 2$.

In the first case you have a K3-surface. Indeed the Riemann-Roch inequality applied to $2K_S$ yields $h^0(2K_S) + h^0(-K_S) \geq 2$. In a similar way as in the previous case, I infer from this that K_S is trivial.

I shall exclude the possibility $q(S) = 1$ and show that S is a torus in the remaining case.

Since $q(S) > 0$, you can find a non-trivial line bundle $\mathcal{O}_S(\tau)$ with $\mathcal{O}_S(2\tau) = \mathcal{O}_S$ (any non-trivial 2-torsion point of the torus $\mathrm{Pic}^0(S)$ gives such a line bundle). If $q = 1$, the Riemann-Roch inequality reads

$$h^0(\mathcal{O}_S(\tau)) + h^0(\mathcal{O}_S(K_S - \tau)) \geq 1$$

and hence $h^0(\mathcal{O}_S(K_S - \tau) \geq 1$. Take $D \in |K_S - \tau|$ and let K be any canonical divisor. One has $2D = 2K$ since $P_2 = 1$ and hence $D = K$, contradicting the fact that $\mathcal{O}_S(\tau) \not\cong \mathcal{O}_S$.

In the second case, you first look at the possible components of the canonical divisor $K = \sum_j m_j C_j$. Since K_S is nef and $(K_S, K_S) = 0$ you find $(K_S, C_i) = 0$. Writing down

$$0 = (K, C_j) = m_j(C_j, C_j) + \sum_{i \neq j} m_i(C_i, C_j)$$

you see that either $(C_j, C_j) = -2$ and C_j is a smooth rational curve, or you have $(C_j, C_j) = 0$ but also $(C_j, C_i) = 0$ for all $i \neq j$. So you can partition the connected components of $\cup C_i$ into two types: unions of smooth rational curves on the one hand and smooth elliptic curves or rational curves with one node on the other hand.

I consider the Albanese map $S \to \mathrm{Alb}\, S$. It either maps to a curve $C \subset \mathrm{Alb}\, S$ or it maps onto the (two-dimensional) Albanese. In the first case, since $q(S) = 2$ the curve C is a genus 2 curve by Proposition 10.6. Let $f : S \to C$ be the resulting fibration. By the preceding analysis, every connected component D of the canonical divisor K is either rational or elliptic. Since such curves cannot map onto a curve of genus 2 these must be contained in some fibre F of f, say over $c \in C$.

Now I need Zariski's lemma:

Lemma 6. (Zariski's Lemma) *Let $f : S \to C$ be a fibration of a surface S to a curve C and let $F = \sum_i m_i C_i$ be a fibre, where C_i is irreducible. Let $D = \sum_i r_i C_i$ be a $\mathbb{Q}$-divisor. Then $(D, D) \leq 0$ and equality holds if and only if $D = rF$ for some $r \in \mathbb{Q}$.*

Proof: Let $F_i = m_i C_i$ so that $F = \sum_i F_i$. Also, set $s_i = r_i/m_i$ so that $D = \sum_i s_i F_i$. One has

$$\begin{aligned}(D,D) &= \sum_i s_i^2(F_i, F_i) + 2\sum_{i<j} s_i s_j (F_i, F_j)\\ &= \sum s_i^2(F_i, F) - \sum_{i<j}(s_i^2 + s_j^2 - 2s_i s_j)(F_i, F_j)\\ &= 0 - \sum_{i<j}(s_i - s_j)^2 (F_i, F_j) \leq 0\end{aligned}$$

with equality if and only if $s_i = s_j = r$ for all i and j, which means $D = r\sum_i F_i = rF$. ■

By Zariski's Lemma $D = a/b \cdot F$ with a, b positive integers. Then $bD = f^*(a[c])$ and hence $h^0(ndD)$ and $h^0(ndK_S)$ grow indefinitely when n tends to infinity. This contradicts $\kappa(S) = 0$. The possibility that K_S is trivial is left. In this case, simply take an unramified cover $C' \to C$ of degree ≥ 2 and pull back your fibration. You get an unramified cover $S' \to S$ of degree ≥ 2 and $K_{S'}$ is still trivial, $\chi(\mathcal{O}_{S'}) = 0$ and hence $q(S') = 2$ by what we have seen. But $q(S') \geq q(C') \geq 3$, a contradiction.

There remains the case that Alb S is a 2-torus and that the Albanese maps surjectively onto it. It is an elementary fact that in this case $\alpha^* : H^2(\mathrm{Alb}\, S) \to H^2(S)$ is injective (dually: every 2-cycle on Alb X is homologous to a cycle which lifts to a 2-cycle on X). Since $b_2(S) = 6$ this then is an isomorphism and so no fundamental cohomology-class of a curve maps to zero. In particular, the Albanese map must be a finite morphism. So, if D is a connected component of the canonical divisor it cannot map to a point and hence it must be an elliptic curve E which maps to an elliptic curve $E' \subset \mathrm{Alb}\, S$. Now form the quotient elliptic curve $E'' = \mathrm{Alb}\, S/E'$ and consider the surjective morphism $S \to E''$. The Stein-factorisation then yields an elliptic fibration and D is contained in a fibre. By the Zariski lemma 14.6 it follows that D is a rational multiple of a fibre and as before one concludes that $\kappa(S) = 1$. It follows that the only possibility is that K_S is trivial, but then, by the formula for the canonical divisor of coverings (Problem 14.2) I conclude that the Albanese map is a finite unramified covering and therefore S itself is a torus. ■

15. The canonical bundle formula for elliptic fibrations

The formula refered to in the section heading will be used in the final step of the classification theorem but is also of independent interest. The proof given here does not use relative duality in contrast with the proof of [B-P-V].

Multiple fibres of a fibration $f : S \to C$ of a surface S onto a curve C are defined in the following way. If $F = \sum_j m_j F_j$ is a singular fibre with F_j irreducible, the *multiplicity* of F is the greatest common divisor m of the numbers m_i and F is called *multiple* if this multiplicity is > 1. If $F = f^{-1}c = mF'$ is such a multiple fibre and $\{ U, z \} \subset C$ a coordinate disk about c, the function $z \circ f$ vanishes of multiplicity m along F'. Since all fibres are connected, all $g \in \mathcal{O}(f^{-1}U)$ are of the form $g = h \circ f$ with $h \in \mathcal{O}(U)$ and so, if g vanishes along F' it must vanish with order $\geq m$.

Now $\mathcal{O}_{f^{-1}U}(mF') = \mathcal{O}_{f^{-1}U}(F)$ is trivial and so $\mathcal{O}_{f^{-1}U}(F')$ has order $\leq m$. Smaller order is impossible, since it would give a function vanishing to order $< m$ along F'. It follows that $\mathcal{O}(F')$ has order precisely m in $\mathrm{Pic}(f^{-1}U)$. I claim that this also holds for the restriction of the bundle to F'.

Lemma 1. *The bundle $\mathcal{O}_{F'}(F')$ has order m in* $\mathrm{Pic}\, F'$.

Proof: The proof depends on the existence of an exponential sequence on F'.

Sublemma 2. *Let $C = \sum_i n_i C_i$ be any effective divisor on a surface S. Let $\mathcal{O}_C^*$ be the sheaf of functions on C which are restrictions to C of functions which are nonzero in a neighbourhood in S of a point of C. There is a commutative diagram*

$$\begin{array}{ccc} \mathcal{O}_S & \xrightarrow{\text{restriction}} & \mathcal{O}_C \to 0 \\ \downarrow{\scriptstyle \exp} & & \downarrow{\scriptstyle \exp} \\ \mathcal{O}_S^* & \xrightarrow{\text{restriction}} & \mathcal{O}_C^* \to 1 \end{array}$$

and the map exp *fits into an exact sequence*

$$0 \to \mathbb{Z}_C \to \mathcal{O}_C \xrightarrow{\exp} \mathcal{O}_C^* \to 1.$$

Proof of sublemma: For simplicity assume that C has connected support. If $g \in \mathcal{I}_C$, from $\exp(g) - 1 = \sum_{m \geq 1} (2\pi i g)^m / m!$ it follows that $e^{2\pi i g} - 1 \in \mathcal{I}_C$. Here you use 'closedness of submodules' [Gr-Re, Chapt. 2 §2.3]. This remark implies that the diagram is commutative. the only non-trivial part in the exactness statement for the exponential sequence is the fact that $\exp(f) = 1$ for $f \in \mathcal{O}_C$ implies that $f|C = n \in \mathbb{Z}$. To see this, let $g \in \mathcal{O}_S$ restrict to f and consider $e^{2\pi i g} - 1 \in \mathcal{I}_C$. Look at $g|C_i$. This function must be equal to some fixed integer n by the usual exponential sequence and connectedness of the support of C. So $g - n$ vanishes along every C_i, say precisely to order m_i. Then also $\exp(g) - 1 = \sum_i (2\pi i g)^m / m!$ vanishes precisely to order m_i along C_i and hence $m_i = n_i$, i.e. $g - n \in \mathcal{I}_C$ and so $f = n$ on C. ∎

Proof of the lemma:

Let me triangulate S such that F' supports a subcomplex. See Appendix A2, Example A2.1 (this treats the case when F' is smooth; in the case at hand, there are finitely many singular points and the same methods apply to yield the desired triangulation; anyway, I advise you to accept these topological fineries upon first reading). In particular it follows that F' is a deformation retract of a neighbourhood, which one may assume to be of the form $f^{-1}U$ with U an open neighbourhood of $c \in C$. Then the restriction maps $H^p(f^{-1}U, \mathbb{Z}) \to H^p(F', \mathbb{Z})$ are isomorphisms. Now consider the exponential sequence on $f^{-1}U$ and its restriction to F'. I find a commutative diagram

$$\begin{tikzcd}
H^1(f^{-1}U,\mathbb{Z}) \arrow[r] \arrow[d, equal] & H^1(\mathcal{O}_{f^{-1}U}) \arrow[r] \arrow[d] & H^1(\mathcal{O}^*_{f^{-1}U}) \arrow[r] \arrow[d] & H^2(f^{-1}U,\mathbb{Z}) \arrow[d, equal] \\
H^1(F',\mathbb{Z}) \arrow[r] & H^1(\mathcal{O}_{F'}) \arrow[r] & H^1(\mathcal{O}^*_{F'}) \arrow[r] & H^2(F',\mathbb{Z}).
\end{tikzcd}$$

Chasing through this diagram now gives the result. ■

Next, an explicit description of the possible multiple singular fibres in an elliptic fibration is needed.

Lemma 3. *If $F = mF'$ is a multiple fibre of an elliptic fibration, F' is either a smooth elliptic curve, a rational curve with one ordinary double point or a cycle of non-singular rational curves.*

Proof: For a smooth fibre $(K, F) = 0$ by the adjunction formula and hence $(K, F') = 0$ which implies that $\chi(\mathcal{O}_{F'}) = 0$ by Lemma 5.7. So, if F' is connected, it is either an elliptic curve or a rational curve with an ordinary node by remark 5.8. If F' is reducible and E_i is a component of the fibre F', Zariski's lemma 14.6 implies that $(E_i, E_i) < 0$ and hence, by minimality and the adjunction formula, E_i must be a smooth rational curve with $(E_i, E_i) = -2$. Again by Zariski's lemma, we see that the intersection number of two distinct components components E_i and E_j is 1 or 2. Now I claim that the topological space F' cannot be simply connected. Indeed, by the previous lemma, there is a non-trivial torsion bundle on F' and so $H^1(F', \mathbb{Z}) \neq 0$. If some intersection number (E_i, E_j) is equal to 2 there must be exactly two components forming a cycle as can be seen as follows. If $F' = \sum_i n_i E_i$ one has

$$0 = (E_i, F') = -2n_i + \sum_{j \neq i} n_j (E_i, E_j).$$

If $n_i \leq n_j$ it follows from $2n_i = 2n_j + \sum_{k \neq i,j} n_k (E_k, E_i)$ that all coefficients n_i must be one and that there are exactly two E_i. If $(E_i, E_j) \leq 1$ and three components meet in a single point one similarly finds that there are no more components, contradicting non-simply connectedness of F' in this case. So there must be a cycle contained in F' and it is easily seen that then there can be no more components. ■

Corollary 4. *The dualising sheaf $\omega_{F'} = \mathcal{O}_{F'}(K_S \otimes \mathcal{O}(F'))$ is trivial.*

Proof: To compute the dualising sheaf recall Proposition 5.9. If F' is elliptic, of course $\omega_{F'} = K_{F'} = \mathcal{O}_{K'}$. Otherwise, there is a global meromorphic one-form α on each component $E_i \cong \mathbb{P}^1$ with poles in two points with opposite residues (the form dz/z has this property with respect to 0 and ∞). If one scales such a form on each of the components suitably, the residues in points belonging to two components cancel. This defines a trivialising section of the sheaf $\omega_{F'}$. ■

Let me come now to the main result.

Theorem 5. *Let S be surface with K_S nef and let $f : S \to C$ be an elliptic fibration. Let F_i, $i = 1, \dots, m$ be the multiple fibres and let m_i be the multiplicity of $F_i = m_i F_i'$. One has*

$$K_S = f^*L + \sum_{i=1}^{m}(m_i - 1)F_i'$$

with L a divisor on C of degree $\chi(\mathcal{O}_S) - \chi(\mathcal{O}_C)$.

Proof:
Step 1.: $K_S = f^*L + \sum_i (n_i - 1)F_i'$ for some divisor L on C.
Take N smooth fibres $G_1, \dots, G_N$ and tensor the exact sequence

$$0 \to \mathcal{O}_S \to \mathcal{O}(\sum_j G_j) \to \oplus_j \mathcal{O}_{G_j}(G_j) \to 0$$

with K_S. Since $\mathcal{O}_{G_j}(K_S + G_j) \cong \mathcal{O}_{G_j}$ by the Adjunction Formula and the fact that G_j is a smooth elliptic curve, one finds

$$0 \to \mathcal{O}(K_S) \to \mathcal{O}(K_S + \sum_j G_j) \to \oplus_j \mathcal{O}_{F_j} \to 0.$$

The long exact sequence in cohomology gives $h^0(K_S + \sum_j G_j) \geq p_g - q + (N-1)$ and so for N large enough, there exists a divisor $D \in |K_S + \sum_j G_j|$. Now $(F, G_j) = 0 = (K_S, F)$ and so $(D, F) = 0$ implies that D consists of linear combinations of fibral components D_i, $i = 1, \dots, M$. Since D is effective, and K_S nef, one has $(D, D_i) = (K_S, D_i) \geq 0$. On the other hand, from Proposition 12.4 one sees that $(K_S, K_S) = 0$ and hence

$$0 = (K_S, K_S) = (D, K_S) = (\sum d_i D_i, K_S).$$

It follows that $(D, D_i) = (K_S, D_i) = 0$ and so $(D, D) = 0$. If $D = \sum_i D_i'$ with D_i' the part supported in exactly one fibre, it follows that $(D_i', D_i') = 0$ and from Zariski's lemma one concludes that D_i' is a rational multiple of a fibre. Any part supported on a multiple fibre F_i can be written as $n_i F_i' + r_i F_i$ with $0 < n_i < m_i$ and $r_i \in \mathbb{Z}$. The parts supported on a non-multiple fibre G_j must be of the form $r_j G_j$ with $r_j \in \mathbb{Z}$ and so $K_S = f^*(L) + \sum_i n_i F_i'$ and since $\mathcal{O}_{F_i'}(K_S + F_i') = \mathcal{O}_{F_i'} = \mathcal{O}_{F_i'}((n_i + 1)F_i')$, by the previous Corollary, one finds that $n_i + 1 = m_i$.
Step 2. deg $L = \chi(\mathcal{O}_S) - \chi(\mathcal{O}_C)$.
Compute $h^0(K_S + \sum_j G_j)$ in two ways. First, note that $\sum_i (m_i - 1)F_i'$ is a fixed component

of $|K_S + \sum_j G_j|$ and so $h^0(K_S + \sum_j G_j)) = h^0(\mathcal{O}_S(f^*L + \sum G_j) = h^0(\mathcal{O}_C(L + \sum_j c_j))$, where $c_j = f(G_j) \in C$. Riemann-Roch then shows that

$$h^0(K_S + \sum_j G_j) = \deg L + N + 1 - g(C)$$

provided N is large enough. On the other hand, the computation from Step 1 shows that

$$h^0(K_S + \sum_j G_j) = \chi(\mathcal{O}_S) + N - 1 + \dim\Big(\mathrm{im}\ H^1(\mathcal{O}_S(K_S)) \to H^1(\mathcal{O}_S(K_S + \sum_j G_j))\Big).$$

The dimension of the image in the preceding formula, by Serre-duality translates into the dimension of the image of $H^1(\mathcal{O}_S(-\sum_j G_j)) \to H^1(\mathcal{O}_S)$, which, by exactness of the usual cohomology sequence, is the dimension of the kernel of $H^1(\mathcal{O}_S) \to \oplus_j H^1(\mathcal{O}_{G_j})$. Now the Hodge decomposition for H^1 tells us that $H^1(\mathcal{O}) = \overline{H^0(\Omega^1)}$ and so, the dimension I am after is equal to the dimension of the kernel of $H^0(\Omega^1_S) \to \oplus_j H^0(\Omega^1_{G_j})$. But this kernel consists precisely of the holomorphic one forms which are pull backs of 1-forms on C and these form a space of dimension $g(C)$. See Problem 17.3.

Combining everything, you get

$$h^0(K_S + \sum_j G_j) = \deg L + N + 1 - g(C) = \chi(\mathcal{O}_S) + N - 1 + g(C)$$

and so $\deg L = \chi(\mathcal{O}_S) - 2 + 2g(C) = \chi(\mathcal{O}_S) - \chi(\mathcal{O}_C)$.

■

16. Two technical results and the final step

For the final step of the proof of the classification theorem, an easy topological result about fibrations of surfaces over curves is needed but also an involved result about isotrivial fibrations. Then the proof of the classification theorem can be completed by proving Proposition 14.1.

First I state and prove the fact about the topology of fibrations.

Proposition 1. *Let S be a surface with K_S nef and let $f : S \to C$ be a fibration onto a curve. Let $\delta(f) \subset C$ be the (finite) set of critical values of f, i.e. $c \in \delta(f)$ if and only if at some $s \in f^{-1}(c)$ the map $df(s)$ vanishes. Let F be a smooth fibre and let $F_c = f^{-1}(c)$. One has*

$$e(S) = e(C)e(F) + \sum_{b \in \delta(f)} (e(F_b) - e(F)).$$

Furthermore, $e(F_b) - e(F) \geq 0$ with equality if and only if F_b supports a smooth elliptic curve.

Proof: You triangulate C in such a way that $\delta(f)$ becomes a subcomplex L' of the resulting complex K'. Likewise you triangulate $f^{-1}L$ and extend the triangulation to S Let (K, L) be the resulting pair of complexes. Then $e(K) = e(K \setminus L) + e(L) = e(K' \setminus L')e(F) + \sum_{b\in\delta(f)} e(F_b)$. The last equality follows, since f is topologically a locally trivial fibration over the set of non-critical values of f. See Problem 17.1. So $e(S) = e(C)e(F) - \sum_{b\in\delta(f)} e(F) + \sum_{b\in\delta(f)} e(F_b)$, which proves the first statement. For the second statement I need

Lemma 2. *Let $C = \sum_i C_i$ be any curve on a surface, where C_i are the irreducible components. Then $e(C) \geq 2\chi(\mathcal{O}_C)$ with equality if and only if C is smooth.*

Proof: (of the Lemma) Let $\nu : \tilde{C} \to C$ be the normalisation of the curve C. There is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & \mathbb{C}_C & \to & \nu_*\mathbb{C}_{\tilde{C}} & \to & \delta & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow{\scriptstyle j} & & \\ 0 & \to & \mathcal{O}_C & \to & \nu_*\mathcal{O}_{\tilde{C}} & \to & \Delta & \to & 0. \end{array}$$

From the diagram one finds that

$$e(\tilde{C}) = e(C) + h^0(\delta)$$
$$\chi(\mathcal{O}_{\tilde{C}}) = \chi(\mathcal{O}_C) + h^0(\Delta)$$

and since $e(\tilde{C}) = 2\chi(\mathcal{O}_{\tilde{C}})$, one finds

$$e(C) = 2\chi(\mathcal{O}_C) + 2h^0(\Delta) - h^0(\delta).$$

One checks that j is injective and hence $h^0(\delta) \leq h^0(\Delta)$ and so $e(C) \geq 2\chi(\mathcal{O}_C)$ with equality if and only if $h^0(\Delta) = 0$. ■

Now I can finish the proof of the Proposition. For a singular fibre $F_b = \sum_i m_i C_i$, put $F' = \sum C_i$. The lemma says that $e(F') \geq 2\chi(\mathcal{O}_{F'})$. On the other hand $2\chi(\mathcal{O}_{F'}) = -(F', F') - (F', K_S)$ by the Adjunction Formula. The first term is ≥ 0 by Zariski's Lemma. For the second term write $-\sum_i (C_i, K_S) \geq \sum_i -m_i(C_i, K_S)$ (since K_S is nef)$= -(F_b, K_S) = -(F, K_S) = e(F)$). It follows that $e(F_b) \geq e(F)$ and equality implies that F' is smooth, and hence $F_b = nF'$. Closer inspection of the preceding computation then reveals that $e(F') = 1/n \cdot e(F)$. Now f cannot be a fibration with rational fibres, since K_S is nef. So $e(F) \leq 0$ and hence $e(F') = e(F) = 0$, i.e. F' is a smooth elliptic fibre. ■

Secondly, I need a result about families of curves over a base curve of low genus. To this end I introduce the notion of isotrivial fibration.

Definition 3. *A fibration $f : X \to Y$ between projective manifolds is called isotrivial if there exists a finite unramified covering $g : Y' \to Y$ such that the pull back $f' : X' = X \times_Y Y' \to Y'$ of f is isomorphic to a product-fibration $X' \cong Y' \times F$, for some projective manifold F.*

Example 4. Let G be a finite group which is the quotient of $\pi_1(Y)$ and which acts on a manifold F. Let $g : Y' \to Y$ be the covering defined by G and consider the product action of G on $Y' \times F$. The quotient manifold $(Y' \times F)/G$ admits an isotrivial fibration onto Y. Conversely, any fibre bundle $X \to Y$ such that the fibre F has a finite group of automorphisms arises in this way. See Problem 4.

The main result in this section is:

Proposition 5. *Suppose that $f : S \to C$ is a fibration of a surface onto a curve of genus 0 or 1 and suppose that f has everywhere maximal rank. Then f is isotrivial.*

Before I give a proof of this proposition I make a few comments on the situation arising in this proposition. Any proper surjective morphism $f : X \to Y$ between complex manifolds (not necessarily compact) which is everywhere of maximal rank is called a *family of complex manifolds.* Note that all fibres in a family are compact manifolds. By Ehresmann's theorem (Problem 1), the family is differentiably locally trivial. I can therefore assume that the sheaf $\cup_{y \in Y} H_1(X_y, \mathbb{Z})$ is locally constant. In case the fibres are curves, this is a locally constant sheaf of $\mathbb{Z}$-modules of rank $2g$, where g is the genus of the fibre. Now consider $f_* K_Y$. This is a sheaf whose fibres are $H^0(K_{X_y})$ and hence have constant dimension. It is a non-trivial fact that $f_* K_Y$ is locally trivial on Y and has rank g. This follows for example from deep results of Grauert. See [Gr-Re, Chapter 10, §5] or [Ha, p. 288] for a proof in the slightly easier algebraic setting. In the case at hand this implies that you can choose a basis $\omega_1(y), \ldots, \omega_g(y)$ for $H^0(K_{X_y})$ depending holomorphically on y. Now you take any small open subset $U \subset Y$ over which the family is differentiably trivial and so there is a constant basis $\gamma_1, \ldots, \gamma_{2g}$ for $H_1(X_y, \mathbb{Z})$ over U such that the periods matrices

$$\begin{pmatrix} \int_{\gamma_1} \omega_1(y) & \cdots & \int_{\gamma_{2g}} \omega_1(y) \\ \vdots & \ddots & \vdots \\ \int_{\gamma_1} \omega_g(y) & \cdots & \int_{\gamma_{2g}} \omega_g(y) \end{pmatrix}$$

are all normalised (as defined in Example 10.5).

For the rest of the proof I assume that the reader is acqainted with some more advanced topics from the theory of curves, for instance with the material presented in [G-H, Chapter 2, §6-7].

Any normalised period matrix Ω is a symmetric matrix whose imaginary part is positive definite and hence defines a point in the Siegel upper half space

$$\mathfrak{h}_g = \{ \Omega \in \mathbb{C}^{g,g} \mid \Omega = \Omega^T, \operatorname{Im}(\Omega) > 0 \}.$$

Note that different choices of symplectic bases for H_1 give a different normalised period matrix Ω and one can compute that these are in the same Γ_g-orbit, where Γ_g is the symplectic group modulo its center $\pm \mathbb{1}_g$ acting by means of fractional linear transformations: if

$$\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

the action is given by $\gamma(\Omega) = (A\Omega + B)(C\Omega + D)^{-1}$.

Fixing one symplectic basis for the first homology of curves over U, you get a holomorphic map

$$U \to \mathfrak{h}_g,$$

the *local period map* associated to the family over U. In general, one cannot extend this map to a uni-valent map $Y \to \mathfrak{h}_g$, due to the monodromy action on $H_1(X_y, \mathbb{Z})$. This action can be trivialised by passing to the universal cover $\tilde{Y}$ of Y. The holomorphic map $p : \tilde{Y} \to \mathfrak{h}_g$ thus obtained is the *period map*.

Let me now sketch the proof of the Proposition.

Proof: (Sketch) The universal cover of C is $\mathbb{P}^1$ or $\mathbb{C}$. In both cases the period map then has to be constant since $\mathfrak{h}_g$ is isomorphic to a bounded domain (see Problem 5). Now apply Torelli's theorem:

Theorem (Torelli's theorem [G-H,p. 359]) *Let C and C' be two smooth curves of genus g such that their Jacobians together with their natural polarisations are isomorphic, or, equivalently such that their normalised period matrices are in the same Γ_g-orbit. Then C and C' are isomorphic.*

It follows that all fibres of f are isomorphic. Then you can for instance apply the local-triviality theorem of Grauert-Fischer [F-G] which says that f must be a locally trivial fibre bundle in this case. If the genus of the fibre is ≥ 2, the automorphism group of the fibre is always finite and hence f is isotrivial by Example 16.4. If the fibre is elliptic, this is also the case, but slightly more involved. Consider the action of the fundamental group of C on the group of n-torsion points of a fibre F. It can be trivialised by passing to a finite unramified covering. In particular you have the zero-section globally. Since the group of automorphisms of an elliptic curve preserving the origin is finite, it follows that after taking some unramified covering, the family becomes trivial. ∎

Now, finally, the proof of Proposition 14.1 can be given. Let me recall it before giving the proof.

Proposition 6. *Suppose S is a surface with K_S nef and $(K_S, K_S) = 0$, $q = 1$ and $p_g = 0$. Then $\kappa(S) = 0$ or 1 and $\kappa(S) = 0$ if and only if S is bielliptic.*

Proof: Since $q(S) = 1$, the Albanese of S is an elliptic curve C and by Lemma 10.6 the Albanese mapping $\alpha : S \to C = \operatorname{Alb} S$ has connected fibres. Recall (Proposition 12.4) that $b_2(S) = 2$ and hence $e(S) = 2 - 2b_1(S) + b_2(S) = 0$. Now apply the topological lemma 16.1 to conclude that $\alpha : S \to C = \operatorname{Alb} S$ is either a genus g fibration with $g \geq 2$ and α everywhere of maximal rank, or an elliptic fibration with only smooth fibres, some of which are possibly multiple. In the first case, apply Proposition 16.5 to conclude that $\alpha : S \to C = \operatorname{Alb} S$ is isotrivial, so that there exists a finite unramified covering $\hat{S} \to S$ which is a product. By Proposition 9.9 the Kodaira-dimension does not change under finite unramified covers and so $\kappa(S) = 1$ in this case. So S is not bielliptic. If $\alpha : S \to C = \operatorname{Alb} S$ is elliptic and has multiple fibres, an application of the elliptic bundle formula 15.5 shows that $\kappa(S) = 1$ in this case too. There remains the possibility that $\alpha : S \to C = \operatorname{Alb} S$ is an elliptic fibration which is everywhere of maximal rank and then, again by Proposition 16.5 one has an isotrivial fibration. There exists therefore an elliptic curve F, an elliptic curve

E, a group of automorphisms G of F acting as translations on E such that $S = E \times F/G$. Since $p_g(S) = 0$, by definition S is bi-elliptic. So only in the case $\kappa(S) = 0$ you get a bi-elliptic surface, in all other cases $\kappa(S) = 1$.

■

Problems.

16.1. Let $f : X \to Y$ be a surjective differentiable map between differentiable manifolds which is everywhere of maximal rank and which is proper. Show that f is locally a differentiably trivial fibre bundle. This is sometimes called *Ehresmann's Theorem*. Hint: see e.g. [We, Chapter V, Proposition 6.4].

16.2. Let $f : X \to Y$ be a locally trivial fibre bundle with compact fibres $X_y = f^{-1}y$. The fundamental group $\pi_1(Y, y_0)$ acts on $H^k(X_y, \mathbb{Q})$ as follows (*monodromy representation*). Choose a loop $\gamma : I \to Y$ based at y and choose a differentiable trivialisation of the pull back of f to γ. So there are diffeomorphisms $g_t : X_{\gamma(0)} \to X_{\gamma(t)}$ and hence an induced isomorphism g_1^* on $H^k(X_y, \mathbb{Q})$.
Show that it is independent of the choosen trivialisation and that g_1^* only depends on the class of γ in the fundamental group.
The groups $H^k(X_y, \mathbb{Q}), y \in Y$ form a locally constant sheaf on Y which is nothing but $R^k f_* \mathbb{Q}$ and any class a_y invariant under monodromy yields a global section of this sheaf.
Let $a \in H^k(X, \mathbb{Q})$ be given and consider the restrictions $a_y \in H^k(X_y, \mathbb{Q})$ of a to the fibres. Show that this yields a global section of $R^k f_* \mathbb{Q}$.

16.3. Let $f : S \to C$ be an ellipic fibration and let ω be a holomorphic 1-form on S which restricts to zero on a smooth fibre F. Show that $\omega = f^*\omega'$ where ω' is a holomorphic 1-form on C. Hint: Use the previous exercise to see that ω restricts to zero on all smooth fibres. Now fix a regular value $x_0 \in S$ for f and define $g(x) = \int_{x_0}^{x} \omega$. This yields a well-defined function on $f^{-1}U$ where U is a suitable neighbourhood of x_0. It is constant on any fibre and so is of the form $g = f^*h$ with $h \in \mathcal{O}_C(U)$ and $\omega = f^*(dh)$ on $f^*(U)$. The local forms dh define a global meromorphic form ω' with $\omega = f^*\omega'$. Show that ω' must be holomorphic.

16.4. Show that any fibre bundle is an isotrivial fibration if the fibre has a finite group of automorphisms. Hint: any fibre bundle determines a homomorphism of the fundamental group of the base manifold to the group of automorphisms of the fibre ('monodromy') and the bundle is trivial if and only if this homomorphism is trivial.

16.5. Prove that $\mathfrak{h}_g$ is isomorphic to the bounded domain

$$\{ U \in \mathbb{C}^{g,g} \mid \mathbb{1}_g - U\bar{U}^T \geq 0 \}.$$

Hint: consider the map $\mathfrak{h}_g \ni Z \mapsto (i\mathbb{1}_g + Z)(i\mathbb{1}_g - Z)^{-1}$.