Cours de l'institut Fourier

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Appendices

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A 1. Appendix: Some algebra

A1.1 Direct limits of modules

In this section R is any commutative ring with unit.

One starts with a partially ordered set I with partial ordering $<$. Assume that the partial ordering is directed. This means that every two elements $i, j \in I$ have a common upperbound k, i.e. $i < k$ and $j < k$. Assume furthermore that for every $i \in I$ there is some R-module M_i and for any pair $i, j \in I$ with $i < j$ there are homomorphisms $h^i_j : M_i \to M_j$ which satisfy a cocycle relation $h^j_k \circ h^i_j = h^i_k$.

A direct limit of this system of modules is a module M together with homomorphisms $h^i : M_i \to M$ with $h^j \circ h^i_j = h^i$ whenever $i < j$ and such that the usual universality property holds: Given any module N with homomorphisms $k^i : M_i \to N$ which also satisfy $k^j \circ h^i_j = k^i$ whenever $i < j$ there is a unique homomorphism $k : M \to N$ such that $k^i = k \circ h^i$ for all $i \in I$.

It follows that any two direct limits are isomorphic by a unique isomorphism and it makes sense to speak of the direct limit denoted by

$$\operatorname*{dirlim}_I M_i.$$

There is the following standard construction of the direct limit. One takes the direct product $\prod_{i \in I} M_i$ and identifies $m \in M_i$ (viewed as submodule of the product) with $m' \in M_j$ whenever there is some k with $i < k$ and $j < k$ such that $h^i_k(m) = h^j_k(m')$. The quotient module M and the natural maps $h^i : M_i \to M$ then satisfy the properties needed for direct limit as one may readily verify.

A useful remark is that in forming the direct limit one need not take the entire set I. Any subset $J \subset I$ which itself directed under $<$ and which is co-final in it will do. This means that for any $i \in I$ there is some $j \in J$ with $i < j$. So, the remark is that the homomorphism resulting from the universal property of direct limits

$$\operatorname*{dirlim}_J M_j \to \operatorname*{dirlim}_I M_i$$

is indeed an isomorphism. It is straightforward to see that this map is surjective because of the fact that J is cofinal. That it is injective is slightly more involved, and is left to the reader.

A1.2 Some basic commutative algebra

The following concepts and theorems are used freely. For background and proofs see [Reid, p.48-49].

A commutative ring R with unit is *Noetherian* if every ideal in it is finitely generated. Equivalently, every ascending chain of ideals in R becomes stationary. A basic fact is:

Theorem *(Hilbert's basis theorem) If R is Noetherian, then so is $R[X]$.*

The concept of *localisation* of a ring R is used throughout. One starts with a multiplicative set $S \subset R$, i.e. $1 \in S$ and if $f, g \in S$ then $fg \in S$. Then one considers the equivalence relation on $R \times S$ given by $(r, s) \equiv (r', s')$ if and only $s''(rs' - r's) = 0$ for some $s'' \in S$. The equivalence class of (r, s) is denoted r/s. The equivalence classes form a ring R_S, the localisation of R in S. The map which sends $r \in R$ to $r/1$ is a homomorphism $R \to R_S$. Important special cases are when $S = \{$ non-zero divisors in $R \}$ or when $S = \{ f^n \mid n \in \mathbb{Z}_{\geq 0} \}$ with f a non-zero divisor. In the first case you get the ring of fractions $Q(R)$ of R and in the second case you get a ring denoted by R_f. If R is an integral domain (i.e. there are no zero-divisors except 0), the ring $Q(R)$ is a field, the field of fractions and R embeds in it and if S does not contain 0 the localisation R_S also embeds naturally in the field of fractions.

If R is Noetherian, any localisation is.

The concepts of Noetherian goes over to R-modules M by replacing 'ideal' with 'submodule' of M if appropriate in the above. Also, by replacing R with M one can define the localisation M_S of M in S. It is in a natural way an R_S-module and homomorphisms between R-modules induce homomorphism between their localisations.

A1.3 Normalisation of rings

Let there be given a commutative ring R with unit 1 and let S be a subring. An element $r \in R$ is called integral over S if it satisfies an equation of the form

$$x^n + a_{n-1}x^{n-1} + \dots a_1 x + a_0 = 0, \quad a_j \in S.$$

The integral elements form a subring of R containing S, the integral closure of S in R. This is not entirely trivial. See [Ma §9]. If it coincides with S, the ring S is said to be integrally closed in R. A ring which is integrally closed in its quotient field is called *normal.*

It is easy to see that if an integral domain is normal, all of its localisations are normal. The converse is also quite easy and in fact already follows as soon as the localisations in all maximal ideals are normal. Indeed, any element in the field of fractions of a given integral domain R which is integral over the locatisation in a maximal ideal then must belong to this localisation and the intersection of these localisations is precisely R (this is a nice exercise).

The following fact is less elementary. The proof uses Galois theory. See for instance [Ii, §2.2].

Theorem *Let R be a normal Noetherian ring with K its field of fractions. Let L be a finite separable field extension. Then the integral closure of R in L is a finite R-module.*

A 2. Appendix: Algebraic Topology

A2.1 Chain complexes

Let me start out with some commutative ring R with a unit and a collection of R-modules K_i. A *chain complex* is a sequence of R-module homomorphisms

$$K_\bullet = \{ \cdots \to K_{i-1} \xrightarrow{d_{i-1}} K_i \xrightarrow{d_i} K_{i+1} \to \cdots \}$$

with the property that $d_i \circ d_{i-1} = 0$. The homology groups are defined by

$$H_p(K_\bullet) = \frac{\ker\left(K_i \xrightarrow{d_i} K_{i+1}\right)}{\operatorname{im}\left(K_{i-1} \xrightarrow{d_{i-1}} K_i\right)}.$$

If preceding maps increase the index-degree you have a *cochain complex.* Usualy one uses upper-indices in this case. It should be clear what is meant by a homomorphism $f : K^\bullet \to L^\bullet$ of cochain complexes. These induce maps $H^p(f) : H^p(K^\bullet) \to H^p(L^\bullet)$ in cohomology. For a short exact sequence

$$0 \to K'^\bullet \xrightarrow{f} K^\bullet \xrightarrow{g} K''^\bullet \to 0$$

of cochain complexes one can define coboundary maps $\delta : H^p(K''^\bullet) \to H^{p+1}(K'^\bullet)$ such that the resulting cohomology sequence

$$\ldots \to H^p(K'^\bullet) \xrightarrow{H^p(f)} H^p(K^\bullet) \xrightarrow{H^p(g)} H^p(K''^\bullet) \xrightarrow{\delta} H^{p+1}(K'^\bullet) \to \ldots$$

is exact.

A2.2 Polyhedra, cell-complexes

Classically, (co)-homology groups were first defined for polyhedra. These are spaces built up from linear simplices which make computation of (co)-homology an almost mechanical task.

The basic building blocks are the (linear) p-simplices, i.e. the convex hulls of $p+1$ independent points in some $\mathbb{R}^n$. Each p-simplex has a boundary consisting of $p-1$-simplices and there are $p+1$ of them. These form the $(p-1$*-faces.* By induction one defines the q-*faces* of a p-simplex for $q < p$. The 0-dimensional faces are also called *vertices.* A (compact) *polyhedron* is a topological space X which admits a triangulation, i.e. a homeomorphism $t : K \to X$, where K is a *simplicial complex*, i.e. a closed subset of $\mathbb{R}^n$ which is the finite union of simplices such that two simplices have at most an entire face in common. A *simplicial map* between simplicial complexes K and K' is a homeomorphism $K \to K'$ which maps every simplex of K in an affine-linear way to a simplex of K'. It should be clear what is meant by a subcomplex of a given simplicial complex and a compact polyhedral pair (X, A) of topological spaces.

Example A2 1. Any compact differentiable manifold X with a compact submanifold A is a polyhedral pair. In fact one may choose a differentiable triangulation for the pair (X, A). See [Mun, Problem 10.8].

Order the vertices occurring in a given simplicial complex K once and for all. Now given any simplex any ordering of its set of vertices $\{ P_0, \dots, P_q \}$ can be compared with the fixed order and hence determines a sign. In this way one can unambiguously speak of *ordered simplices* in a given complex and one can define a q-chain as a finite formal linear combination of ordered q-simplices. These form an abelian group $C_q(K)$. The 'dual' group $\mathrm{Hom}_{\mathbb{Z}}(C_q(K), \mathbb{Z})$ is called the group of q-cochains and denoted $C^q(K)$. There is the boundary homomorphism $\delta_p : C_p(K) \to C_{p-1}(K)$ defined for an ordered p-simplex $[P_0, \dots, P_p]$ by $\delta([P_0, \dots, P_p]) = \sum_{q=0}^{p} (-1)^q [P_0, \dots, \widehat{P_q}, \dots, P_p]$ and then extended by linearity. The coboundary $\partial^{p-1} : C^{p-1}(K) \to C^p(K)$ is its transpose. One verifies that $\delta_{p-1} \circ \delta_p = 0$ and so this gives a complex with homology group

$$H_p(K) = H_p(C_\bullet(K)) = \frac{\ker (C_p(K) \to C_{p-1}K)}{\operatorname{im} (C_{p+1}(K) \to C_p(K))}.$$

Similarly one has the cohomology groups

$$H^p(K) = H^p(C^\bullet) = \frac{\ker (C^p(K) \to C^{p+1}(K))}{\operatorname{im} (C^{p-1}(K) \to C^p(K))}.$$

Clearly, simplicial maps $f : K \to K'$ induce homomomorphisms $C_q(f)$ between the groups of p-chains compatible with the boundaries and likewise for the p-cochains. So there are induced maps $H_q(f) : H_q(K) \to H_q(K')$ in homology and $H^q(f) : H^q(K') \to H^q(K)$ in cohomology with the obvious functoriality properties ($H^q(\mathrm{Id}) = \mathrm{Id}$ and $H^q(f \circ g) = H^q(g) \circ H^q(f)$).

If L is a subcomplex of K with inclusion $i : L \to K$, define

$$C^p(K, L) = \frac{C^p(K)}{C^p(i)(C^p(L))}$$

and let

$$j : C^p(K) \to C^p(K, L)$$

be the natural projection. From Appendix A2.1 one concludes that there are coboundary homomorphisms $H^q(L) \xrightarrow{\delta^q} H^{q+1}(K, L)$ fitting into a long exact sequence

$$\cdots \xrightarrow{\delta^{q-1}} H^q(K, L) \xrightarrow{H^q(j)} H^q(K) \xrightarrow{H^q(i)} H^q(L) \xrightarrow{\delta^q} H^{q+1}(K, L) \to \cdots$$

Similar assertions hold for homology.

If one considers a polyhedron X with triangulation $t : K \to X$, it is by no means clear that the groups $H_q(K)$ are intrinsically attached to X. This however is true, and I'll come back to this in section A2.3 where singular (co)-homology is introduced. For the moment, assuming this fact, note that the homology groups of many topological spaces now can be computed 'by hand' by choosing an appropriate triangulation.

Examples A2 2. 1. The sphere S^n. One easily finds that $H_q(S^n) = H^q(S^n) = 0$ unless $q = 0$ or $q = n$ in which case these groups are infinite cyclic.

2. A compact Riemann surface of genus g. One finds that $H_0 \cong H^0 \cong H_2 \cong H^2 \cong \mathbb{Z}$ and $H_1 \cong H^1 \cong \mathbb{Z}^{2g}$, where g is the genus of the Riemann-surface.

Disappointingly, even for relatively simple spaces such as a torus, one needs a lot of simplices to triangulate. For this reason one needs larger building blocks, so called *cells.* A singular q-cell inside X is the continuous image in X of the closed q-ball by means of a continuous map, the *characteristic map* which restricts to a homeomorphism from the open q-ball onto its image. A (finite) *cell complex, or CW-complex* is a compact Hausdorff space which is the union of a finite number of (singular) cells such that the boundary of a cell is a union of cells of strictly lower dimension and two cells have no interior points in common. The union of the q-cells is called the *q-skeleton* and for a $q+1$-cell, the characteristic map restricted to the boundary sphere is a continuous map of the q-sphere onto the q-skeleton and is called the *attaching map* of the q-cell.

As in the case of a triangulation one could define homology-groups for cellular complexes, but the definition is a bit more involved. See [M-S, p. 260–263] for details. Instead of carrying this out, in Appendix A2.3 it is indicated how one can use these building blocks to compute homology using the Mayer-Vietoris sequence.

Examples A2 3.

1. The n-sphere is the standard example consisting of one n-ball and one 0-ball.

2. The complex projective space $\mathbb{P}^n$ is the union of $\mathbb{C}^n$ and $\mathbb{P}^{n-1}$. One can take the standard closed $2n$-ball $B_{2n} \subset \mathbb{C}^{2n}$ and define a surjective continuous map $f_n : B_{2n} \to \mathbb{P}^n$ by setting $f_n(z_1, \ldots, z_n) = (\sqrt{1 - |z_1|^2 - \ldots - |z_n|^2}, z_1, \ldots, z_n)$. This map restricts to the boundary sphere as the restriction to S^{2n-1} of the defining projection $\mathbb{C}^n \setminus \{ 0 \} \to \mathbb{P}^{n-1}$. This is the *Hopf fibration* $S^{2n-1} \to \mathbb{P}^{n-1}$. Then $\mathbb{P}^n$ is also the union of the open cell $f_n(\overset{\circ}{B}_{2n})$ and $\mathbb{P}^{n-1}$.

Inductively one can define a cell-complex by letting $\mathbb{P}^0$ be the complex consisting of one point and use f_i, $i = 1, 2, \ldots, n$ to attach successively cells of dimensions $2i$. The resulting complex has only even-dimensional cells, one in each even dimension.

3. The direct product of two cell-complexes is again a cell-complex.

A2.3 The axiomatic approach.

Let R be a commutative ring with 1 (mostly $\mathbb{Z}$ or $\mathbb{R}$ or $\mathbb{C}$). Consider a collection of pairs of topological spaces (X, A) (this means $A \subset X$) and certain continuous maps $f : (X, A) \to (Y, B)$ between them (this means a continuous $f : X \to Y$ with $f(A) \subset B$. A cohomology theory with coefficients R assigns to each such pair (X, A) R-modules $H^q(X, A)$, $q \in \mathbb{Z}$ (the cohomology-groups) and to each of the allowed continuous $f : (X, A) \to (Y, B)$ R-module homomorphisms $H^q(f) : H^q(Y, B) \to H^q(X, A)$ (the induced maps in cohomology) such that the usual functorial properties hold:

1. $H^q(f \circ g) = H^q(g) \circ H^q(f)$,
2. $H^q(\mathrm{Id}) = \mathrm{Id}_{H^q}$.

Furthermore the following axioms are to hold:

3. (Homotopy Axiom) Homotopic continuous maps induce the same map in cohomology.

4. (Exactness Axiom) For every pair (X, A) and every $q \in \mathbb{Z}$ there are R-module homomorphisms $\delta^q : H^q(A) \to H^{q+1}(X, A)$ (the *coboundary-operators*) such that the following sequence is exact ($i : A \to X$ and $j : X \to (X, A)$ are the obvious inclusions)

$$\xrightarrow{\delta^{q-1}} H^q(X,A) \xrightarrow{H^q(j)} H^q(X) \xrightarrow{H^q(i)} H^q(A) \xrightarrow{\delta^q} H^{q+1}(X,A) \to \dots$$

5. (Naturality of the Coboundary) If $f : (X, A) \to (Y, B)$ a continuous map there are commutative diagrams

$$\begin{tikzcd} H^q(A) \arrow[r, "\delta^q"] & H^{q+1}(X,A) \\ H^q(B) \arrow[u] \arrow[r, "\delta^q"] & H^{q+1}(Y,B), \arrow[u] \end{tikzcd}$$

where the vertical arrows are induced by f.

6. (Excision Axiom) Let (X, A) be a pair and $U \subset X$ open with closure contained in the interior of A, then the excision map $(X - U, A - U) \subset (X, U)$ induces isomorphisms in cohomology.

7. (Dimension Axiom) Let p be a point. Then $H^0(p) = R$ and $H^q(p) = 0$ for $q \neq 0$.

Examples A2 4.

1. Simplicial Cohomology. Restrict to complexes $K, K', \dots$ and and simplicial maps $K \to K', \dots$ between them. The cohomology groups $H^q(K)$ and induced maps give an example of a cohomology theory, which is called *simplicial cohomology*. The verification of the axioms is done for instance in [Sp,Chapter 4] by relating these groups to the singular cohomology groups, which are introduced next.

2. Singular Cohomology. The singular cohomology groups are defined for all pairs (X, A) of topological spaces. To define them a couple of concepts are needed.

The standard p-simplex. This is the convex hull in $\mathbb{R}^{p+1}$ of the $p+1$ standard unit-vectors:

$$\Delta_p = \{ (x_1, \dots, x_{p+1}) \mid x_i \geq 0, \sum_i x_i = 1 \}.$$

The boundary of Δ_p consists of the $p-1$-simplices $\Delta_p^q = \Delta_p \cap \{ x_q = 0 \}$, $q = 1, \dots, p+1$. There are natural embeddings $i^q : \Delta_p \to \Delta_p^q$.

A singular p-simplex in a topological space X is a continuous map of the standard p-simplex to X.

A singular p-chain in X is a formal finite linear combination of singular p-simplices with coefficients in R. These form a R-module $S_p(X; R)$. A *singular p-cochain* is a R-module homomorphism $S_p(X; \mathbb{Z}) \to R$. The singular p-cochains form the R-module $S^p(X; R)$.

The boundary homomorphism $\delta_p : S_p(X;R) \to S_{p-1}(X;R)$ is defined first for a singular simplex $\sigma : \Delta_p \to X$ by $\delta(\sigma) = \sum_{q=1}^{p+1}(-1)^{q+1}\sigma \circ i^q$ and then extending it as an R-module-homomorphism. The singular chains and their boundary maps form a chain complex with homology groups $H_p(X;R)$. Explicitly

$$H_p(X;R) = \frac{\ker(S_p(X;R) \to S_{p-1}(X;R)}{\operatorname{im}(S_{p+1}(S;R) \to S_p(X;R))}.$$

The *coboundary* $\partial^{p-1} : S^{p-1}(X;R) \to S^p(X;R)$ is the R-dual or transpose of the boundary map, yielding a cochain complex $\{ S^\bullet, \partial^\bullet \}$. Its p-th cohomology by definition is the p-th singular cohomology with coefficients in R, notation $H^p(X;R)$.

If $f : X \to Y$ is continuous, there are obvious R-module homomorphisms $S_q(f) : S_q(X) \to S_q(Y)$ resp. $S^q(f) : S^q(Y) \to S^q(X)$ compatible with the boundary, resp. coboundary maps and which induce R-module homomorphisms $H_q(f) : H_q(X;R) \to H_q(Y;R)$ and $H^q(f) : H^q(Y;R) \to H^q(X;R)$. For the latter the axioms 1. and 2. are obvious. They imply cohomological invariance.

The singular cohomology is a topological invariant.

Special Case I want to mention that $H_1(X)$ is the fundamental group modulo its commutator subgroup. See [Gr, section 12]. In particular, simply connected spaces have no first homology groups. Also, if T is a g-torus, $H_1(T) \cong \mathbb{Z}^{2g}$.

For a pair (X,A) the inclusion $i : A \to X$ induces $S(i_q)$ and the cokernel is denoted $S_q(X,A;R)$ and its R-dual by $S^q(X,A;R)$. One verifies that the boundaries, resp. the coboundaries give $S_\bullet(X,A;R)$ resp. $S^\bullet(X,A;R)$ the structure of a chain, resp. cochain complexe and by definition $H_q(X,A;R) = H_q(S_\bullet(X,A;R))$ resp. $H^q(X,A;R) = H^q$ $(S^\bullet(X,A;R))$. The usual theory of complexes then shows that the axioms 4. and 5. are valid. Also the dimension axiom is almost trivial. The remaining two axioms however require some work. See [Gr, sections 11,15].

Later I shall compare homology and cohomology with coefficients in a ring R. First note that the tautological pairing $S^q(X,A;R) \times S_q(X,A;R) \to R$ is compatible with boundary and coboundary and hence one gets a pairing (Kronecker pairing)

$$H^q(X,A;R) \times H_q(X,A;R) \to R.$$

This pairing will be denoted by $\langle\, , \rangle$, so that

$$\langle [f], [c] \rangle = f(c) \quad f \text{ a } q\text{-cocycle, } c \text{ a } q\text{-cycle,}$$

and the square brackets denote the corresponding classes in (co)-homology. This pairing induces the Kronecker homomorphism

$$H^q(X,A) \to \operatorname{Hom}_R(H_q(X,A),R).$$

For a principal ideal ring R this map is surjective. In fact [Gr, section 23]:

Proposition A2 5.

1. *If R is a field, the Kronecker homomorphism is an isomorphism.*

2. *For $R = \mathbb{Z}$ the Kronecker homomorphism is surjective. If $H^q(X, A)$ and $H_{q-1}(X, A)$ are finitely generated, the kernel of the Kronecker map (which is precisely the torsion subgroup of $H^q(X, A)$) is isomorphic to the torsion subgroup of $H_{q-1}(X, A)$.*

In the sequel I shall omit R in case $R = \mathbb{Z}$, hence $H^q(X, A)$ denotes the singular relative cohomology group with integral coefficients. The *Universal coefficient theorem* gives a recipe to determine the (co)-homology groups with coefficients in any principal ideal domain R from the groups with values in $\mathbb{Z}$. See [Gr, Section 29]. I only need the result for fields:

Proposition A2 6. *Let R be a field. For any topological space, the natural homomorphism*

$$H_n(M) \otimes R \to H_n(M; R)$$

is an isomorphism.

For a polydron $t : K \to X$ there is an obvious map of complexes $C_\bullet(K) \to C_\bullet(X)$. This assignment extends to polyhedral pairs and maps between them in an obvious way and induces isomorphisms between simplicial (co)-homology and singular (co)-homology [Sp, 4.6, Theorem 8]. There are some useful consequences.

Proposition-Definition A2 7. *The cohomology groups of compact polyhedral pairs are finitely generated abelian groups. The rank of $H_q(X)$ (here X is a polyhedron) is called the Betti-number $b_q(X)$. The alternating sum $e(X) = \sum_j (-1)^q b_q(X)$ is called the Euler number. It is equal to the alternating sum $\sum_q (-1)^q n_q$ of the number of q-simplices n_q.*

Corollary A2 8. *If $f : X \to Y$ is an unramified covering between polyhedra of degree d one has $e(Y) = d \cdot e(X)$.*

There is a way to compute the cohomology of a product of polyhedra, or more generally, cell complexes, from the cohomology groups of the factors. This is expressed by the *Künneth formula*. I only give the result for the ranks of the cohomology groups and refer to [Gr, 29.11] for the full statement.

Proposition A2 9. *Let X and Y be finite cell complexes. Then $X \times Y$ is a finite cell compex and one has*

$$b_n(X \times Y) = \sum_{i=0}^{n} b_i(X) \cdot b_{n-i}(Y).$$

From the axioms one can derive the Mayer-Vietoris sequence (see [Sp, 4.6 and 5.4] which tells one how to compute the (ho)mology of a union of (suitable sets) from its parts. Since many spaces are cell-complexes, this is very useful. Suppose that a topological space X is a union $X = X_1 \cup X_2$ of two parts for which the inclusion maps $(X_1, X_1 \cap X_2) \to (X, X_2)$ and $(X_2, X_1 \cap X_2) \to (X, X_1)$ induce isomorphisms in the cohomology. The most important cases are when X_1 and X_2 are open in X or when X is a simplicial complex and X_1 and X_2 are subcomplexes or when $X_1 = X_{q-1}$ is the $q-1$-skeleton of a cell-complex and X_2 a given q-cell. With $i_k : X_k \to X$ and $j_k : X_1 \cap X_2 \to X_k$ the inclusion maps, the exact Mayer-Vietoris sequence reads as follows

$$\cdots \to H^q(X) \xrightarrow{(H^q(i_1),H^q(i_2))} H^q(X_1) \oplus H^q(X_2) \xrightarrow{j_1 - j_2} H^q(X_1 \cap X_2) \to H^{q+1}(X) \to \cdots$$

Example A2 10. One easily computes the cohomology of $\mathbb{P}^n$ using the previous description of it as a cell-complex. You find that $H^q(\mathbb{P}^n) = 0$ when q is odd or $q < 0$ or $q > 2n$ and the remaining cohomology groups are all infinite cyclic. See [Gr, section 19] for details.

A2.4 Manifolds

If M is any n-dimensional topological manifold one defines an orientation sheaf O_M, as the locally constant sheaf of $\mathbb{Z}$-modules defined by the presheaf $U \to H_n(M, M \setminus U)$. By the Excision theorem these groups are isomorphic to $H_n(U, U \setminus \{ x \}) \cong \mathbb{Z}$, where x is any point of a coordinate ball U. So O_M is locally free of rank 1. M is *orientable* if O_M is constant. An orientation is a choice of one of the two generators of $\Gamma(O_M) \cong \mathbb{Z}_M$. One can easily see that M is orientable if and only if one can orient the tangents spaces $T_x(M)$ in a coherent way, i.e. if the line bundle $\det T_M$ is trivial. In the differentiable context this is equivalent to the existence of a nowhere zero differential form of maximal degree n. One can show [Gr, section 22]:

Lemma-Definition A2 11. *A connected compact manifold M of dimension n is orientable if and only if $H_n(M) \cong \mathbb{Z}$. The choice of a generator is equivalent to choosing an orientation. The generator corresponding to a chosen orientation is called the fundamental class of M, denoted $o_M \in H_n(M, \mathbb{Z})$,*

Corollary A2 12. *For any compact complex manifold of dimension n one has $H_{2n} \cong \mathbb{Z}$.*

Next I state the topological version of the Poincaré-duality theorem for compact manifolds, which is more refined than the version for differentiable manifolds in terms of the de Rham cohomology.

I first say a few words about the cup products in singular cohomology. Introduce a product on $\oplus_p S^p(X)$ (now X is an arbitrary topological space) as follows. Define $f \cup g$ for $f \in S^p(X)$ and $g \in S^q(X)$ by evaluating it on a singular $p+q$-simplex $\sigma : \Delta \to X$. Define σ_p to be the singular p-simplex obtained by restricting σ on the standard-simplex spanned by the first $p+1$ unit vectors and σ_q by restricting to the 'complementary face' spanned by the last $q+1$ unit vectors. Then define

$$f \cup g(\sigma) = f(\sigma_p) \cdot g(\sigma_q).$$

One shows that this cup-product induces a (non)-commutative ring structure on the direct sum $H^*(X) = \oplus_q H^q(X)$. This ring has a unit $1 \in H^0(X)$ given by the constant cochain $x \mapsto 1$ for any point $x \in X$. The ring is skew-commutative in that

$$a \cup b = (-1)^{pq} b \cup a, \quad a \in H^p(X), b \in H^q(X).$$

If M is a compact connected oriented n-dimensional manifold, one can use the cup product also to define the duality homomorphism. Let me follow [Gr, section 22]. Define the duality homomorphism

$$D_M : H^q(M) \to H_{n-q}(M)$$

by demanding that

$$\langle b, D_M a\rangle = \langle a \cup b, o_M\rangle,$$

where the Kronecker-pairing and the fundamental class o_M are used.

Theorem A2 13. (Poincaré-duality) *The duality homomorphism*

$D_M : H^q(M) \to H_{n-q}(M)$ *is an isomorphism for all* $q \in \mathbb{Z}$.

Combining this with A2.5 one immediately gets:

Corollary A2 14. *There is only cohomology for* $q = 0, \ldots, n = \dim M$. *For the Betti-numbers one has* $b_q = b_{n-q}$ *and the torsion subgroup of* H_q *is isomorphic to the torsion subgroup of* H_{n-q-1}.

Poincaré-duality also shows that the cup-product pairing between cohomology groups of complentary degrees is perfect.

Corollary A2 15. *The cup-product pairing*

$$\begin{array}{rcl} I^q : H^q(M,\mathbb{Z}) \times H^{n-q}(M,\mathbb{Z}) & \longrightarrow & \mathbb{Z} \\ (a,b) & \longmapsto & \langle a \cup b, o_M\rangle = \langle D_M a, b\rangle \end{array}$$

is perfect in the sense that if $I^q(a,b) = 0$ *for all* $a \in H^q(M,\mathbb{Z})$ *then* b *is torsion and similarly if* $I^q(a,b) = 0$ *for all* $b \in H^{n-q}(M,\mathbb{Z})$ *then* a *is torsion.*

If $n = 2m$ is even, there is the cup product pairing on $H^m(M,\mathbb{Z})$ and Poincaré-duality says that it is unimodular. Recall that this means the following. Choose a basis for $H^m(M,\mathbb{Z})$ mod torsion. Then the Gram matrix of the cup product pairing is integral with determinant ± 1. If m is odd, this pairing is skew-symmetric and it is an easy exercise in linear algebra to show that the rank of $H^m(M,\mathbb{Z})$ must be even, say $2g$ and that one can find a basis so that the Gram matrix becomes the standard symplectic form

$$J_g := \begin{pmatrix} 0_g & -\mathbb{1}_g \\ \mathbb{1}_g & 0_g \end{pmatrix}.$$

One says that the form is *isometric* to J_g.

Example A2 16.

1. Compact Riemann-surface. The preceding considerations show that $H^1 \cong H_1 \cong \mathbb{Z}^{2g}$ the cup-product form is isometric to J_g.

2. Compact complex surfaces. Let T be the torsion subgroup of H_1. Then T is isomorphic to the torsion subgroup in H_2, in H^2 and H^3, whereas H^1 and H_3 are free modules. Moreover $b_1 = b_3$ so that $e = 2 - 2b_1 + b_2$.

One can introduce a dual pairing on $H_m(M,\mathbb{Z})$, the intersection pairing, by setting $I(a,b) = I^m(D_M^{-1}b, D_M^{-1}a)$. I shall give a more geometric description of this pairing for special classes, coming from submanifolds. Recall that for any compact oriented p-dimensional manifold P there is a fundamental class $o_P \in H_p(P,\mathbb{Z})$. If now $i : P \hookrightarrow M$ realises P as a submanifold of M, I define $h(P) = i_*(o_P) \in H_p(M,\mathbb{Z})$. This class is called the *fundamental homology class of P*. Its Poincaré-dual class in $H^{n-p}(M,\mathbb{Z})$ is called the *fundamental co-homology class* and denoted by $c(M)$.

Suppose that M is a compact differentiable oriented manifold of dimension $n = 2m$ as before and that P and P' are two submanifolds of dimension m intersecting transversally in a finite number of points. Any choice of an orientation for P and P' makes it possible to define the homology classes $h(P)$ and $h(P')$. Now at an intersection point m of P and P' taking first the induced orientation of T_xP and then of T_xP' yields an orientation of T_xM which may or may not be compatible with the given orientation of M. In the first case set $(P,P')_x = +1$ and in the second case set $(P,P')_x = -1$. Then set $(P,P') = \sum_x (P,P')_x$. This pairing is the *geometric intersection pairing.*

Claim A2 17. *In the preceding set-up* $(P,P') = I(h(P),h(P')) = \langle c(P) \cup c(P'), o_M\rangle$.

There are various ways to prove this. See [G-H, Chapter 0.4] for a proof. In rough outline this goes as follows. Consider a compact differentiable manifold M and fix a smooth triangulation. First observe that one can assign a fundamental class to topological manifolds, in particular to to piecewise smooth submanifolds. Hence also the geometric intersection pairing for submanifolds can be extended to piecewise smooth submanifolds meeting transversally. Now any p-cycle a can be represented by a linear combination of smooth p-simplices and one can then see that there is a piecewise smooth submanifold A such that a is a multiple of $h(A)$. Next, one shows that a given $n-p$-piecewise linear submanifold B' is homologous to a piecewise linear submanifold B meeting A transversally and one extends the geometric intersection pairing by setting $(A,B') = (A,B)$. Now one shows that Poincaré duality can be given by

$$\langle a,b\rangle = (D_M a, b), \quad a \in H^{n-p}(M,\mathbb{Z}), b \in H_{n-p}(M,\mathbb{Z}).$$

From this formula one easily sees that $I(h(A),h(B)) = (A,B)$ for any two piecewise linear submanifolds of complementary dimension. Indeed $I(h(A),h(B)) = \langle D_M^{-1}h(B) \cup D_M^{-1}h(A), o_M\rangle$ by definition, while $(A,B) = (D_M \circ D_M^{-1}h(A), h(B)) = \langle D_M^{-1}h(A), h(B)\rangle = \langle D_M^{-1}h(B) \cup D_M^{-1}h(A), o_M\rangle$ by the previous formula and the meaning of the Poincaré duality isomorphism D_M.

It should be remarked that in [G-H] the preceding formula for the Poincaré duality isomorphism is derived using differential forms and that De Rham's theorem is used implicitly. To treat this properly, introduce the groups $S^p_\infty(M)$ of singular smooth p-cochains which are simply functionals on the free group S_p^∞ on the smooth singular p-simplices on M. These form a subcomplex $S^\bullet_\infty(M)$ of $S^\bullet(M)$ and one shows ([Wa, 5.31, 5.32]) that there are canonical isomorphisms

$$H^p(\mathbb{R}_M) \cong H^p(S^\bullet_\infty(M,\mathbb{R})) \cong H^p(M;\mathbb{R}).$$

By the De Rham theorem $H^p_{DR}(M)$ is isomorphic to $H^p(\mathbb{R}_M)$ and so there is an isomorphism from the p-th De Rham group to the p-th (smooth) singular cohomology group with real coefficients. There is a homomorphism

$$\mathcal{E}^p(M) \longrightarrow S^p_\infty(M,\mathbb{R})$$

given by integrating p-forms over smooth p-simplices. Stokes' theorem implies that it induces a homomorphism in cohomology

$$\int^{[p]} : H^p_{DR}(M) \to H^p(M;\mathbb{R}).$$

There is the following refined version of the theorem of De Rham, a proof of which can be found in [Wa, p. 205-214].

Theorem A2 18. (Explicit Form of De Rham) *The integration map*

$$\int^{[p]} : H^p_{DR}(M) \to H^p(M;\mathbb{R}).$$

is an isomorphism. If you endow $\oplus_p H^p_{DR}(M)$ with the ring structure coming from the wedge product of differential forms and put a ring structure on $\oplus_p H^p(M;\mathbb{R})$ by means of the cup product, the isomorphism $\oplus_p \int^{[p]}$ becomes an isomorphism of (graded) rings.

Finally, I can reformulate Poincaré duality in terms of this isomorphism.

Corollary A2 19. *Let M be a compact oriented manifold of dimension n with orientation class o_M. There is a commutative diagram*

$$\begin{array}{ccccc}
H^p(M,\mathbb{R}) & \times & H^{n-p}(M,\mathbb{R}) & \xrightarrow{t} & \mathbb{R} \\
\downarrow & & \downarrow & & \| \\
H^p_{\mathrm{DR}}(M) & \times & H^{n-p}_{\mathrm{DR}}(M) & \xrightarrow{t_{\mathrm{DR}}} & \mathbb{R}.
\end{array}$$

Here $t(a,b) = \langle a \cup b, o_M\rangle$ and $t_{\mathrm{DR}}(\alpha,\beta) = \int_M \alpha \wedge \beta$

A.2.5 Lefschetz theory

Here I review the theory of hyperplane sections and its consequences for hypersurfaces in $\mathbb{P}^n$. An excellent treatment, using Morse theory, can be found in [Mi].

Theorem A2 20. (Lefschetz theorem on hyperplane sections) *Let $X \subset \mathbb{P}^N$ be an $(n+1)$-dimensional projective variety and let H be a hyperplane which contains the singular points of X. The inclusion $X \cap H \to X$ induces isomorphisms for the integral homology groups of degree $\leq n-1$ and a surjection in degree n. A similar result holds for the homotopy groups. In particular, if X is connected and $n \geq 1$, $X \cap H$ is connected. If X is simply connected and $n \geq 2$, also $X \cap H$ is simply connected.*

This can be applied in the following way. Let $X \subset \mathbb{P}^{n+1}$ be a smooth hypersurface of degree d and consider the d-uple Veronese embedding $\mathbb{P}^{n+1} \hookrightarrow \mathbb{P}^N$ using the polynomials of degree d. The hyperplanes in $\mathbb{P}^N$ correspond to the hypersurfaces of degree d in $\mathbb{P}^n$ and the preceding theorem can be applied. More generally one has

Corollary A2 21. *Let $X \subset \mathbb{P}^{n+k}$ be a complete intersection manifold of dimension n. If $n \geq 2$ the manifold X is connected and simply connected. One has $H_m(X,\mathbb{Z}) = 0$ for m odd and $m \leq n-1$, $H_m(X,\mathbb{Z}) \cong \mathbb{Z}$ for m even and $m \leq n-1$.*

A 3. Appendix: Hodge Theory and Kähler manifolds

A3.1 Hodge theory and consequences

Let M be a compact complex n-dimensional manifold with a Hermitian metric h. By definition, this is a smooth section in the bundle $T(M) \otimes \overline{T}(M)$. Taking the anti-symmetric part gives a real $(1,1)$-form ω_h, the metric form associated to h. To fix the normalisation, if in local coordinates h is given by

$$h = \sum_{i,j} h_{i\bar{j}} dz_i \otimes d\bar{z}_j,$$

the form is given by

$$\omega_h = \frac{1}{2}\sqrt{-1} \sum_{i,j} h_{i\bar{j}} dz_i \wedge d\bar{z}_j.$$

Let me outline how Hodge theory works. The metric h, which is a metric on the tangent bundle, induces metrics on all the associated tensor bundles. For instance, one obtains pointwise metrics $(\, ,\,)_m$, $m \in M$ on the bundle $\mathcal{E}^{p,q}(M)$, the bundle of complex-valued (p,q)-forms. Now using the volume-form

$$vol_h := \underbrace{\omega_h \wedge \ldots \wedge \omega_h}_{n}$$

global inner products, the *Hodge inner products* can be defined

$$(\, ,\,) := \int_M (\, ,\,)_m vol_h.$$

With this inner-product, $\mathcal{E}^m_{\mathbb{C}}(M) = \oplus_{p+q=m} \mathcal{E}^{p,q}(M)$ is an an orthogonal splitting ([We, Chapt V, Prop. 2.2]).

The Hodge *-operators

$$* : \wedge^m T^\vee_m M \to \wedge^{2n-m} T^\vee_m M$$

are defined by the formula $\alpha \wedge * \beta = (\alpha, \beta) vol_h(m)$. They induce linear operators on $\mathcal{E}^m(M)$ and $\mathcal{E}^m_{\mathbb{C}}(M)$. The corresponding conjugate linear operator

$$\bar{*} : \mathcal{E}^{p,q}(M) \to \mathcal{E}^{n-q,n-p}(M)$$

is defined by $\bar{*}(\alpha) = *\bar{\alpha}$.

The d, ∂ and $\overline{\partial}$-operators have formal adjoints d^*, ∂^* and $\overline{\partial}^*$ with respect to these inner products and one can form the associated Laplacians:

$$\begin{aligned}\Delta_d &= dd^* + d^*d,\\ \Delta_\partial &= \partial\partial^* + \partial^*\partial,\\ \Delta_{\overline{\partial}} &= \overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}.\end{aligned}$$

The m-forms that satisfy the Laplace equation $\Delta_d = 0$ are called *d-harmonic* and denoted $\mathrm{Harm}^m(M)$. Likewise for the $\overline{\partial}$-harmonic (p,q)-forms. These constitute $\mathrm{Harm}^{p,q}(M)$.

To motivate the Hodge theorems, let me assume that the spaces $\mathcal{E}^p(M)$ are finite dimensional. Look at the short complex

$$\mathcal{E}^{p-1}(M) \xrightarrow{d} \mathcal{E}^p(M) \xrightarrow{d} \mathcal{E}^{p+1}(M).$$

Standard linear algebra yields orthogonal sum decompositions $\mathcal{E}^p(M) = \ker d \oplus \operatorname{im} d^* = \operatorname{im} d \oplus \ker d^*$ and since $\operatorname{im} d \subset \ker d$ and $\operatorname{im} d^* \subset \ker d^*$, the direct sum decomposition

$$\mathcal{E}^p(M) = \ker d \cap \ker d^* \oplus \operatorname{im} d \oplus \operatorname{im} d^*$$

follows. Now the first summand consists precisely of the d-harmonic forms and the d-Laplacian is an isomorphism on the other two summands. In particular one sees that the p-th De Rham group can be canonically identified with the space of harmonic forms. Now in general the spaces $\mathcal{E}^p(M)$ are infinite dimensional, but the results still hold. This is the content of the Hodge theorem.

Theorem A3 1. (Hodge Theorem) *Let M be a compact differentiable manifold equipped with a hermitian metric. Then*

1. $\dim \mathrm{Harm}^m(M) < \infty$.
2. *Let*
$$H : \mathcal{E}^m(M) \to \mathrm{Harm}^m(M)$$
be orthogonal projection onto the harmonic forms. There is a unique operator
$$G : \mathcal{E}^m(M) \to \mathcal{E}^m(M)$$
with kernel containing the harmonic forms and which satisfies
$$\mathrm{Id} = H + \Delta_{\overline{\partial}} \cdot G.$$
In particular, one has a direct sum decomposition
$$\mathcal{E}^m(M) = \mathrm{Harm}^m(M) \oplus dd^*G\mathcal{E}^m(M) \oplus d^*dG\mathcal{E}^m(M)$$
and H induces an isomorphism
$$H^m_{\mathrm{DR}}(M) \xrightarrow{\cong} \mathrm{Harm}^m(M).$$

There is a similar version for complex manifolds. In fact the theorem is valid in a more general context, that of elliptic complexes. See [We Chapt. IV, Theorem 5.2]. In particular, one can apply it to (p,q)-forms with values in a vector bundle E with a hermitian metric h_E, replacing $\mathcal{E}^{p,q}$ by $\mathcal{E}^{p,q}(E) := \mathcal{E}^{p,q} \otimes E$. To introduce a Hodge metric on $\mathcal{E}^m_{\mathbb{C}} \otimes E$, first choose a conjugate linear isomorphism $\tau : E \to E^\vee$ and define

$$\bar{*}_E : \mathcal{E}^{p,q}(E) \to \mathcal{E}^{n-q,n-p}(E^\vee)$$

by $\bar{*}_E(\alpha \otimes e) = \bar{*}\alpha \otimes \tau(e)$. Then one defines the Hodge metric on $\mathcal{E}^{p,q}(E)$ by

$$(-,-') = \int_M - \wedge \bar{*}_E -' .$$

Let us now state the version needed in the text.

Theorem A3 2. (Hodge Theorem - Second Version) *Let M be a compact complex manifold with hermitian metric and let E be a vector bundle equipped with an hermitian metric.*

1. *$\dim \mathrm{Harm}^{p,q}(E) < \infty$.*
2. *Let*

$$H : \mathcal{E}^{p,q}(E) \to \mathrm{Harm}^{p,q}(E)$$

be orthogonal projection onto the harmonic forms. There is a unique operator

$$G : \mathcal{E}^{p,q}(E) \to \mathcal{E}^{p,q}(E)$$

with kernel containing the harmonic forms and which satisfies

$$\mathrm{Id} = H + \triangle_{\overline{\partial}} \cdot G.$$

In particular, one has a direct sum decomposition

$$\mathcal{E}^{p,q}(E) = \mathrm{Harm}^{p,q}(E) \oplus dd^*G\mathcal{E}^{p,q}(E) \oplus d^*dG\mathcal{E}^{p,q}(E)$$

and H induces an isomorphism

$$H^{p,q}_{\overline{\partial}}(E) \xrightarrow{\cong} \mathrm{Harm}^{p,q}(E)$$

where

$$H^{p,q}_{\overline{\partial}}(E) := \frac{\overline{\partial}\text{-closed } (p,q)\text{-forms with values in } E}{\overline{\partial}\mathcal{E}^{p,q-1}(E)} \xrightarrow{\cong} \mathrm{Harm}^{p,q}(E).$$

Combining the last part of this theorem with the Dolbeault-isomorphism 3.3 one finds that the groups $H^q(\Omega^q(E))$ are finite dimensional.

Next, note that the operator $\bar{*}_E$ commutes with the Laplacian $\Delta_{\bar\partial}$ as acting on $\mathcal{E}^{p,q}(E)$ and hence harmonic (p,q)-forms with values in E go to harmonic $(n-p, n-q)$-forms with values in $E^\vee$. In particular $\mathrm{Harm}^{p,q}(E)$ and $\mathrm{Harm}^{n-q,n-p}(E^\vee)$ are conjugate-linearly isomorphic. The following classical consequence then follows.

Corollary A3 3. (Serre Duality) *The operator $*_E$ defines an isomorphism*

$$H^q(M, \Omega^p(E)) \xrightarrow{\cong} H^{n-p}(M, \Omega^{n-p}(E^\vee))^\vee.$$

A3.2 Kähler metrics and the Hodge decomposition theorem

A metric h is called *Kähler* if the associated form is closed. Such a form is called a *Kähler form.* Any manifold admitting a Kähler metric is called Kähler manifold.

Examples A3 4.

1. Any hermitian metric on a Riemann surface is Kähler.
2. The Fubini-Study metric on $\mathbb{P}^n$ is Kähler. It is defined by the $\mathbb{C}^*$-invariant form $\frac{1}{2\pi}\partial\bar\partial \log ||Z||^2$ on $\mathbb{C}^{n+1} \setminus \{ 0 \}$.
3. Any submanifold of a Kähler manifold is Kähler. Indeed, the restriction of the Kähler form restricted to the submanifold is a Kähler form on this submanifold. An important special case are the projective manifolds.

Let now h be a Kähler metric, and ω_h its associated $(1,1)$-form. Let L denote the operator defined by multiplication against the Kähler form: $L(\alpha) = \omega_h \wedge \alpha$. Let Λ denote its formal adjoint. These operators are of types $(1,1)$ and $(-1,-1)$, respectively. I use the square brackets to denote commutators of operators: $[A,B] = AB - BA$. This said, there are the fundamental *Kähler identities* [We, Chapt V, Coroll. 4.10]

$$\begin{aligned}\partial^* &= \sqrt{-1}[\Lambda, \bar\partial] \\ \bar\partial^* &= -\sqrt{-1}[\Lambda, \partial].\end{aligned}$$

If you introduce the real operator

$$d_c := -\sqrt{-1}(\partial - \bar\partial)$$

with formal adjoint d_c^* these can be rewritten as

$$[\Lambda, d] = -d_c^*.$$

The adjoint relation is also useful and reads as follows.

$$[L, d^*] = d_c.$$

In [We] these relations are derived more or less together with the following Claim, using representation theory of $SL(2,\mathbb{C})$. I have separated the latter from the Kähler identities for clarity. Also, the proof given here is somewhat shorter than the proof in [We].

From these identities one derives:

Claim *One has $\triangle_d = 2\triangle_{\overline{\partial}}$ and hence in particular*

1. *The Laplacian is real,*
2. *The Laplacian preserves types*
3. *The L-operator preserves harmonic forms.*

Proof: One has $\partial\overline{\partial}^* + \overline{\partial}^*\partial = 0$ since

$$\begin{aligned}\sqrt{-1}(\partial\overline{\partial}^* + \overline{\partial}^*\partial) =& \partial[\Lambda, \partial] + [\Lambda, \partial]\partial \\ =& \partial\Lambda\partial - \partial\Lambda\partial = 0.\end{aligned}$$

Then

$$\begin{aligned}\triangle_d =& (\partial + \overline{\partial})(\partial^* + \overline{\partial}^*) + (\partial^* + \overline{\partial}^*)(\partial + \overline{\partial}) \\ =& \triangle_\partial + \triangle_{\overline{\partial}}.\end{aligned}$$

Next observe that

$$\begin{aligned}-\sqrt{-1}\triangle_\partial =& -\partial[\Lambda, \overline{\partial}] - [\Lambda, \overline{\partial}\,]\partial \\ =& -\partial\Lambda\overline{\partial} + \partial\overline{\partial}\Lambda - \Lambda\overline{\partial}\partial + \overline{\partial}\Lambda\partial \\ =& \ \overline{\partial}\Lambda\partial + \partial\overline{\partial}\Lambda - \Lambda\overline{\partial}\partial - \partial\Lambda\overline{\partial} \\ =& \ \overline{\partial}[\Lambda, \partial] + [\Lambda, \partial]\overline{\partial} \\ =& -\sqrt{-1}\triangle_{\overline{\partial}}.\end{aligned}$$

Finally, if α is harmonic, one has $d\alpha = d^*\alpha = 0$. So $dL\alpha = d(\omega \wedge \alpha) = \omega \wedge d\alpha = 0$ and if one assumes that α has pure type (this is allowed because of part (2) of the Consequences) one also has $d^*L\alpha = -[L, d^*]\alpha = d_c^*\alpha = 0$. ■

From the preceding consequences one derives immediately:

Theorem A3 5. (Hodge Decomposition) *Let M be a compact Kähler manifold. There is a direct sum decomposition*

$$H^m_{\mathrm{DR}}(M) \otimes \mathbb{C} = \oplus_{p+q=m} H^{p,q}_{\overline{\partial}}.$$

Moreover $H^{p,q}_{\overline{\partial}} = \overline{H^{q,p}_{\overline{\partial}}}$.

Proof: Since the Laplacian preserves types, there is a homomorphism

$$\mathrm{Harm}^m(M) \to \oplus^{p,q}\,\mathrm{Harm}^{p,q}(M).$$

which is clearly injective and surjective. Since the Laplacian is real, the last statement follows also. ■

This theorem allows us to see that the Hopf manifolds are not Kähler and hence a fortiori not projective.

Example A3 6. The Hopf manifolds are not Kähler . To see this, recall that a Hopf manifold is homeomorphic to $S^1 \times S^{2n-1}$ and so $b_1(S^1 \times S^{2n-1}) = b_1(S^1) + b_1(S^{2n-1}) = 1$ since $n > 1$. Here the Künneth formulas are used, see Proposition A2.9. On the other hand, b_1 must be even for any Kähler manifold.

A3.3 Implications for Riemann surfaces

In this section I prove that any compact Riemann surface can be embedded in some projective space and hence, by Chow's Theorem, is projective.

From Hodge theory you know that for any line bundle L on a compact Riemann surface M the space of sections $H^0(\mathcal{O}_M(L))$ as well as the space $H^1(\mathcal{O}_M(L))$ is finite dimensional. It is an elementary observation that L can have no holomorphic sections if deg $L < 0$. Indeed, any holomorphic section of L would vanish in a divisor D which either is zero or effective and hence deg $L =$ deg $D \geq 0$. Now Serre-duality implies that dually $H^1(\mathcal{O}_M(L)) = 0$ if deg $L > 2g(M) - 2$.

Let me start with a divisor D of degree $> 2g(M)$ and let $L = \mathcal{O}_M(D)$ be the corresponding line bundle. I claim that the corresponding meromorphic map $\varphi_L : C \dashrightarrow \mathbb{P}^N$ gives an embedding. By 4.20 it is sufficient to show that $H^1(M, \mathfrak{m}_x \cdot L) = 0$ for all $x \in M$ and that $H^1(M, \mathfrak{m}_x \cdot \mathfrak{m}_y \cdot L) = 0$ for all pairs of points $x, y \in M$. Now $\mathfrak{m}_x = \mathcal{O}_M(-x)$ since you are on a Riemann surface and the bundles involved all have degree $> 2g(M) - 2$ and so, by the previous remark, the desired groups vanish and map φ_L is an embedding.

A3.4 First Chern class

Let M be a compact manifold and let L be a holomorphic line bundle on M with a hermitian metric h. The form

$$c(L, h) := -\frac{i}{2\pi}\partial\overline{\partial}\log h$$

is is a closed $(1,1)$-form and is called the *Chern*-form of the metric h. Any other metric h' on the line bundle is related to h by a relation $h' = e^{\varphi}h$ with φ some C^∞ function on the manifold. It follows that $c(L, h') = c(L, h) + d(\overline{\partial}\varphi)$ and hence the class of $c_1(L)$ in $H^2_{\mathrm{DR}}(M)$ is independent of the chosen metric. It is called the *first Chern class* of the line bundle L and denoted $c_1(L)$. For the proof of the following proposition see [We, Chapt. III, Theorem 4.5].

Proposition A3 7. *Consider the exponential sequence*

$$0 \to \mathbb{Z}_M \to \mathcal{O}_M \xrightarrow{exp} \mathcal{O}^*_M \to 0,$$

where "exp" means the map $f \mapsto \exp(2\pi i f)$. Let $\delta : H^1(\mathcal{O}^*_M) \to H^2(\mathbb{Z}_M)$ be the coboundary map and let $i : H^2(M, \mathbb{Z}) \to H^2_{\mathrm{DR}}(X)$ be the natural map. Then $i \circ \delta(\mathcal{O}(L)) = c_1(L)$.

For a divisor D the class $\delta(\mathcal{O}(D))$ is the fundamental cohomology class as defined in Appendix 2:

Proposition A3 8. *Let M be a compact complex manifold and let D be a smooth hypersurface in M. The fundamental cohomology class of D in $H^2(M, \mathbb{Z})$ coincides with $\delta(\mathcal{O}(D))$, where δ is the coboundary map $H^1(\mathcal{O}^*_M) \to H^2(M, \mathbb{Z})$ from the exponential sequence.*

Proof: (Sketch). With help of the explicit version of the Poincaré-duality isomorphism for De Rham cohomology, the fundamental cohomology class of D can be interpretated as the class $[c(D)]$ for which

$$\int_D i^*\alpha = \int_M c(D) \cup \alpha,$$

where $i : D \hookrightarrow M$ is the embedding. To prove the preceding proposition, it suffices to prove the formula

$$\int_D i^*\alpha = -\frac{i}{2\pi}\int_M \partial\overline{\partial}\log h(s,s) \wedge \alpha,$$

where h is a hermitian metric on the line bundle $\mathcal{O}(D)$ and s is the section of $\mathcal{O}(D)$ defining D. If one takes the example of a point in $\mathbb{P}^1$ this reduces to the residue theorem and the general case is similar. ■

See [G-H, Chapter 1, p. 141] for the details.

A3.5 Kodaira-Vanishing

For details of the following discussion see [We, Chapt. VI, §2].

A line bundle is called *positive* if for some metric h the Chern-form is positive.

Examples A3 9.

1. A line bundle L on a Riemann surface M is positive if and only if its degree is positive. Indeed, the generator of $H^2(M,\mathbb{Z})$ is represented by a positive form, which is a positive multiple of the volume form of any hermitian metric.

2. The bundle $\mathcal{O}(1)$ on $\mathbb{P}^n$ with the metric coming from the Fubini-Study metric. It follows that any ample line bundle is positive.

Theorem A3 10. (Kodaira-Nakano Vanishing) *Let L be a positive line bundle on a compact complex manifold M of dimension n. Then*

$$H^q(M, \Omega^p_M \otimes \mathcal{O}_M(L)) = 0, \quad \textit{if}\ p+q>n.$$

The version for $p = n$ yields:

Corollary A3 11. (Kodaira Vanishing) *Let L be an ample line bundle on a projective manifold M then*

$$H^q(M, \mathcal{O}(K_M + L)) = 0 \quad \textit{if}\ q > 0.$$

A 4. The GAGA Theorems

In this appendix I gather the various GAGA-type theorems from [Se].

Let me start with a projective variety $X \subset \mathbb{P}^n$. It can naturally be regarded as a complex subvariety X_h of $\mathbb{P}^n$. Any morphism $f : X \to Y$ between projective varieties X and Y can be regarded as a holomorphic map $f_h : X_h \to Y_h$ between the associated complex varieties.

If $\mathcal{F}$ is a coherent sheaf on X, one defines a coherent analytic sheaf $\mathcal{F}_h$ in the following manner. Locally (for the Zariski topology) the sheaf $\mathcal{F}$ is a cokernel

$$\mathcal{O}_U^n \xrightarrow{\varphi} \mathcal{O}_U^m \to \mathcal{F}|U \to 0.$$

Now φ is given by a matrix of regular functions on U and these are holomorphic functions on U, which is open in the ordinary topology. So one can define $\mathcal{F}_h$ on U by the cokernel

$$\mathcal{O}_{U_h}^n \xrightarrow{\varphi} \mathcal{O}_{U_h}^m \to \mathcal{F}_h|U \to 0,$$

where $\mathcal{O}_{U_h}$ is the sheaf of germs of holomorphic functions on U.

One can furthermore compare the cohomology groups $H^p(X,\mathcal{F})$ and $H^p(X_h,\mathcal{F}_h)$. The identity map $f : X_h \to X$ is continuous and it induces a natural map $f^{-1}\mathcal{O}_X \to \mathcal{O}_{X_h}$ (you simply regard a regular function on a Zariski-open set as a holomorphic function). Clearly $\mathcal{F}_h \cong f^*\mathcal{F}$ and so there are natural maps of cohomology groups

$$\alpha_p : H^p(X,\mathcal{F}) \to H^p(X_h,\mathcal{F}_h).$$

One now has

Theorem (Serre) *Let X be a projective variety. Then for any coherent analytic sheaf $\mathcal{F}$ on X_h there exists a coherent sheaf $\mathcal{F}^a$ on X such that ${\mathcal{F}^a}_h \cong \mathcal{F}$.*
Furthermore, any homomomorphism $\varphi' : \mathcal{F}' \to \mathcal{G}'$ between coherent analytic sheaves on X_h is induced by a unique homomorphism $\varphi^a : \mathcal{F}^a \to \mathcal{G}^a$ between the corresponding associated sheaves.
The assigment $\varphi \mapsto \varphi^a$ is functorial, i.e. $\mathrm{Id}^a = \mathrm{Id}$ and $(\varphi \circ \psi)^a = \varphi^a \circ \psi^a$.
Finally, the natural maps

$$\alpha_p : H^p(X,\mathcal{F}) \to H^p(X_h,\mathcal{F}_h).$$

are isomorphisms.

The following corollaries should be obvious:

Corollary *If $\mathcal{F}$ and $\mathcal{G}$ are coherent on X and the sheaves $\mathcal{F}_h$ and $\mathcal{G}_h$ are isomorphic on X_h, then $\mathcal{F} \cong \mathcal{G}$.*

Corollary *Let X be projective. The group of holomorphic line bundles up to isomorphism is isomorphic to the group of algebraic line bundles up to isomorphism.*

With little work, one can derive

Corollary (Chow's Theorem) *Any compact subvariety X of $\mathbb{P}^n$ has the structure of a projective variety, i.e. there exists a projective subvariety $X^a \subset \mathbb{P}^n$ such that $X^a_h = X$.*

as well as

Corollary *For any holomorphic map $f : X \to Y$ between projective manifolds, there is a unique morphism $X^a \to Y^a$ inducing f.*