L. S. Bosanquet
A. C. Offord

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by

L. S. Bosanquet and A. C. Offord

London

Suppose \( f(t) \) is integrable \( L \) in \( (-\pi, \pi) \) and periodic outside, and suppose that its Fourier series is

\[
\frac{1}{2}a_0 + \sum_{n=0}^{\infty} (a_n \cos nt + b_n \sin nt) = \sum_{n=0}^{\infty} A_n(t). \quad (1)
\]

Then the allied series is

\[
\sum_{n=1}^{\infty} (b_n \cos nt - a_n \sin nt) = \sum_{n=1}^{\infty} B_n(t). \quad (2)
\]

Let us write

\[
\varphi(t) = \frac{1}{2} \{f(x+t) + f(x-t)\}
\]

\[
\psi(t) = \frac{1}{2} \{f(x+t) - f(x-t)\}
\]

and

\[
s_n = \sum_{m=0}^{n} A_m(x) = \sum_{m=0}^{n} A_m \quad (4)
\]

\[
\bar{s}_n = \sum_{m=1}^{n} B_m(x) = \sum_{m=1}^{n} B_m.
\]

The following theorem was recently given by Hardy \(^1\).

**Theorem A.** If

\[
|\varphi(t)| = o\left(\log \frac{1}{t}\right) \quad (C, 1) \quad (5)
\]

\(^1\) Hardy 5, 108.

\(^2\) We suppose that \( t > 0 \), and say that \( \chi(t) = o\{L(1/t)\} \) \( (C, \alpha) \), \( \alpha > 0 \), as \( t \to 0 \) if

\[
\frac{1}{t^{\alpha}} \int_0^{t^\alpha} (t-u)^{\alpha-1} \chi(u)du = o\{L(1/t)\} \quad (C, \alpha), \quad x > -1, \quad \text{as } n \to \infty \text{ if}
\]

\[
s_n^\alpha = \frac{1}{A_n^\alpha} \sum_{n=0}^{n} A_{n-v}^{\alpha-1}s_v = o\{L(n)\}
\]

as \( n \to \infty \), where \( A_n^\alpha = \frac{I(\alpha+n+1)}{I(\alpha+1)I(n+1)} \); \( s_n^\alpha \) is the Cesàro mean of order \( \alpha \) of \( s_n \).
as $t \to 0$, then a necessary and sufficient condition that
\[ \sum_{v=1}^{n} \frac{s_v}{v} = o(\log n) \] (6)
as $n \to \infty$ is that
\[ \int_{t}^{\pi} \frac{\varphi(u)}{u} \, du = o\left(\log \frac{1}{t}\right) \] (7)
as $t \to 0$.

The problem arises of relaxing conditions (5) and (7). We do this in theorem 1, and at the same time obtain a sharper conclusion than (6).

**Theorem 1.** If
\[ |\varphi(t)| = O\left(\log \frac{1}{t}\right) \] (C, 1) (8)
as $t \to 0$, then a necessary and sufficient condition that
\[ \sum_{v=1}^{n} \frac{s_v}{v} = o \left(\log n\right) \] (C, $-1 + \delta$) (9)
as $n \to \infty$, for any $\delta > 0$, is that
\[ \int_{t}^{\pi} \frac{\varphi(u)}{u} \, du = o \left(\log \frac{1}{t}\right) \] (C, $k$) (10)
as $t \to 0$, for some $k$.

This theorem can be further generalised by replacing the functions $\log \frac{1}{t}$ and $\log n$ by $L\left(\frac{1}{t}\right)$ and $L(n)$ respectively, where $L(x)$ is a logarithmico-exponential function such that $1 < L(x) \leq x$ as $x \to \infty$ 2a). We obtain then

**Theorem 2.** If
\[ |\varphi(t)| = O\left\{L\left(\frac{1}{t}\right)\right\} \] (C, 1) (11)
as $t \to 0$, then a necessary and sufficient condition that
\[ \sum_{v=1}^{n} \frac{s_v}{v} = o\{L(n)\} \] (C, $-1 + \delta$) (12)
as $n \to \infty$, for any $\delta > 0$, is that
\[ \int_{t}^{\pi} \frac{\varphi(u)}{u} \, du = o\left\{L\left(\frac{1}{t}\right)\right\} \] (C, $k$) (13)
as $t \to 0$, for some $k$.

2a) See HARDY 3. We shall suppose throughout the paper that $L(x)$ satisfies these conditions unless the contrary is explicitly stated.
The theorem becomes trivial when \( L(x) = x \), since \( A_n = o(1) \) as \( n \to \infty \). When \( L(x) = 1 \) it remains true if restated as follows.

**Theorem 3.** If

\[
| \varphi(t) | = O(1) \quad (C, 1)
\]

as \( t \to 0 \), then a necessary and sufficient condition that

\[
\sum_{\nu=1}^{n} \frac{s_{\nu}}{\nu}
\]

should be summable \( (C, -1 + \delta) \), for any \( \delta > 0 \), is that

\[
\int_{0}^{\eta} \frac{\varphi(u)}{u} \, du
\]

should exist as a Cesàro integral of some order.

We shall only give the proof of theorem 2. Theorem 1 is included in theorem 2, and the proof of theorem 3 can readily be constructed from that of theorem 2. We employ the following lemmas.

**Lemma 1** 3). If \( x^{\beta - \delta} \leq L(x) \leq x^{\beta + \delta} \) as \( x \to \infty \), for every \( \delta > 0 \), and if \( \alpha + \beta > 1 \), then, as \( t \to 0 \),

\[
\int_{t}^{\eta} u^{-\alpha} L\left( \frac{1}{u} \right) du \sim \frac{t^{1-\alpha}}{\alpha + \beta - 1} L\left( \frac{1}{t} \right).
\]

**Lemma 2.** If \( (11) \) holds, then \( s_n = O(L(n)) \) \( (C, \delta) \), for every \( \delta > 0 \).

We may suppose without loss of generality that \( 0 < \delta < 1 \). We have to show that

\[
I(n) = \int_{0}^{\eta} \varphi(t) \chi_{n}^{\delta}(t) \, dt = O(L(n)),
\]

as \( n \to \infty \), where \( \chi_{n}^{\delta}(t) \) is the \( n \)-th Fejér kernel of order \( \delta \), and \( 0 < \eta \leq \pi \). M. Riesz 4) has shown that

\[
| \varphi(t) | \leq \begin{cases} \leq An & \\
\leq An^{-\delta} t^{-1-\delta} & \end{cases}
\]

for \( n > 0 \), \( 0 < t < \pi \), \( 0 < \delta < 1 \). Write

\[
I(n) = \int_{0}^{1/n} + \int_{1/n}^{\eta} = I_1 + I_2.
\]

3) **Hardy** 3, 37.

4) **Riesz** 10.
Then
\[ |I_1| \leq An \int_0^{1/n} |\varphi(u)| \, du = O(L(n)) \]
by hypothesis, and, if \( \Phi(t) = \int_0^t |\varphi(u)| \, du \),
\[ |I_2| \leq An^{-\delta} \int_0^{\eta} |\varphi(u)| \, u^{-1-\delta} \, du \]
\[ \leq An^{-\delta} \Phi\left(\frac{1}{n}\right) n^{1+\delta} + An^{-\delta} \int_0^{\eta} \Phi(u) u^{-2-\delta} \, du \]
\[ = O(L(n)) + n^{-\delta} \int_0^{\eta} O\left(L\left(\frac{1}{u}\right)\right) u^{-1-\delta} \, du \]
\[ = O(L(n)), \]
by lemma 1 5).

**Lemma 3.** Necessary and sufficient conditions that (12) should hold, for a given \( \delta = \delta_0 > 0 \), are that it should hold for some \( \delta > 0 \) and that \( s_n = o\{L(n)\} \) \( (C, \delta_0) \).

Let \( d_n = \sum_{v=1}^{n} s_v \), and let \( d_n^\alpha \) be the \( n \)-th Cesàro mean of order \( \alpha \) or \( d_n \). Then it is easily verified 6) that, for \( \alpha > 0, n > 0 \),
\[ \alpha(d_n^\alpha - d_n^\alpha) = s_n^\alpha - s_0. \quad (18) \]

Also if \( d_n^\alpha = o\{L(n)\} \) then \( d_n^\beta = o\{L(n)\} \) for \( \beta > \alpha > -1 \). From (18) it then follows by induction that necessary and sufficient conditions that \( d_n^{\delta-1} = o\{L(n)\} \) for \( \delta = \delta_0 \) are that this should hold for some \( \delta \) and that \( s_n^{\delta_0} = o\{L(n)\} \).

**Lemma 4.** If \( s_n = O\{L(n)\} \) \((C, \delta)\), for a given \( \delta > 0 \), and \( s_n = o\{L(n)\} \) \((C, k)\), for some \( k \), then \( s_n = o\{L(n)\} \) \((C, \delta')\), for every \( \delta' > \delta \).

This is a particular case of a theorem of Dixon and Ferrar 7).

**Lemma 5.** A necessary and sufficient condition that
\[ s_n = o\{L(n)\} \quad (C) \quad (19) \]

5) Here \( A \) denotes some constant, not necessarily the same at each occurrence.
7) Dixon and Ferrar 2, theorem II. See also Riesz 11.
as \( n \to \infty \) is that
\[
\varphi(t) = o\left\{ L \left( \frac{1}{t} \right) \right\} \quad (C)
\]

as \( t \to 0 \).

The corresponding result with \( L(x) = 1 \) is due to Hardy and Littlewood \(^8\). The proof of lemma 5 is on the same lines. The properties of \( L(x) \) required have been given by Hardy \(^9\).

Lemma 6 \(^{10}\). A necessary and sufficient condition that
\[
\sum_{v=1}^{n} \frac{s_v}{v} = o\{L(n)\} \quad (C)
\]
as \( n \to \infty \) is that
\[
\int_{t}^{\pi} \frac{\varphi(u)}{u} \, du = o\left\{ L \left( \frac{1}{t} \right) \right\} \quad (C)
\]
as \( t \to 0 \).

Let
\[
\chi(t) = \int_{t}^{\pi} \frac{\varphi(u)}{u} \, du, \quad \chi^{*}(t) = \int_{t}^{\pi} \varphi(u) \frac{1}{2} \cot \frac{\frac{1}{2}u}{u} \, du.
\]

Then, since \( \frac{1}{2} \cot \frac{\frac{1}{2}u}{u} - \frac{1}{u} \) is bounded in \((0, \pi)\), it is easy to see that \( \chi(t) - \chi^{*}(t) \) tends to a limit as \( t \to 0 \). Also \(^{11}\)
\[
\chi^{*}(t) = \frac{1}{2} c_0 + \sum_{n=1}^{\infty} c_n \cos nt,
\]
where, for \( n > 0 \),
\[
c_n = \frac{2}{\pi} \int_{0}^{\pi} \chi^{*}(t) \cos nt \, dt = \frac{s_n}{n} - \frac{1}{2} \frac{A_n}{n}.
\]

Hence
\[
\sum_{v=1}^{n} \frac{s_v}{v} - \sum_{n=1}^{\infty} c_v = \frac{1}{2} \sum_{v=1}^{n} A_v,
\]
and the lemma will follow by applying lemma 5 to \( \chi^{*}(t) \), if we show that (21) and (22) each imply
\[
\sum_{v=1}^{n} \frac{A_v}{v} = o\{L(n)\} \quad (C)
\]
as \( n \to \infty \).

\(^8\) Hardy and Littlewood 6, 70. See also Bosanquet 1.

\(^9\) Hardy 3, 37.

\(^{10}\) The case corresponding to \( L(x) = 1 \), in the modified form of theorem 3, was conjectured by Hardy and Littlewood 7, 242.

\(^{11}\) Hardy 4.
Now, by (18), (21) implies (19), and the first result follows easily by partial summation. Again, (22) implies (20), for, writing $\Phi(t) = \int_0^t \varphi(u) \, du$, we have

$$\chi(t) = \int_0^t \frac{\varphi(u)}{u} \, du = C - \frac{\Phi(t)}{t} + \int_t^\pi \frac{\Phi(u)}{u^2} \, du = o\left(\frac{1}{t}\right)$$

as $t \to 0$. Hence

$$\Phi(t) = \int_0^t u \frac{\varphi(u)}{u} \, du = \left[-u \chi(u)\right]_0^t + \int_0^t \chi(u) \, du = -t \chi(t) + \int_0^t \chi(u) \, du.$$

Hence (22) implies

$$\frac{\Phi(t)}{t} = o\left\{L\left(\frac{1}{t}\right)\right\} \quad (C),$$

which is equivalent to (20). Lemma 5 now gives the second result.

Theorem 2 is an immediate consequence of lemmas 2, 3, 4 and 6.

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The following analogue of theorem 2 is also true.

**Theorem 4.** If

$$| \varphi(t) | = O\left\{L\left(\frac{1}{t}\right)\right\} \quad (C, 1) \quad (23)$$

as $t \to 0$, then a necessary and sufficient condition that

$$\frac{1}{n} \sum_{v=1}^n \frac{\tilde{s}_v}{v} = o\{L(n)\} \quad (C, -1 + \delta) \quad (24)$$

as $n \to \infty$, for any $\delta > 0$, is that

$$\int_t^\pi \cot \frac{1}{2} u \, du \int_{u}^\pi \cot \frac{1}{2} v \, \varphi(v) \, dv = o\left\{L\left(\frac{1}{t}\right)\right\} \quad (C)$$

as $t \to 0$.

We require the following additional lemmas, the proofs of which are analogous to those already given.

**Lemma 7.** If (23) holds, then $nB_n = O\{L(n)\} \quad (C, 1 + \delta)$, for every $\delta > 0$.

**Lemma 8.** Necessary and sufficient conditions that

$$\tilde{s}_n = O\{L(n)\} \quad (C, \delta),$$

for a given $\delta$, are that this be true for some $\delta$ and

$$nB_n = O\{L(n)\} \quad (C, 1 + \delta).$$
Both these lemmas depend on the identity
\[ \tau_n^\alpha = \alpha (\delta_{n-1}^\alpha - \delta_n^\alpha), \]  
(26)
where \( \tau_n^\alpha \) is the \( n \)-th Cesàro mean of order \( \alpha \) of \( nB_n \).

The Fejér kernel for the Allied series is \( \tilde{z}_n^\delta(t) \), where
\[ |\tilde{z}_n^\delta(t)| \leq An \quad \text{and} \quad |\tilde{z}_n^\delta - \frac{1}{2} \cot \frac{1}{2}t| \leq A n^{-\delta} t^{1-\delta}, \]  
for \( n > 0 \), \( 0 < t < \pi \).

**Lemma 9.** A necessary and sufficient condition that
\[ s_n = o\{L(n)\} \]  
(27)
as \( n \to \infty \) is that
\[ \int_0^{\pi} \cot \frac{1}{2}u \psi(u) \, du = o\left( L\left( \frac{1}{t} \right) \right) \]  
(28)
as \( t \to 0 \).

The lemma remains true when \( L(x)=1 \), this case being due to Hardy and Littlewood \(^{12}\).

**Lemma 10.** A necessary and sufficient condition that
\[ \sum_{\nu=1}^{n} \frac{s_{\nu}}{\nu} = o\{L(n)\} \]  
(29)
as \( n \to \infty \) is that
\[ \int_0^{\pi} \cot \frac{1}{2}u \, du \int_u^{\pi} \cot \frac{1}{2}v \psi(v) \, dv = o\left( L\left( \frac{1}{t} \right) \right) \]  
(30)
as \( t \to 0 \).

Theorem 4 follows from lemmas 7, 8 and 10, and lemmas 3 and 4 with \( \tilde{s}_n \) in place of \( s_n \).

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\(^{12}\) Hardy and Littlewood, 7. See also Paley, 9.


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