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Note on Fourier Series

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Suppose $f(t)$ is integrable L in $(-\pi, \pi)$ and periodic outside, and suppose that its Fourier series is

$$\tfrac{1}{2}a_0 + \sum_{n=0}^{\infty} (a_n \cos nt + b_n \sin nt) = \sum_{n=0}^{\infty} A_n(t) . \tag{1}$$

Then the allied series is

$$\sum_{n=1}^{\infty} (b_n \cos nt - a_n \sin nt) = \sum_{n=1}^{\infty} B_n(t). \tag{2}$$

Let us write

$$\left.\begin{aligned} \varphi(t) &= \tfrac{1}{2}\{f(x+t)+f(x-t)\} \\ \psi(t) &= \tfrac{1}{2}\{f(x+t)-f(x-t)\} \end{aligned}\right. \tag{3}$$

and

$$\left.\begin{aligned} s_n &= \sum_{m=0}^{n} A_m(x) = \sum_{m=0}^{n} A_m \\ \bar{s}_n &= \sum_{m=1}^{n} B_m(x) = \sum_{m=1}^{n} B_m. \end{aligned}\right. \tag{4}$$

The following theorem was recently given by Hardy [1].

Theorem A. If

$$|\varphi(t)| = o\left(\log \frac{1}{t}\right) \quad (C, 1) \text{ [2]} \tag{5}$$

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[1] Hardy 5, 108.

[2] We suppose that $t > 0$, and say that $\chi(t) = o\{L(1/t)\}\,(C, \alpha)$, $\alpha > 0$, as $t \to 0$ if $\frac{1}{t^\alpha}\int_0^t (t-u)^{\alpha-1}\chi(u)\,du = o\{L(1/t)\}$ as $t \to 0$. We also say that $s_n = o\{L(n)\}$ (C, α), $\alpha > -1$, as $n \to \infty$ if

$$s_n^\alpha = \frac{1}{A_n^\alpha}\sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1}s_\nu = o\{L(n)\}$$

as $n \to \infty$, where $A_n^\alpha = \dfrac{\Gamma(\alpha+n+1)}{\Gamma(\alpha+1)\Gamma(n+1)}$; s_n^α is the Cesàro mean of order α of s_n.

as $t \to 0$, then a necessary and sufficient condition that

$$\sum_{\nu=1}^{n} \frac{s_\nu}{\nu} = o(\log n) \tag{6}$$

as $n \to \infty$ is that

$$\int_t^\pi \frac{\varphi(u)}{u} du = o\left(\log \frac{1}{t}\right) \tag{7}$$

as $t \to 0$.

The problem arises of relaxing conditions (5) and (7). We do this in theorem 1, and at the same time obtain a sharper conclusion than (6).

Theorem 1. If

$$|\varphi(t)| = O\left(\log \frac{1}{t}\right) \qquad (C,\ 1) \tag{8}$$

as $t \to 0$, then a necessary and sufficient condition that

$$\sum_{\nu=1}^{n} \frac{s_\nu}{\nu} = o\ (\log n) \qquad (C,\ -1+\delta) \tag{9}$$

as $n \to \infty$, for any $\delta > 0$, is that

$$\int_t^\pi \frac{\varphi(u)}{u} du = o\ \left(\log \frac{1}{t}\right) \qquad (C,\ k) \tag{10}$$

as $t \to 0$, for some k.

This theorem can be further generalised by replacing the functions $\log \frac{1}{t}$ and $\log n$ by $L\left(\frac{1}{t}\right)$ and $L(n)$ respectively, where $L(x)$ is a logarithmico-exponential function such that $1 \prec L(x) \preceq x$ as $x \to \infty$ [2a]. We obtain then

Theorem 2. If

$$|\varphi(t)| = O\left\{L\left(\frac{1}{t}\right)\right\} \qquad (C,\ 1) \tag{11}$$

as $t \to 0$, then a necessary and sufficient condition that

$$\sum_{\nu=1}^{n} \frac{s_\nu}{\nu} = o\{L(n)\} \qquad (C,\ -1+\delta) \tag{12}$$

as $n \to \infty$, for any $\delta > 0$, is that

$$\int_t^\pi \frac{\varphi(u)}{u} du = o\left\{L\left(\frac{1}{t}\right)\right\} \qquad (C,\ k) \tag{13}$$

as $t \to 0$, for some k.

[2a]) See Hardy 3. We shall suppose throughout the paper that $L(x)$ satisfies these conditions unless the contrary is explicitly stated.

The theorem becomes trivial when $L(x) = x$, since $A_n = o(1)$ as $n \to \infty$. When $L(x)=1$ it remains true if restated as follows.

Theorem 3. If

$$|\varphi(t)| = O(1) \qquad (C,\ 1) \tag{14}$$

as $t \to 0$, *then a necessary and sufficient condition that*

$$\sum_{\nu=1}^{n} \frac{s_\nu}{\nu} \tag{15}$$

should be summable $(C, -1+\delta)$, *for any* $\delta > 0$, *is that*

$$\int_0^\pi \frac{\varphi(u)}{u} du \tag{16}$$

should exist as a Cesàro integral of some order.

We shall only give the proof of theorem 2. Theorem 1 is included in theorem 2, and the proof of theorem 3 can readily be constructed from that of theorem 2. We employ the following lemmas.

Lemma 1[3]. *If* $x^{\beta-\delta} \lesssim L(x) \lesssim x^{\beta+\delta}$ *as* $x \to \infty$, *for every* $\delta > 0$, *and if* $\alpha + \beta > 1$, *then, as* $t \to 0$,

$$\int_t^\eta u^{-\alpha} L\left(\frac{1}{u}\right) du \sim \frac{t^{1-\alpha}}{\alpha+\beta-1} L\left(\frac{1}{t}\right). \tag{17}$$

Lemma 2. If (11) *holds, then* $s_n = O\{L(n)\}$ (C, δ), *for every* $\delta > 0$.

We may suppose without loss of generality that $0 < \delta < 1$. We have to show that

$$I(n) = \int_0^\eta \varphi(t) \varkappa_n^\delta(t) dt = O\{L(n)\},$$

as $n \to \infty$, where $\varkappa_n^\delta(t)$ is the n-th Fejér kernel of order δ, and $0 < \eta \leqq \pi$. M. Riesz[4]) has shown that

$$|\varkappa_n^\delta(t)| \begin{cases} \leqq An \\ \leqq An^{-\delta} t^{-1-\delta} \end{cases}$$

for $n > 0$, $0 < t < \pi$, $0 < \delta < 1$. Write

$$I(n) = \int_0^{1/n} + \int_{1/n}^\eta = I_1 + I_2.$$

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[3]) Hardy 3, 37.

[4]) Riesz 10.

Then

$$|I_1| \leqq An\int_0^{1/n} |\varphi(u)|du = O\{L(n)\}$$

by hypothesis, and, if $\Phi(t) = \int_0^t |\varphi(u)|du$,

$$\begin{aligned} |I_2| &\leqq An^{-\delta}\int_{1/n}^{\eta} |\varphi(u)|\, u^{-1-\delta}\, du \\ &\leqq An^{-\delta}\left|\Phi\left(\frac{1}{n}\right)\right| n^{1+\delta} + An^{-\delta}\int_{1/n}^{\eta} \Phi(u)u^{-2-\delta}\, du \\ &= O\{L(n)\} + n^{-\delta}\int_{1/n}^{\eta} O\left\{L\left(\frac{1}{u}\right)\right\}u^{-1-\delta}\, du \\ &= O\{L(n)\}, \end{aligned}$$

by lemma 1 [5]).

Lemma 3. Necessary and sufficient conditions that (12) *should hold, for a given* $\delta = \delta_0 > 0$, *are that it should hold for some* $\delta > 0$ *and that* $s_n = o\{L(n)\}$ (C, δ_0).

Let $d_n = \sum_{\nu=1}^{n} \frac{s_\nu}{\nu}$, and let d_n^α be the n-th Cesàro mean of order α or d_n. Then it is easily verified [6]) that, for $\alpha > 0$, $n > 0$,

$$\alpha(d_n^{\alpha-1} - d_n^\alpha) = s_n^\alpha - s_0. \tag{18}$$

Also if $d_n^\alpha = o\{L(n)\}$ then $d_n^\beta = o\{L(n)\}$ for $\beta > \alpha > -1$. From (18) it then follows by induction that necessary and sufficient conditions that $d_n^{\delta-1} = o\{L(n)\}$ for $\delta = \delta_0$ are that this should hold for some δ and that $s_n^{\delta_0} = o\{L(n)\}$.

Lemma 4. If $s_n = O\{L(n)\}(C, \delta)$, *for a given* $\delta > 0$, *and* $s_n = o\{L(n)\}$ (C, k), *for some* k, *then* $s_n = o\{L(n)\}$ (C, δ'), *for every* $\delta' > \delta$.

This is a particular case of a theorem of Dixon and Ferrar [7]).

Lemma 5. A necessary and sufficient condition that

$$s_n = o\{L(n)\} \quad (C) \tag{19}$$

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[5]) Here A denotes some constant, not necessarily the same at each occurrence.

[6]) Cf. Kogbetliantz 8, 30.

[7]) Dixon and Ferrar 2, theorem II. See also Riesz 11.

as $n \to \infty$ is that

$$\varphi(t) = o\left\{L\left(\frac{1}{t}\right)\right\} \quad (C) \tag{20}$$

as $t \to 0$.

The corresponding result with $L(x) = 1$ is due to Hardy and Littlewood [8]). The proof of lemma 5 is on the same lines. The properties of $L(x)$ required have been given by Hardy [9]).

Lemma 6 [10]). *A necessary and sufficient condition that*

$$\sum_{\nu=1}^{n} \frac{s_\nu}{\nu} = o\{L(n)\} \qquad (C) \tag{21}$$

as $n \to \infty$ is that

$$\int_t^\pi \frac{\varphi(u)}{u} du = o\left\{L\left(\frac{1}{t}\right)\right\} \qquad (C) \tag{22}$$

as $t \to 0$.

Let

$$\chi(t) = \int_t^\pi \frac{\varphi(u)}{u} du, \qquad \chi^*(t) = \int_t^\pi \varphi(u)\, \tfrac{1}{2}\cot \tfrac{1}{2}u\, du\,.$$

Then, since $\frac{1}{2}\cot\frac{1}{2}u - \frac{1}{u}$ is bounded in $(0, \pi)$, it is easy to see that $\chi(t) - \chi^*(t)$ tends to a limit as $t \to 0$. Also [11])

$$\chi^*(t) = \tfrac{1}{2}c_0 + \sum_{n=1}^{\infty} c_n \cos nt\,,$$

where, for $n > 0$,

$$c_n = \frac{2}{\pi}\int_0^\pi \chi^*(t) \cos nt\, dt = \frac{s_n}{n} - \frac{1}{2}\frac{A_n}{n}\,.$$

Hence

$$\sum_{\nu=1}^{n} \frac{s_\nu}{\nu} - \sum_{\nu=1}^{n} c_\nu = \tfrac{1}{2}\sum_{\nu=1}^{n} \frac{A_\nu}{\nu}\,,$$

and the lemma will follow by applying lemma 5 to $\chi^*(t)$, if we show that (21) and (22) each imply

$$\sum_{\nu=1}^{n} \frac{A_\nu}{\nu} = o\{L(n)\} \quad (C)$$

as $n \to \infty$.

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8) Hardy and Littlewood 6, 70. See also Bosanquet 1.

9) Hardy 3, 37.

10) The case corresponding to $L(x) = 1$, in the modified form of theorem 3, was conjectured by Hardy and Littlewood 7, 242.

11) Hardy 4.

Now, by (18), (21) implies (19), and the first result follows easily by partial summation. Again, (22) implies (20), for, writing $\Phi(t) = \int_0^t \varphi(u)\,du$, we have

$$\chi(t) = \int_t^\pi \frac{\varphi(u)}{u} du = C - \frac{\Phi(t)}{t} + \int_t^\pi \frac{\Phi(u)}{u^2} du = o\left(\frac{1}{t}\right)$$

as $t \to 0$. Hence

$$\Phi(t) = \int_0^t u\frac{\varphi(u)}{u} du = [-u\chi(u)]_0^t + \int_0^t \chi(u)\,du = -t\chi(t) + \int_0^t \chi(u)\,du\,.$$

Hence (22) implies

$$\frac{\Phi(t)}{t} = o\left\{L\left(\frac{1}{t}\right)\right\} \qquad (C),$$

which is equivalent to (20). Lemma 5 now gives the second result.

Theorem 2 is an immediate consequence of lemmas 2, 3, 4 and 6.

ALLIED SERIES.

The following analogue of theorem 2 is also true.

Theorem 4. If

$$|\,\psi(t)\,| = O\left\{L\left(\frac{1}{t}\right)\right\} \qquad (C,\ 1) \tag{23}$$

as $t \to 0$, then a necessary and sufficient condition that

$$\sum_{\nu=1}^{n} \frac{\bar{s}_\nu}{\nu} = o\{L(n)\} \qquad (C,\ -1+\delta) \tag{24}$$

as $n \to \infty$, for any $\delta > 0$, is that

$$\int_t^\pi \cot \tfrac{1}{2}u\, du \int_u^\pi \cot \tfrac{1}{2}v\, \psi(v)\, dv = o\left\{L\left(\frac{1}{t}\right)\right\} \qquad (C)$$

as $t \to 0$.

We require the following additional lemmas, the proofs of which are analogous to those already given.

Lemma 7. If (23) *holds, then* $nB_n = O\{L(n)\}$ $(C,\ 1+\delta)$, *for every* $\delta > 0$.

Lemma 8. Necessary and sufficient conditions that

$$\bar{s}_n = O\{L(n)\} \qquad (C,\ \delta),$$

for a given δ, are that this be true for some δ and

$$nB_n = O\{L(n)\} \qquad (C,\ 1+\delta).$$

Both these lemmas depend on the identity

$$\tau_n^{\alpha} = \alpha(\bar{s}_n^{\alpha-1} - \bar{s}_n^{\alpha}), \tag{26}$$

where τ_n^{α} is the n-th Cesàro mean of order α of nB_n.

The Fejér kernel for the Allied series is $\bar{\varkappa}_n^{\delta}(t)$, where $|\bar{\varkappa}_n^{\delta}(t)| \leqq An$ and $|\bar{\varkappa}_n^{\delta} - \frac{1}{2}\cot\frac{1}{2}t| \leqq A\, n^{-\delta}\, t^{-1-\delta}$, for $n > 0$, $0 < t < \pi$.

Lemma 9. A necessary and sufficient condition that

$$\bar{s}_n = o\{L(n)\} \qquad (C) \tag{27}$$

as $n \to \infty$ *is that*

$$\int_t^{\pi} \cot\tfrac{1}{2}u\, \psi(u)\, du = o\left\{L\left(\frac{1}{t}\right)\right\} \qquad (C) \tag{28}$$

as $t \to 0$.

The lemma remains true when $L(x)=1$, this case being due to Hardy and Littlewood [12]).

Lemma 10. A necessary and sufficient condition that

$$\sum_{\nu=1}^{n} \frac{\bar{s}_\nu}{\nu} = o\{L(n)\} \qquad (C) \tag{29}$$

as $n \to \infty$ *is that*

$$\int_t^{\pi} \cot\tfrac{1}{2}u\, du \int_u^{\pi} \cot\tfrac{1}{2}v\, \psi(v)\, dv = o\left\{L\left(\frac{1}{t}\right)\right\} \qquad (C) \tag{30}$$

as $t \to 0$.

Theorem 4 follows from lemmas 7, 8 and 10, and lemmas 3 and 4 with $\bar{s}_n$ in place of s_n.

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[12]) Hardy and Littlewood, 7. See also Paley, 9.

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