Compositio Mathematica

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Compositio Mathematica, tome 1 (1935), p. 248-253
<http://www.numdam.org/item?id=CM_1935__1__248_0>

On Some Arithmetical Results in the Geometry of Numbers

by

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Let a function $f(x_1, x_2, \ldots, x_n)$, or say f for brevity, of the n variables $x_1, x_2, \ldots, x_n$, be defined for all real $x_1, x_2, \ldots, x_n$, and have the following properties:

(A). For all real $t > 0$,

$$f(tx_1, tx_2, \ldots, tx_n) = t^\delta f(x_1, x_2, \ldots, x_n), \tag{1}$$

where $\delta \geqq 0$ is a constant independent of the x's and t, and the positive arithmetical value of t^δ is taken.

(B).

$$f(x_1 - y_1, \ldots, x_n - y_n) \leqq k\{f(x_1, x_2, \ldots, x_n) + f(y_1, y_2, \ldots, y_n)\}, \tag{2}$$

where $k > 0$ is a constant independent of the x's and y's.

(C). The number, N, of lattice points, that is, sets of integers $x_1, x_2, \ldots, x_n$, such that

$$f(x_1, x_2, \ldots, x_n) \leqq G, \tag{3}$$

where $G > 0$ is sufficiently large, satisfies the inequality

$$N > JG^{n/\delta}, \tag{4}$$

where $J > 0$ is independent of G.

Then integer values of the x's not all zero exist such that

$$f(x_1, x_2, \ldots, x_n) \leqq 2kJ^{-\delta/n}. \tag{5}$$

It may be supposed in (C) that N is finite for bounded G.

When the hypersolid S defined by

$$f(x_1, x_2, \ldots, x_n) \leqq 1, \tag{6}$$

has a volume $V > 0$, (and Minkowski has considered the question of the existence of V subject to conditions of the type (A) and (B)), it is clear by taking hypercubes, centres at the lattice points and sides of length unity, that as $G \to \infty$,

(7) $$N/VG^{n/\delta} \to 1.$$

Hence if G is large, (5) holds for $J < V$, and then also for $J = V$. For on making $J \to V$, at least one and at most a finite number of sets of values of the x's exist satisfying (5). Hence on taking the limit of both sides of (5), it follows that for at least one of these sets,

(8) $$f(x_1, x_2, \ldots, x_n) \leqq 2kV^{-\delta/n}.$$

The last result is of a well known type, which in the case $\delta = k = 1$, with slightly different conditions was introduced by Minkowski [1]) into the theory of numbers in which it is known to be of great importance.

My proof is completely arithmetical and even simpler than Minkowski's geometric proof. It has its origin in my [2]) recent arithmetical demonstration of Minkowski's theorem for linear homogeneous forms.

It is an immediate consequence of the obvious fact that if M is any positive integer, there exists only M^n sets of incongruent residues for a set of n integers $x_1, x_2, \ldots, x_n$. For the hypersolid,

$$f(x_1, x_2, \ldots, x_n) \leqq gM^\delta$$

will for sufficiently large g and M contain at least M^n+1 lattice points. From (4), it suffices to take g such that

$$J(gM^\delta)^{n/\delta} > M^n,$$

(9) or $$g > J^{-\delta/n}.$$

For two of these, say the sets

$$(y_1, y_2, \ldots, y_n), \qquad (z_1, z_2, \ldots, z_n),$$

(10) $$f(y_1, y_2, \ldots, y_n) \leqq gM^\delta, \qquad f(z_1, z_2, \ldots, z_n) \leqq gM^\delta,$$

and

(11) $$y_r - z_r = Mx_r, \qquad (r = 1, 2, \ldots, n)$$

where $x_1, x_2, \ldots, x_n$ are integers which are not all zero.

From (B)

$$f(y_1 - z_1, \ldots, y_n - z_n) \leqq k\{f(y_1, \ldots, y_n) + f(z_1, \ldots, z_n)\},$$

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[1]) Geometrie der Zahlen (1910), 76.

[2]) Journal London Math. Society 8 (1933), 179—182.

and so from (11), (10), (1), (9)

$$f(x_1, x_2, \ldots, x_n) \leqq 2kg.$$

This gives (5) with $J^{-\delta/n}$ replaced by g. By making $g \to J^{-\delta/n}$, it is clear by the argument following (7), that (5) follows immediately.

The proof shows that if S, instead of being given by $f(x_1, x_2, \ldots, x_n) \leqq 1$, is defined by any number of inequalities,

$$f_r(x_1, x_2, \ldots, x_n) \leqq \varepsilon_r \qquad (r = 1, 2, \ldots) \tag{12}$$

where ε_r is 0 or ± 1, and each f satisfies (A), (B), and (C) with (3), (6) replaced by

$$f_r(x_1, x_2, \ldots, x_n) \leqq G\varepsilon_r,$$

$$f_r(x_1, x_2, \ldots, x_n) \leqq \varepsilon_r, \qquad (r = 1, 2, \ldots, n)$$

respectively, then (5), (8) still hold.

It is also clear that if $\varphi(\xi, \eta)$ is a function of the real variables ξ, η satisfying the conditions

$$\varphi(t\xi, t\eta) = t\varphi(\xi, \eta)$$

for $t > 0$, and

$$\varphi(\xi, \eta) \leqq \varphi(\xi', \eta')$$

for

$$\xi \leqq \xi', \quad \eta \leqq \eta',$$

the condition (B) can be replaced by

$$f(x_1 - y_1, \ldots, x_n - y_n) \leqq \varphi\{f(x, \ldots, x), f(y, \ldots, y)\},$$

and then $2k$ in (5), (8) must be replaced by $\varphi(1, 1)$.

The conditions (A), (B) are really different from those of Minkowski. He assumes first that the solid S has a centre, i. e.,

$$f(-x_1, -x_2, \ldots, -x_n) = f(x_1, x_2, \ldots, x_n).$$

By including in (12) inequalities such as $\pm x_1^\delta \leqq 0$, (e. g., when $n = 3$, and δ is an odd integer > 0, the inequalities $x_1^\delta \leqq 0$, $x_2^\delta \leqq 0$, $x_3^\delta \leqq 0$, mean that S will lie in one octant), this is seen not to be necessary, but no essentially new results arise.

He assumes next that the hypersolid S is convex, and then proves that if a convex n dimensional solid has centre at the origin O, and has a volume $\geqq 2^n$, then at least one lattice point in addition to O must lie within S, i.e., be an interior or boundary point of S. The convexity condition really means that if P, Q, are two points within S, then $P + Q$ lies within $2S$, i.e. the point whose coordinates are the sum of those of P and Q lies within the solid derived from S by increasing the coordinates of all its points in the ratio $2:1$.

Suppose, however, S is a semi-convex solid, that is, a constant k exists such that $P+Q$ lies within kS, so that $k \geqq 2$ is a sort of measure of the lack of convexity of S. Then my theorem includes the result, that if S has a centre at O and a volume $V \geqq k^n$, it contains within it at least one lattice point in addition to the origin. I give the proof[3]) ab initio.

If M is any positive integer, the hypersolid $\frac{MS}{k}$ has a volume $\frac{M^n V^n}{k^n}$, and so as $M \to \infty$, N the number of lattice points within $\frac{MS}{k}$, satisfies

$$N \sim \frac{M^n V}{k^n},$$
$$> M^n$$

if $V > k^n$. Hence two of these lattice points, say P, Q will have coordinates which are unequal and congruent mod M. Hence, since $P-Q$ is $P+Q'$ where Q' is the image of Q in O, $\frac{P-Q}{M}$ is a lattice point lying within $k\left(\frac{M}{k}\right)\frac{S}{M}$ or S. This proves the result for $V > k^n$. It follows for $V = k^n$ by the argument leading to (5).

My form also simplifies some of the applications.

Thus take

$$f_r(x_1, x_2, \ldots, x_n) = \left|\sum_{s=1}^{n} a_{rs} x_s\right|, \qquad (r = 1, 2, \ldots, n)$$

then (A), (B) are satisfied with $\delta = k = 1$, since

$$|X+Y| \leqq |X| + |Y|;$$

also

$$V = \iint \ldots \int dx_1 dx_2 \ldots dx_n$$

taken over $|f_r| \leqq 1$. $\qquad (r = 1, 2, \ldots, n)$

Thus if all the a's are real, on putting

$$\xi_r = \sum_{s=1}^{n} a_{rs} x_s,$$

$$\Delta = \begin{vmatrix} a_{11} \ldots a_{1n} \\ \ldots\ldots \\ a_{n1} \ldots a_{nn} \end{vmatrix},$$

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[3]) The theorem and method of proof still hold if S has no centre and $P-Q$ lies within kS.

then

$$V=\frac{1}{|\Delta|}\int_{-1}^{1}\int_{-1}^{1}\dots\int_{-1}^{1}d\xi_1 d\xi_2\dots d\xi_n=\frac{2^n}{|\Delta|}.$$

Hence the well known result,

$$\left|\sum_{s=1}^{n}a_{rs}x_s\right|\leqq|\Delta|^{1/n},\qquad (r=1,2,\dots,n)$$

where $x_1, x_2, \dots, x_n$ are integers not all zero.

Again suppose that p is any number $\geqq 1$, and that $\alpha, \beta, \gamma, \dots$ are any given integers $\geqq 0$ whose sum is n.

Take

$$f_1=\sum_{r=1}^{\alpha}\left|\sum_{s=1}^{n}a_{rs}x_s\right|^p=\xi_1^p+\xi_2^p+\dots+\xi_\alpha^p,$$

say,

$$f_2=\sum_{r=\alpha+1}^{\alpha+\beta}\left|\sum_{s=1}^{n}a_{rs}x_s\right|^p=\eta_1^p+\eta_2^p+\dots+\eta_\beta^p,$$

etc., where ξ_1^p etc. denote the positive values.

Then (A), (B) are satisfied with $\delta=p$, $k=2^{p-1}$, as is clear since

$$|X+Y|^p\leqq(|X|+|Y|)^p\leqq 2^{p-1}(|X|^p+|Y|^p).$$

Also
$$V=\iint\dots\int dx_1 dx_2\dots dx_n$$

is easily evaluated for the general case of complex a_{rs} when in each f, complex linear forms occur in conjugate pairs.

When all the a's are real,

$$V=\frac{2^n}{|\Delta|}\iint\dots\int d\xi_1\dots d\xi_\alpha d\eta_1\dots d\eta_\beta\dots,$$

where
$$\xi_1^p+\xi_2^p+\dots+\xi_\alpha^p\leqq 1,\quad \xi_1\geqq 0,\quad \xi_2\geqq 0,\ \dots$$
$$\eta_1^p+\eta_2^p+\dots+\eta_\beta^p\leqq 1,\quad \eta_1\geqq 0,\quad \eta_2\geqq 0,\ \dots$$

etc.

These are Dirichlet's integrals, whence

$$V=\frac{2^n}{|\Delta|}\,\frac{\Gamma\left(1+\frac{1}{p}\right)^n}{\Gamma\left(1+\frac{\alpha}{p}\right)\Gamma\left(1+\frac{\beta}{p}\right)\dots},$$

$$2kV^{-\delta/n}=|\Delta|^{p/n}\frac{\left[\Gamma\left(1+\frac{\alpha}{p}\right)\Gamma\left(1+\frac{\beta}{p}\right)\dots\right]^{p/n}}{\left[\Gamma\left(1+\frac{1}{p}\right)\right]^p},$$

$$=\lambda,\ \text{say}.$$

Hence integer values of the x's not all zero exist such that

$$\begin{aligned}
&|f_1|^p+\dots \quad +|f_\alpha|^p &\leqq \lambda,\\
&|f_{\alpha+1}|^p+\dots \quad +|f_{\alpha+\beta}|^p &\leqq \lambda,\\
&|f_{\alpha+\beta+1}|^p+\dots+|f_{\alpha+\beta+\gamma}|^p &\leqq \lambda,
\end{aligned}$$

etc., where the coefficients of the linear forms f_r are all real, and α, β, γ, ... are any integers $\geqq 0$, with sum n.

When $\alpha=n$, $\beta=\gamma=\dots=0$, this becomes Minkowski's result

$$|f_1|^p+\dots+|f_n|^p \leqq |\Delta|^{p/n}\left[\Gamma\left(1+\frac{n}{p}\right)\right]^{p/n}\Big/\left[\Gamma\left(1+\frac{1}{p}\right)\right]^p.$$

The proof of this by his theorem requires the consideration of

$$(|f_1|^p+\dots+|f_n|^p)^{1/p},$$

and his now well known inequality

$$[(\xi_1+\eta_1)^p+\dots+(\xi_n+\eta_n)^p]^{1/p} \leqq [\xi_1^p+\dots+\xi_n^p]^{1/p}+[\eta_1^p+\dots+\eta_n^p]^{1/p}$$

for positive ξ, η, but which is not so simple as

$$\xi^p+\eta^p \leqq 2^{p-1}(\xi+\eta)^p$$

used above.

(Received, November 2nd, 1933.)