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A Remark on Fourier Transforms and Functions Analytic in a Half-Plane

by

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Let $\mathfrak{L}_p$ be the class of functions $f(x)$ measurable over $(-\infty, \infty)$ such that the integral

$$(1) \qquad \int_{-\infty}^{\infty} |f(x)|^p dx, \qquad p \text{ fixed}, \ p \geqq 1,$$

is finite. It is well known that $\mathfrak{L}_p$ becomes a complete linear metric vector space if we define its metric by

$$(2) \qquad \| f(x) \|_p = \| f \|_p = \left[\int_{-\infty}^{\infty} |f(x)|^p dx \right]^{1/p}.$$

The value $p = \infty$ will also be admitted, with the agreement that $\mathfrak{L}_\infty$ is the class of functions $f(x)$ continuous and bounded over $(-\infty, \infty)$ with the metric

$$(3) \qquad \| f \|_\infty = \sup_{-\infty < x < \infty} |f(x)|.$$

Let $g(x) \subset \mathfrak{L}_p$, $1 \leqq p < \infty$. Put

$$(4) \qquad G(u\,;\ a) = (2\pi)^{-\frac{1}{2}} \int_{-a}^{a} g(x) e^{-iux} dx.$$

If there exists a function $G(u) \subset \mathfrak{L}_q$, $1 \leqq q \leqq \infty$, such that

$$(5) \qquad \| G(u\,;\ a) - G(u) \|_q \to 0 \text{ as } a \to \infty,$$

then we say that $G(u)$ is the Fourier transform of $g(x)$ in $\mathfrak{L}_q$ and write

$$(6) \qquad G(u) = \underset{a \to \infty}{L^q} G(u\,;\ a).$$

Such a function is known to exist when $1 \leqq p \leqq 2$ and $q = p' = \dfrac{p}{p-1}$.

Let $f(z)$ be a function of the complex variable $z = x + iy$, analytic in the upper half-plane $y > 0$. If, for almost all x,

$f(z)$ tends to a definite limit, $f(x)$, as $z \to x$ along all non-tangential paths, the function $f(x)$ so defined is said to be the limit-function of $f(z)$. We shall assume that $f(x) \subset \mathfrak{L}_p$, $1 \leqq p < \infty$.

In the present note we are concerned with the class $\mathfrak{A}_p$ of functions $f(z)$, analytic in the half-plane $y > 0$, each possessing a limit-function $f(x) \subset \mathfrak{L}_p$, $1 \leqq p < \infty$, and representable by its Cauchy integral,

$$f(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{f(\xi)d\xi}{\xi - z} \equiv I(z\,;\,f), \tag{7}$$

or, what is equivalent [1], by its Poisson integral,

$$f(z) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{yf(\xi)d\xi}{(\xi - x)^2 + y^2} \equiv Q(z\,;\,f). \tag{8}$$

The following problem presents itself naturally: find necessary and sufficient conditions which must be satisfied by a given function $f(x) \subset \mathfrak{L}_p$ in order that $f(x)$ be the limit-function of a function $f(z) \subset \mathfrak{A}_p$. In a recent note [2] we gave a solution of this problem under the assumption that $f(x)$ possesses a Fourier transform (in a certain generalized sense). It turns out that this transform must vanish for negative values of its argument. The purpose of the present note is to investigate the same problem under a different assumption, that $f(x)$ itself is the Fourier transform in $\mathfrak{L}_p$ of a function $g(u) \subset \mathfrak{L}_q$, $1 \leqq q \leqq \infty$.

Theorem. Let $f(x)$ be the Fourier transform in $\mathfrak{L}_p$ of a function $\varphi(u) \subset \mathfrak{L}_q$, $1 \leqq q \leqq \infty$. In order that $f(x)$ be the limit-function of a function $f(z) \subset \mathfrak{A}_p$ it is necessary and sufficient that $\varphi(u)$ vanish for $u > 0$.

Proof. For convenience we replace $\varphi(u)$ by $g(-u)$ and set

$$f(\xi) = \underset{a\to\infty}{L^p} F(\xi;\ a), \tag{9}$$

where

$$F(\xi\,;\,a) = \int_{-a}^{a} g(u)e^{i\xi u}du, \tag{10}$$

and $g(u) \subset \mathfrak{L}_q$. We first assume that $1 < q < \infty$, and proceed to the computation of the Poisson integral of $f(\xi)$,

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[1] Cf. HILLE and TAMARKIN, On a theorem of Paley and Wiener [Annals of Math. (2) 34 (1933), 606—614].

[2] Loc. cit. [1] For a special case see N. WIENER, The operational calculus [Math. Ann. 95 (1926), 557—584; (580)].

$$(11)\quad Q(z;f)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{yf(\xi)d\xi}{(\xi-x)^2+y^2}=\lim_{N\to\infty}\frac{1}{\pi}\int_{-N}^{N}\frac{yf(\xi)d\xi}{(\xi-x)^2+y^2},$$

which obviously converges absolutely. In view of (9) we have

$$(12)\quad \int_{-N}^{N}\frac{yf(\xi)d\xi}{(\xi-x)^2+y^2}=\int_{-\infty}^{\infty}g(u)du\int_{-N}^{N}\frac{ye^{e\xi u}d\xi}{(\xi-x)^2+y^2}.$$

On the other hand, by a direct computation,

$$(13)\quad \frac{y}{(\xi-x)^2+y^2}=\frac{1}{2}\int_{-\infty}^{\infty}e^{i(x-\xi)t}e^{-y|t|}dt.$$

Hence

$$(14)\quad \begin{aligned}\int_{-N}^{N}\frac{ye^{i\xi u}d\xi}{(\xi-x)^2+y^2}&=\frac{1}{2}\int_{-\infty}^{\infty}e^{ixt-y|t|}\,dt\int_{-N}^{N}e^{i\xi(u-t)}d\xi=\\&=\int_{-\infty}^{\infty}e^{ixt-y|t|}\,\frac{\sin N(u-t)}{u-t}\,dt.\end{aligned}$$

On substituting into (12) and interchanging the order of integration, which is clearly permissible, we get

$$(15)\quad \begin{aligned}\frac{1}{\pi}\int_{-N}^{N}\frac{yf(\xi)d\xi}{(\xi-x)^2+y^2}&=\int_{-\infty}^{\infty}e^{ixt-y|t|}\,dt\int_{-\infty}^{\infty}g(u)\frac{1}{\pi}\frac{\sin N(u-t)}{u-t}du=\\&=\int_{-\infty}^{\infty}e^{ixt-y|t|}\,dt\int_{-\infty}^{\infty}g(u)\,\mathfrak{D}_N(u-t)\,du,\end{aligned}$$

where

$$\mathfrak{D}_N(u)=\frac{1}{\pi}\frac{\sin Nu}{u}$$

is the classical Dirichlet kernel. Now put

$$g_N(t)=\int_{-\infty}^{\infty}g(u)\,\mathfrak{D}_N(u-t)du.$$

It is known [1]) that

$$\|g_N(t)-g(t)\|_q\to 0\ \text{ as }\ N\to\infty.$$

Consequently

$$(16)\quad \begin{aligned}Q(z;\,f)&=\int_{-\infty}^{\infty}e^{ixt-y|t|}\,g(t)dt=\\&=\int_{0}^{\infty}e^{izt}g(t)dt+\int_{-\infty}^{0}e^{i\bar{z}t}g(t)dt\equiv\Omega_1(z)+\Omega_2(\bar{z}),\end{aligned}$$

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[1]) See, for instance, Hille and Tamarkin, On the theory of Fourier transforms [Bulletin of the Amer. Math. Soc. 39 (1933)].

where $\bar{z} = x - iy$. Here $\Omega_1(z)$ is analytic in z and $\Omega_2(\bar{z})$ is analytic in $\bar{z}$, while

$$\Omega_1(z) \to 0, \ \Omega_2(\bar{z}) \to 0 \text{ as } |x| \to \infty.$$

Hence the vanishing of $\Omega_2(\bar{z})$ is a necessary and sufficient condition for the analyticity of $Q(z; f)$. Now, if $f(z) \subset \mathfrak{A}_p$, then $f(z)$ is represented by its Poisson integral $Q(z; f)$, and $\Omega_2(\bar{z}) \equiv 0$; conversely, if $\Omega_2(\bar{z}) \equiv 0$, $Q(z; f)$ is analytic and represents a function $f(z) \subset \mathfrak{A}_p$ whose limit-function is precisely $f(x)$. We see therefore that the condition $\Omega_2(\bar{z}) \equiv 0$ is necessary and sufficient in order that $f(x)$ be the limit-function of a function $f(z) \subset \mathfrak{A}_p$. In view of the uniqueness theorem for Fourier integrals, however, the condition $\Omega_2(\bar{z}) \equiv 0$ is equivalent to the condition $g(t) = 0$ for $t < 0$; which is the desired result.

The treatment of the cases $q = 1$, $q = \infty$ is slightly more complicated, but formula (16) and all the subsequent conclusions will still be valid. When $q = 1$ or $q = \infty$, we apply the method of arithmetic means to evaluate the integral (11). Thus

$$\frac{1}{N}\int_0^N dn \frac{1}{\pi}\int_{-n}^{n} \frac{y f(\xi) d\xi}{(\xi - x)^2 + y^2} = \lim_{a \to \infty} \int_{-a}^{a} g(u)\, du \, \frac{1}{N}\int_0^N dn \frac{1}{\pi}\int_{-n}^{n} \frac{y e^{i\xi u} d\xi}{(\xi - x)^2 + y^2} =$$

$$= \lim_{a \to \infty} \int_{-a}^{a} g(u) du \frac{1}{N}\int_0^N dn \int_{-\infty}^{\infty} e^{ixt - y|t|} \frac{1}{\pi} \frac{\sin n(u - t)}{u - t}\, dt =$$

$$= \lim_{a \to \infty} \int_{-a}^{a} g(u)\, du \int_{-\infty}^{\infty} e^{ixt - y|t|} \frac{1}{2N\pi} \left[\frac{\sin \frac{N(u - t)}{2}}{\frac{u - t}{2}} \right]^2 dt = \tag{17}$$

$$= \lim_{a \to \infty} \int_{-\infty}^{\infty} e^{ixt - y|t|}\, dt \int_{-a}^{a} g(u) \mathfrak{F}_N(u - t) du = \int_{-\infty}^{\infty} e^{ixt - y|t|}\, dt \int_{-\infty}^{\infty} g(u) \mathfrak{F}_N(u - t) du$$

where

$$\mathfrak{F}_N(u) = \frac{1}{2N\pi} \left[\frac{\sin \frac{Nu}{2}}{\frac{u}{2}} \right]^2$$

is the Fejér kernel. Now put

$$g_N'(t) = \int_{-\infty}^{\infty} g(u) \mathfrak{F}_N(u - t) du.$$

When $N \to \infty$, the left-hand member of (17) tends to $Q(z; f)$

since the integral (11) converges. On the other hand, when $q = 1$,

$$\| g'_N(t) - g(t) \|_1 \to 0$$

while, when $q = \infty$, $g'_N(t) \to g(t)$ boundedly, and in fact uniformly over every finite range. Hence allowing $N \to \infty$, we obtain (16) again, and finish the proof as above. This method of course might have been used in the case $1 < q < \infty$ as well.

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