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The Cayley-Spottiswoode coordinates of a conic in 3-space.

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If a conic in a three-dimensional projective space is defined by the quadratic complex of lines which meet it, the coefficients in the equation of the complex may be regarded as coordinates of the conic. The coordinates thus defined, analogous to the Plücker coordinates of a line, are due essentially to Cayley [1]), but were defined independently and differently by Spottiswoode [2]). They were employed recently by J. A. Todd [3]) to represent the conics of 3-space by points of 19-space, for which purpose he introduced a symmetrical and concise notation which, with certain modifications, is used in the present paper. Each conic has twenty-one distinct homogeneous coordinates which satisfy certain identical relations; these are obtained below by a method which shows that the symbolic calculus [4]) employed by Todd admits of a geometrical interpretation. A variety of other formulae are also established, expressing the condition that two conics should intersect, that they should be coplanar, and so on.

Todd's notation is extended so as to be brought into conformity with that of tensor, or rather of spinor, analysis. The theory of four-component spinors [5]) is really that of three-dimensional projective geometry, and the associated calculus has a power and conciseness which makes it a valuable instrument in the analytical treatment of ordinary projective geometry. It may however be added that the present paper is not concerned with the dif-

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1) Cayley [Quart. J. of Math. **3** (1860), 225].

2) Spottiswoode [Proc. London Math. Soc. (1) **10** (1879), 185].

3) Todd [Proc. London Math. Soc. **36** (1933), 172].

4) See Grace and Young, Algebra of Invariants (Cambridge 1903).

5) The theory of four-component spinors, and the notation adopted in this paper, is outlined in a series of papers by Veblen and others in Proc. Nat. Acad. Sci. **19** (1933) and **20** (1934).

ferential aspects of the spinor theory, since it deals with a single 3-space and not with the infinity of such spaces (each associated with a point of an "underlying space") which appear in spin-geometry proper.

§ 1. *Notation and preliminaries.*

The points of the projective 3-space are represented, in a given system of reference, by four homogeneous coordinates $X^A \equiv (X^1, X^2, X^3, X^4)$. Capital letters A, B, C, ... used as suffixes will always take the values 1, 2, 3, 4, and the summation convention for repeated suffixes will be employed throughout. A transformation of coordinates (or, alternatively, a collineation) is given by a linear relation of the type.

$$\varrho X'^A = T^A_B X^B, \tag{1.1}$$

where ϱ is an arbitrary factor of homogeneity and T^A_B is a square matrix of rank 4. ϱ will be used generally to denote an arbitrary factor, and will not necessarily be the same from formula to formula.

A plane whose equation in the first system of coordinates is $\varphi_A X^A = 0$ transforms into $\varphi'_A X'^A = 0$, where

$$\varrho \varphi'_A = t^B_A \varphi_B, \tag{1.2}$$

t^B_A being the matrix reciprocal to T^A_B, so that

$$t^A_B T^B_C = \delta^A_C. \tag{1.3}$$

Here δ^A_C is the Kronecker symbol having the value unity when $A = C$ and zero when $A \neq C$. The coordinates of a point are therefore represented by a central letter with a single upper (contravariant) suffix, and those of a plane by a central letter bearing a single lower (covariant) suffix.

If X^A, Y^A are two points of the space, the Plücker coordinates p^{AB} of the line joining them are defined by

$$\varrho p^{AB} = X^A Y^B - X^B Y^A. \tag{1.4}$$

If φ_A, ψ_A are any two distinct planes through this line, the dual set of Plücker coordinates is given by

$$\varrho p_{AB} = \varphi_A \psi_B - \varphi_B \psi_A. \tag{1.5}$$

Of course $p^{AB} = -p^{BA}$ and $p_{AB} = -p_{BA}$. The two sets of Plücker coordinates are connected by the relation

$$\varrho p_{AB} = \varepsilon_{ABCD} p^{CD}, \tag{1.6}$$

where ε_{ABCD} is defined to have the value 1 or -1 if $ABCD$ is respectively an even or odd permutation of 1234, and to be zero if two or more suffixes are equal; ε^{ABCD}, used below, is defined in the same way. Both ε-symbols are therefore skew-symmetric for an interchange of any pair of suffixes.

Under the transformation (1.1), p^{AB} becomes p'^{AB}, where $\varrho p'^{AB} = T^A_C T^B_D p^{CD}$, and p_{AB} transforms into p'_{AB}, where $\varrho p'_{AB} = t^C_A t^D_B p_{CD}$. In general the mode in which the various quantities transform will be indicated by the position of their suffixes, the matrix T^A_B being used with contravariant and the matrix t^A_B with covariant. So for example ε_{ABCD} transforms according to

$$\begin{aligned} \varrho\varepsilon'_{ABCD} &= \varepsilon_{EFGH} t^E_A t^F_B t^G_C t^H_D \\ &\equiv t\varepsilon_{ABCD} \end{aligned}$$

where t is the determinant $|t^A_B|$. If the arbitrary factor ϱ is chosen [6]) to be equal to t, we obtain $\varepsilon'_{ABCD} = \varepsilon_{ABCD}$, so that ε_{ABCD} may be said to transform into itself; similarly ε^{ABCD} transforms into itself.

In the theory of spinors the two sets of Plücker coordinates of a line are normalized in terms of one another (that is, a particular choice is made of their factors of proportionality) according to the formula

$$p^{AB} = \tfrac{1}{2}\varepsilon^{ABCD} p_{CD}, \tag{1.7}$$

from which it at once follows that

$$p_{AB} = \tfrac{1}{2}\varepsilon_{ABCD} p^{CD}. \tag{1.8}$$

This normalization renders definite in any given coordinate system the operations of raising and lowering pairs of skew-symmetric suffixes, these operations being in fact defined by (1.7) and (1.8) respectively.

With this notation, the identical relation satisfied by the Plücker coordinates of a line may be written in any of the forms [7])

(2.11) $$\varepsilon_{ABCD} p^{AB} p^{CD} \equiv 0, \quad \varepsilon^{ABCD} p_{AB} p_{CD} \equiv 0, \quad p^{AB} p_{AB} \equiv 0. \tag{1.9}$$

[6]) This choice of the arbitrary factor ϱ is equivalent to attaching a "weight" to the "spinor" ε_{ABCD}. For present purposes this, like the normalization defined by (1.7) and (1.8), is not really necessary, but it seemed desirable that the notation of this paper should be kept as consistent as possible with that of Veblen. The normalizations introduced in §§ 2, 3 are however made for the purpose of adding conciseness to the algebra. They do not affect the homogeneity of the formulae, and therefore do not destroy the projective character of the geometry. (But see Note 12 below.)

[7]) Small numbers to the left of formulae in this paper refer to the corresponding formulae of Todd's paper (Note 3 above.)

The condition that two lines p^{AB}, q^{AB} should intersect is

$$p^{AB}q_{AB} = 0 \text{ or } p_{AB}q^{AB} = 0. \tag{1.10}$$

The point where a plane φ_A meets a line p^{AB} has coordinates $p^{AB}\varphi_B$, and the plane through a point X^A and a line p_{AB} has coordinates $p_{AB}X^B$. The plane φ_A passes through the line p_{AB} if

$$p^{AB}\varphi_B = 0, \tag{1.11}$$

and the point X^A lies on it if

$$p_{AB}X^B = 0; \tag{1.12}$$

and conversely.

The line whose Plücker coordinates are p_{AB} (or p^{AB}) will be referred to as *the line p*. Also it will be convenient later to use the notation (pq) for the inner product $p_{AB}q^{AB}$ of the coordinates of two lines. So

$$(pq) \equiv p_{AB}q^{AB} \equiv (qp). \tag{1.13}$$

The following simple theorems will be of frequent use:

Theorem I. *If h_{AB}, k_{AB} are any pair of square skew-symmetric matrices of order* 4, *not necessarily satisfying* (1.9) *or* (1.10), *then* [8])

$$h_{AB}k^{BC} + k_{AB}h^{BC} \equiv -\tfrac{1}{2}h_{EF}k^{EF}\delta_A^C. \tag{1.14}$$

The proof consists of writing the relations (1.14) in full, giving the free suffixes A, C particular values.

If p, q are lines which intersect, (1.10) and (1.14) give

$$p_{AB}q^{BC} + q_{AB}p^{BC} = 0, \tag{1.15}$$

which is therefore an equivalent form of the condition (1.10). Putting $q = p$ in the last equation, we at once obtain the identity (1.9) satisfied by the coordinates of a line, but in a slightly different form, namely

$$p_{AB}p^{BC} \equiv 0. \tag{1.16}$$

Theorem II. *If p, q are two lines, and*

$$p_{AB}q^{BC} = 0, \tag{1.17}$$

the lines coincide.

For if (1.17) is true, then

$$p_{AB}q^{BC}\varphi_C = 0 \tag{1.18}$$

for all planes φ_C. But $q^{BC}\varphi_C$ is the point where the plane φ_A meets

8) Cf Weitzenböck, Komplex-Symbolik [Leipzig 1908], 8.

the line q, and (1.18) states that this point lies on the line p (Cf (1.12)). Since φ_C is any plane, this means that all points on q lie on p. Hence p, q are the same line.

THEOREM III. *The necessary and sufficient conditions that three lines p, q, r should be concurrent are*

$$p_{AB}q^{BC}r_{CD} = 0, \quad q_{AB}r^{BC}p_{CD} = 0, \quad r_{AB}p^{BC}q_{CD} = 0. \tag{1.19}$$

The conditions are necessary. For suppose that the lines meet in the point X^A. Then q^{BC} is given by

$$q^{BC} = X^B Q^C - X^C Q^B$$

where Q^A is any other point on the line q. Hence

$$\begin{aligned} p_{AB}q^{BC}r_{CD} &= p_{AB}(X^B Q^C - X^C Q^B)r_{CD} \\ &= 0 \end{aligned}$$

since X^A lies on both lines p, r and consequently $p_{AB}X^B$ and $r_{CD}X^C$ are both zero. So the first of equations (1.19) is satisfied, and similarly the others.

To establish the sufficiency of the conditions, suppose that p, q, r are three lines satisfying (1.19), and assume for the moment that q, r do not coincide. Let X^A be any point. Then the first of the equations (1.19) gives

$$p_{AB}q^{BC}r_{CD}X^D = 0. \tag{1.20}$$

But $r_{CD}X^B$ is the plane through the point X^A and the line r; $q^{BC}r_{CD}X^D$ is the point where this plane meets q, and by (1.20) this point lies on p. Since X^A is any point, this means that p, q meet on all planes through r. Hence either (I) p, q, r are concurrent; or (II) p, r coincide and q meets them; or (III) p, q coincide but do not necessarily meet r. If (I) or (II) is true, the theorem is proved. If (III) is true, it quickly follows from the second of equations (1.19) (using (1.14) and (1.16)) that the coincident lines p, q do meet r. Similarly in the hitherto excluded case when q, r coincide, the fact that p meets them is an almost immediate consequence of the third equation (1.19).

COROLLARY a. *If two lines p^{AB}, q^{AB} intersect, and if $p_{AB}q^{BC}r_{CD} = 0$, then the line q passes through their point of intersection.*

COROLLARY b. *If two lines p, q intersect, then $p_{AB}q^{BC}p_{CD} = 0$, $q_{AB}p^{BC}q_{CD} = 0$. Conversely, if either of these relations is satisfied by two lines, they intersect.*

THEOREM IV. *The necessary and sufficient conditions that three lines p, q, r should be coplanar are*

$$p^{AB}q_{BC}r^{CD} = 0, \quad q^{AB}r_{BC}p^{CD} = 0, \quad r^{AB}p_{BC}q^{CD} = 0. \tag{1.21}$$

This is the dual of Theorem III.

COROLLARY a. *If the two lines p, r intersect and* $p^{AB}q_{BC}r^{CD}=0$ *then q lies in their plane.*

COROLLARY b. *If two lines p, q intersect, then* $p^{AB}q_{BC}p^{CD}=0$, $q^{AB}p_{BC}q^{CD}=0$. *Conversely, if either of these relations is satisfied by two lines, then they intersect.*

§ 2. *The coordinates of a conic.*

Let now

(2.21) $$d_{ABCD}p^{AB}p^{CD}=0 \tag{2.1}$$

be the equation of the quadratic complex of lines which meet a given conic. Since p^{AB} is skew-symmetrical in its suffixes, the coefficients d_{ABCD} may be defined as skew-symmetrical in A, B and also in C, D. That is,

$$d_{ABCD}=-d_{BACD}=d_{BADC}=-d_{ABDC}. \tag{2.2}$$

Also, since the left-hand side of (2.1) may be written $d_{CDAB}p^{CD}p^{AB}$, we may take

$$d_{CDAB}=d_{ABCD}. \tag{2.3}$$

So d_{ABCD} is skew-symmetrical in the first pair and in the last pair of suffixes, but is symmetrical for an interchange of these pairs. Todd indicates this by placing a comma between the pairs in question, thus $d_{AB,CD}$, but it is more convenient for present purposes to omit the comma.

On account of (1.9) it may be assumed that d_{ABCD} satisfies the linear identity

(2.22) $$\varepsilon^{ABCD}d_{ABCD}\equiv 0. \tag{2.4}$$

Further, from (2.2), (2.3), (2.4), we have

$$d_{ABCD}+d_{ACDB}+d_{ADBC}\equiv 0, \tag{2.5}$$

which may also be written

$$\varepsilon^{ABCD}d_{EBCD}\equiv 0. \tag{2.5a}$$

The coefficients d_{ABCD} thus defined are the Cayley-Spottiswoode coordinates of the conic. Because of (2.2) and (2.3) only twenty-one of them are distinct, and these satisfy the identity (2.4). As remarked by Todd [9]), d_{ABCD} has the properties of sym-

[9]) TODD, loc. cit., 205.

metry and skew-symmetry with regard to its suffixes possessed by the curvature tensor R_{ABCD} of Riemannian geometry.

Spottiswoode defined the conic as the intersection of a quadric

(2.31) $$g_{AB}X^A X^B = 0 \tag{2.6}$$

and a plane

(2.32) $$\lambda_A X^A = 0, \tag{2.7}$$

and defined the coordinates of the conic by formulae equivalent to

(2.33) $$d_{ABCD} \equiv g_{AC}\lambda_B\lambda_D + g_{BD}\lambda_A\lambda_C - g_{AD}\lambda_B\lambda_C - g_{BC}\lambda_A\lambda_D, \tag{2.8}$$

or $$d_{ABCD} \equiv \delta^{EF}_{AB}\,(g_{EC}\lambda_F\lambda_D - g_{ED}\lambda_F\lambda_C), \tag{2.8a}$$

where δ^{EF}_{AB} is the generalised Kronecker symbol equal to $\delta^E_A\delta^F_B - \delta^E_B\delta^F_A$. The coefficients g_{AB} are of course assumed to be symmetrical in the suffixes, so $g_{AB} = g_{BA}$. The d_{ABCD} of (2.8) are easily [10]) shown to be the same as those previously defined.

We shall in general suppose that the quadric g_{AB} is non-degenerate, so that the determinant $g \equiv |g_{AB}|$ is not zero. Let g^{AB} be the matrix reciprocal to g_{AB}, so that

$$g^{AB}g_{BC} = \delta^A_C. \tag{2.9}$$

Then of course $g^{AB}\varphi_A\varphi_B = 0$ is the tangential equation of the quadric.

If φ_A is a plane, its pole with respect to the quadric has coordinates

$$\varphi^A \equiv g^{AB}\varphi_B \tag{2.10}$$

and if X^A is a point, its polar plane with respect to the quadric has coordinates

$$X_A \equiv g_{AB}X^B. \tag{2.11}$$

φ^A and X_A could of course be multipied by any factor, but it is convenient to normalize them in relation to φ_A and X^A, and also in relation to the coefficients g_{AB}, according to the last two formulae; these formulae then give a method of raising and lowering single suffixes belonging to symbols representing planes and points. It is perhaps not altogether desirable to introduce a method of raising and lowering suffixes in addition to that defined by (1.7) and (1.8), but the possibility of confusion to which it leads is small, while the gain in conciseness is great.

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[10]) TODD, loc. cit., 183.

It should be noted that a conic may be defined by its plane (2.7) and any quadric (2.6) which passes through it. But in any given problem it will be assumed that a definite quadric is chosen, so that the process of raising and lowering suffixes is unambiguous.

If p_{AB} is a line, its polar line with respect to the quadric has coordinates $g^{AC}g^{BD}p_{CD}$. This may not be denoted by p^{AB}, since in accordance with (1.7) the latter symbol represents the dual coordinates of the same line p_{AB}, which does not in general coincide with its polar. Similarly the dual coordinates $g_{AC}g_{BD}p^{CD}$ of the polar line may not be denoted by p_{AB}.

§ 3. *Special forms for the coordinates of a conic.*

For the moment it will be assumed that the conic is non-degenerate, and that it is defined as the intersection of a plane λ_A and a non-degenerate quadric g_{AB}.

Let $h^A_{(P)} \equiv (h^A_{(1)}, h^A_{(2)}, h^A_{(3)}, h^A_{(4)})$ be the vertices of any tetrahedron which is self-polar with respect to the quadric. The ordinal (scalar) suffixes (P), (Q), (R), ..., which are bracketed to distinguish them from the coordinate (tensor) suffixes A, B, C, ..., take the values 1, .., 4 and, when repeated, imply a summation. Then with a proper choice of the unit point, the equation of the quadric with $h^A_{(P)}$ as tetrahedron of reference

is $$(X^{(1)})^2 + (X^{(2)})^2 + (X^{(3)})^2 + (X^{(4)})^2 = 0$$

or $$\delta_{(P)(Q)} X^{(P)} X^{(Q)} = 0,$$

where $\delta_{(P)(Q)}$ is a Kronecker delta and the coordinates $X^{(P)}$ are connected with the original system by the formula

$$X^A = h^A_{(P)} X^{(P)}, \tag{3.1}$$

if the $h^A_{(P)}$ are properly normalized. Hence

$$g_{AB} h^A_{(P)} h^B_{(Q)} = \delta_{(P)(Q)}, \tag{3.2}$$

which may be written

$$h^A_{(P)} h_{(Q)A} = \delta_{(P)(Q)}. \tag{3.3}$$

The matrices $h^A_{(P)}$, $h_{(Q)A}$ are therefore reciprocal to one another, so

$$h^A_{(P)} h_{(P)B} = \delta^A_B \tag{3.4}$$

(summation with respect to (P)). Lowering the suffix A, we get

(2.41) $$h_{(P)A} h_{(P)B} = g_{AB}. \tag{3.5}$$

The coefficients $h_{(P)A}$ for $(P) = 1, 2, 3, 4$ are of course the (normalized) coordinates of the faces of the tetrahedron, since $h_{(P)A}$ is the polar plane of $h^A_{(P)}$. The above analysis may be compared with that belonging to the theory of orthogonal ennuples in Riemannian geometry.

Equation (3.5) corresponds to the symbolic equation (2.41) of Todd's paper. Symbolically $g_{AB} = l_A l_B$, so it is evident that, in order to pass from the symbolic treatment of the subject to one in which the algebraic operations admit of a geometrical interpretation, it is merely necessary to replace l by $h_{(P)}$ and to allow a repetition of the subscript (P) to imply a summation from 1 to 4. So instead of writing $g_{AB} = l_A l_B = m_A m_B = \dots$ in the manner of Todd, I write $g_{AB} = h_{(P)A} h_{(P)B} = h_{(Q)A} h_{(Q)B} = \dots$.

That $h^A_{(P)}$ are the vertices of *any* self-polar tetrahedron means that $h^A_{(P)}$ are replaceable by $h'^A_{(P)}$, where

$$h'^A_{(P)} = k_{(P)(Q)} h^A_{(Q)}, \tag{3.6}$$

the coefficients $k_{(P)(Q)}$ being such that

$$k_{(P)(Q)} k_{(P)(R)} = \delta_{(Q)(R)}. \tag{3.7}$$

It is however convenient to limit the choice of the self-polar tetrahedron by requiring that the plane of the conic shall be one of its faces. So we write

$$h_{(4)A} \equiv \lambda_A \tag{3.8}$$

and denote $h_{(1)A}$, $h_{(2)A}$, $h_{(3)A}$ by h_{aA} $(a = 1, 2, 3)$. Then (3.5) becomes

$$g_{AB} = h_{aA} h_{aB} + \lambda_A \lambda_B. \tag{3.9}$$

The ordinal suffixes $a, b, \dots$ will run from 1 to 3 and, when repeated, will sum over that range.

By (3.3) and (3.4) we get

$$h^A_a h_{bA} = \delta_{ab}, \tag{3.10}$$

$$h^A_a \lambda_A = 0, \tag{3.11}$$

$$\lambda^A h_{aA} = 0, \tag{3.12}$$

$$\lambda^A \lambda_A = 1, \tag{3.13}$$

$$h^A_a h_{aB} + \lambda^A \lambda_B = \delta^A_B. \tag{3.14}$$

Hence, raising the suffix B in the last equation,

$$h^A_a h^B_a + \lambda^A \lambda^B = g^{AB}. \tag{3.15}$$

The h^A_a, $(a = 1, 2, 3)$, are the vertices of a triangle self-polar with respect to the conic, and (3.11) states that each of these lies in the plane λ_A. The point λ^A is the pole of λ_A with respect to the quadric, and (3.12) states that this lies on each of the planes h_{aA}, which are planes through the respective sides of the self-polar triangle.

Substitute from (3.9) in (2.8). We get

(2.43) $$d_{ABCD} = \lambda_{aAB}\lambda_{aCD}, \tag{3.16}$$

where

(2.42) $$\lambda_{aAB} = \lambda_A h_{aB} - \lambda_B h_{aA}. \tag{3.17}$$

The λ_{aAB} are the normalized coordinates of the sides of any triangle self-polar with respect to the conic, and (3.16) is an expression for the coordinates of the conic in terms of the sides of such a triangle.

The line λ_{aAB} is the side of the triangle joining the points h^A_b, h^A_c, where $a \neq b \neq c$. So

$$\begin{aligned} \lambda^{AB}_a &\equiv \tfrac{1}{2}\varepsilon^{ABCD}\lambda_{aCD} \\ &= m(h^A_b h^B_c - h^B_b h^A_c), \quad (a \neq b \neq c), \end{aligned}$$

where m is some number. Actually it can be shown that, with the normalizations already adopted, $m = \sqrt{g}$. The last relation may therefore be written

$$\lambda^{AB}_a = \sqrt{g}\,\varepsilon_{abc}h^A_b h^B_c, \tag{3.18}$$

where $\varepsilon_{abc} = 1$ or -1 according as abc is an even or odd permutation of 123, and is zero otherwise.

The following relations will be useful. By (3.17), (3.13) and (3.12),

$$\lambda^B\lambda_{aAB} \equiv -\lambda^B\lambda_{aBA} = -h_{aA}. \tag{3.19}$$

By (3.17), (3.10) and (3.11),

$$h^B_a\lambda_{bAB} \equiv -h^B_a\lambda_{bBA} = \delta_{ab}\lambda_A. \tag{3.20}$$

Conseqnently, by (3.16),

$$h^B_a d_{ABCD} \equiv h^B_a d_{CDAB} = \lambda_A\lambda_{aCD}. \tag{3.20a}$$

Since the line λ_{aAB} lies in the plane λ_A,

$$\lambda_A\lambda^{AB}_a = 0. \tag{3.21}$$

By (3.18), (3.10),

$$\lambda^{AB}_a h_{bA} = \sqrt{g}\,\varepsilon_{adc}\delta_{bd}h^B_c,$$

whence

$$\lambda^{AB}_a h_{bA} \equiv -\lambda^{BA}_a h_{bA} = \sqrt{g}\,\varepsilon_{abc}h^B_c. \tag{3.22}$$

We shall write in accordance with the notation described in § 1,

$$d_{AB\cdot\cdot}^{\cdot\cdot CD} = \tfrac{1}{2}\varepsilon^{CDEF} d_{ABEF} = \lambda_{aAB}\lambda_a^{CD}, \tag{3.23}$$

$$d^{AB\cdot\cdot}_{\cdot\cdot CD} = \tfrac{1}{2}\varepsilon^{ABEF} d_{EFCD} = \lambda_a^{AB}\lambda_{aCD} = d_{CD\cdot\cdot}^{\cdot\cdot AB}, \tag{3.24}$$

$$d^{ABCD} = \tfrac{1}{4}\varepsilon^{ABEF}\varepsilon^{CDGH} d_{EFGH} = \lambda_a^{AB}\lambda_a^{CD}. \tag{3.25}$$

Obviously d^{ABCD} has properties of symmetry and skew-symmetry similar to those defined by (2.2) and (2.3) for d_{ABCD}, and satisfies the identity

$$d^{ABCD} + d^{ACDB} + d^{ADBC} \equiv 0 \tag{3.26}$$

similar to (2.5).

From (3.18) it follows that

$$\begin{aligned} d^{ABCD} &= g\varepsilon_{abc}\varepsilon_{ade} h_b^A h_c^B h_d^C h_e^D \\ &= g(\delta_{bd}\delta_{ce} - \delta_{be}\delta_{cd}) h_b^A h_c^B h_d^C h_e^D \end{aligned}$$

by a well-known property of the ε-symbols [11]), so that

$$d^{ABCD} = g(h_b^A h_b^C h_c^B h_c^D - h_b^A h_b^D h_c^B h_c^C) \tag{3.27}$$

or by (3.15),

(4.41) $$\begin{aligned} d^{ABCD} = g[(g^{AC} - \lambda^A\lambda^C)(g^{BD} - \lambda^B\lambda^D) - \\ - (g^{AD} - \lambda^A\lambda^D)(g^{BC} - \lambda^B\lambda^C)] \end{aligned} \tag{3.28}$$

or, more concisely,

$$d^{ABCD} = g\,\delta_{EF}^{AB}\gamma^{EC}\gamma^{FD}, \tag{3.29}$$

where $$\gamma^{AB} \equiv g^{AB} - \lambda^A\lambda^B. \tag{3.30}$$

Now the equation of the cone having λ^A as vertex and touching the quadric g_{AB} where it is met by the plane λ_A is, according to the usual formula,

$$(g_{AB}X^A X^B)(g_{CD}\lambda^C\lambda^D) - (g_{AB}X^A\lambda^B)^2 = 0,$$

or by (3.13) and the fact that $g_{AB}\lambda^B = \lambda_A$,

$$g_{AB}X^A X^B - (\lambda_A X^A)^2 = 0,$$

that is,

$$(g_{AB} - \lambda_A\lambda_B)X^A X^B = 0. \tag{3.31}$$

The polar of this cone with respect to the quadric is the conic itself; by (3.31) this has the tangential equation

$$\gamma^{AB} X_A X_B = 0. \tag{3.32}$$

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[11]) See, e.g., Veblen, Invariants of Quadratic Differential Forms (Cambridge Tract 24) Ch. I, equations (3.4) and (3.3).

So (3.28) or (3.29) gives the coordinates of the conic in terms of the coefficients of its tangential equation, and shows how the conic, regarded as a degenerate quadric envelope, is related to the conic defined in terms of the quadratic complex of lines which meet it.

Until now the conic has been assumed to be non-degenerate. The degenerate cases can be included in the formula (3.16) if certain restrictions are made. Thus if the range of the subscript a is limited to $a = 1, 2$ (that is, if we make $\lambda_{3AB} = 0$), the conic is a pair of lines harmonically conjugate with respect to the two lines λ_{aAB}. If we also make $\lambda_{2AB} = 0$, then $d_{ABCD} = \lambda_{1AB}\lambda_{1CD}$, and the conic reduces to a repeated line. There are no other degenerate forms of the conic as defined by (2.1) or (2.8).

For a pair of distinct lines equations (3.19), (3.20) and (3.20a) remain true ($a = 1,2$); in this case h_{1A}, h_{2A} are any planes through the lines λ_{1AB}, λ_{2AB} respectively, h_2^A, h_1^A are respectively any points upon them, and λ^A is any point on the line of intersection of h_{1A}, h_{2A} (other than the point of intersection of this line with the plane λ_A).

One other form of the coordinates d_{ABCD} will be of use. Let the lines l_{AB}, m_{AB} be two tangents to the conic, and let n_{AB} be the chord of contact. Through l_{AB}, m_{AB}, n_{AB} draw any three planes l_A, m_A, n_A respectively. Then the equation of any cone touching l_A, m_A along the lines where they are met by n_A is of the form

$$2\varrho(l_A X^A)(m_B X^B) + \sigma(n_A X^A)^2 = 0,$$

where the parameter ϱ/σ defines the particular cone. That is,

$$(2\varrho\, l_A m_B + \sigma n_A n_B) X^A X^B = 0,$$

or

$$g_{AB} X^A X^B = 0,$$

where

$$g_{AB} = g_{BA} = \varrho(l_A m_B + l_B m_A) + \sigma n_A n_B.$$

Of course g_{AB} is now of determinant zero, so there is no reciprocal matrix g^{AB}. The conic is the intersection of one such cone and the plane λ_A; suppose that this is the cone ϱ/σ. Substituting in (2.8), we get

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[12]) The non-homogeneous appearance of equations (3.28), (3.30) is due to the normalization of λ^A (i.e., $\lambda^A \lambda_A = 1$). They may be rendered homogeneous in the g's and λ's by writing $\gamma^{AB} \equiv g^{AB}\lambda^C\lambda_C - \lambda^A\lambda^B$.

$$d_{ABCD} = \varrho(l_{AB}m_{CD} + l_{CD}m_{AB}) + \sigma n_{AB}n_{CD}, \tag{3.33}$$

since of course $l_{AB} = \lambda_A l_B - \lambda_B l_A$, with similar formulae for m_{AB} and n_{AB}. In dealing with any particular conic touching the lines l, m where they are met by n, we can absorb the factors ϱ and σ in the coordinates l_{AB}, m_{AB}, n_{AB}, so that (3.33) assumes the simple normalized form

$$d_{ABCD} = l_{AB}m_{CD} + l_{CD}m_{AB} + n_{AB}n_{CD}. \tag{3.34}$$

If the conic is the pair of lines l, m, we may put $n_{AB} = 0$, and obtain

$$d_{ABCD} = l_{AB}m_{CD} + l_{CD}m_{AB}. \tag{3.35}$$

If the conic is a repeated line we may put $m_{AB} = l_{AB}$. Doing so, and dividing by 2, we get

$$d_{ABCD} = l_{AB}l_{CD}. \tag{3.36}$$

Other forms for the coordinates d_{ABCD} can easily be found. For instance, they can be expressed in terms of the sides of triangles inscribed or circumscribed to the conic. In all cases d_{ABCD} has the form $g_{ab}l_{aAB}l_{bCD}$, where l_{aAB} $(a = 1, 2, 3)$ are the sides of the triangle and g_{ab} are coefficients depending on the nature of the triangle and on the normalizations.

§ 4. *Identities.*

We take the coordinates of the conic in the form (3.16). Since each of the lines λ_{aAB} meets the other two, we have by (1.15),

$$\lambda_a^{AB}\lambda_{bAC} + \lambda_{aAC}\lambda_b^{AB} \equiv 0. \tag{4.1}$$

Putting $C = B$ and summing, the two terms become the same. Then putting $a = b$, we get

$$d^{AB\cdot\cdot}{}_{\cdot\cdot AB} \equiv 0, \tag{4.2}$$

which is the linear identity (2.4) already given. Multiplying by $\lambda_a^{DE}\lambda_{bFG}$ (and of course summing with respect to the repeated indices), we obtain a quadratic identity:

$$d^{ABDE}d_{ACFG} + d_{AC}{}^{\cdot\cdot DE}{}_{\cdot\cdot}\, d^{AB}{}_{\cdot\cdot FG}{}^{\cdot\cdot} \equiv 0. \tag{4.3}$$

If in this we put $C = B$, the two terms become the same, and we obtain the quadratic identity in the form given by Todd, viz.

(2.45) $$d^{ABDE}d_{ABFG} \equiv 0. \tag{4.4}$$

Since the lines λ_{aAB} are coplanar, it follows from Theorem IV of § 1 that

$$\lambda_a^{AC}\lambda_{bAB}\lambda_c^{BD} \equiv 0. \tag{4.5}$$

Multiplying by $\lambda_a^{EF}\lambda_{bGH}\lambda_c^{KL}$, we get the cubic identity

$$d^{ACEF}d_{ABGH}d^{BDKL} \equiv 0. \tag{4.6}$$

Some of these equations must be independent of the linear and quadratic identities, since the latter were deduced from the fact that the lines λ_{aAB} meet two by two (which is true of concurrent as well as of coplaner lines), while (4.6) was deduced from the fact that the lines are coplanar. It may however be noticed that if for example we put $E = G$, $F = H$ in (4.6), we obtain an identity which is a consequence of (4.4).

It is easy to show that, if in (4.6) we take only those equations for which the suffixes C, D have equal values, we obtain the cubic identities given by Todd (his formula (2.47)). The cubic identities given by him do not therefore by themselves form a covariantive set, but are part of the larger set (4.6) which is completely covariantive.

It may be remarked that the quadratic identity obtainable from (4.5) by putting $b = c$ and multiplying by λ_a^{EF} is deducible from the linear identity (2.5*a*).

A set of quartic identities satisfied by d_{ABCD} may be obtained by eliminating λ_A in all possible ways from four of the equations $\lambda_A\lambda_a^{AB} = 0$, which express the fact that the lines λ_a^{AB} all lie in the plane λ_A. Since however we have already used the fact that the lines are coplanar, it is to be expected that these quartic identities will be deducible from the identities already obtained, and so contain nothing essentially new (cf. Todd, *loc. cit.*, page 186).

§ 5. *Degenerate conics.*

The present and subsequent sections of this paper contain a series of theorems on conics as defined by their Cayley-Spottiswoode coordinates, many of these theorems being interpretable in terms of the 19-dimensional representation of Todd.

THEOREM 1. *The necessary and sufficient condition that the conic d_{ABCD} should be a pair of lines is*

$$d_{ACEF}d^{ABGH}d_{BDKL} = 0. \tag{5.1}$$

The condition is necessary. For if the conic is a pair of lines, then

$$d_{ACEF} = \lambda_{aAC}\lambda_{aEF}, \tag{5.2}$$

the summation with respect to a being from 1 to 2. Now by (1.16) and Theorem III (§ 1) Cor. b,

$$\lambda_{aAC}\lambda_b^{AB}\lambda_{cBD} = 0 \tag{5.3}$$

since the suffixes a, b, c, which take the values 1, 2 only, cannot all be different. Multiply by $\lambda_{aEF}\lambda_b^{GH}\lambda_{cKL}$, and we obtain (5.1).

To prove the sufficiency, we assume that (5.1) is true for d given by (5.2), where the suffix a sums from 1 to 3; that is, we assume that the conic may be of a general form. Lowering the suffixes GH in (5.1), multiplying by $h_a^F h_b^H h_c^L$ and using (3.20a), we quickly get (5.3), (a, b, c = 1, 2, 3). So by Theorem III the three lines λ_{aAB} are concurrent as well as coplanar. Consequently λ_{3AB} is of the form $p\lambda_{1AB} + q\lambda_{2AB}$ and d is of the form

$$d_{ABCD} = \lambda_{1AB}\lambda_{1CD} + \lambda_{2AB}\lambda_{2CD} + (p\lambda_{1AB} + q\lambda_{2AB})(p\lambda_{1CD} + q\lambda_{2CD}).$$

It is easy to show that this is reducible to

$$d_{ABCD} = \lambda'_{1AB}\lambda'_{1CD} + \lambda'_{2AB}\lambda'_{2CD},$$

where λ'_{1AB}, λ'_{2AB} are linear combinations of λ_{1AB}, λ_{2AB}. The conic is therefore a pair of lines.

THEOREM 2. *The necessary and sufficient condition that the conic d should be a repeated line is*

$$d_{ABCD}d^{AEFG} = 0. \tag{5.4}$$

For, if the conic is a repeated line, say λ_{AB}, then

$$d_{ABCD} = \lambda_{AB}\lambda_{CD}.$$

Since $\lambda_{AB}\lambda^{AE} = 0$ by (1.16), the necessity of (5.4) at once follows.

The condition is certainly not true if the conic is of a more general form, as may be seen by substituting from (3.35) in (5.4) and using (1.16) and (1.15); we get

$$l_{AB}m^{AE}(l^{FG}m_{CD} - l_{CD}m^{FG}) = 0,$$

which is not a true relation if l and m are different.

An equivalent form of the condition is [13])

$$d_{ABCD}d_{EFGH} = d_{ABGH}d_{EFCD}. \tag{5.4a}$$

§ 6. *Relations between points, lines, planes and conics.*

The condition that a line p_{AB} should meet the conic d_{ABCD} has already been given in (2.1).

[13]) I owe to Dr. TODD the remark that (5.4a) is an alternative form of the condition (5.4).

THEOREM 3. *The necessary and sufficient condition that a point X^A should lie in the plane of the conic d is*[14])

$$d_{ABCD}d^{AEFG}X^B = 0. \tag{6.1}$$

The condition is necessary. For, if X^A lies in the plane of the conic, then $\lambda_{aAB}X^B$, which is the plane through the point X^A and the line λ_{aAB}, and is therefore itself the plane of the conic, contains the line λ_b^{AE}. So $\lambda_{aAB}X^B\lambda_b^{AE} = 0$. Multiplying by $\lambda_{aCD}\lambda_b^{FG}$, we get (6.1).

To establish the sufficiency, assume the truth of (6.1). Lower the suffixes FG, multiply by $h_a^D h_b^G$ and use (3.20a). On dividing by $\lambda_C\lambda_F$ we get $\lambda_{aAB}\lambda_b^{AE}X^B = 0$; that is, the plane $\lambda_{aAB}X^A$ through X^A and any one of the lines λ_a contains all the lines λ_a. So X^A is in the plane of the conic.

THEOREM 4. *The necessary and sufficient condition that two points X^A, Y^A should be conjugate with respect to the conic is*

$$d_{ABCD}X^A Y^C = 0. \tag{6.2}$$

Take the conic in the form (3.34), and let l, m be the tangents from X^A to the conic. Then if X^A, Y^A are conjugate, n passes through Y^A. The necessity of (6.2) follows at once from (1.12).

The condition is also sufficient. For (6.2) may be written

$$\lambda_{aAB}\lambda_{aCD}X^A Y^C = 0, \tag{6.3}$$

so, assuming for the moment that the conic is non-degenerate, we get on multiplying by $\lambda^B\lambda^D$ and using (3.19), (3.9),

$$(g_{AC} - \lambda_A\lambda_C)X^A Y^C = 0,$$

so the points are conjugate with respect to the cone (3.31). Multiply (6.3) by $h_b^B h_c^D$ and use (3.20). We get

$$\delta_{ab}\lambda_A\delta_{ac}\lambda_C X^A Y^C = 0,$$

so either $\lambda_A X^A = 0$ or $\lambda_C Y^C = 0$, that is, either X^A or Y^A lies in the plane of the conic. Suppose X^A does so. Multiplying (6.3) by $\lambda^B h_b^D$, using (3.19) and (3.20), we obtain $h_{aA}\delta_{ab}\lambda_C X^A Y^C = 0$. So either $h_{bA}X^A = 0$ or $\lambda_C Y^C = 0$. The former equation would require X^A to lie in all three of planes h_{aA} and therefore coincide with λ^A, which is not the case since it lies in the plane of the conic. So the latter is true and Y^A also lies in the plane of the conic.

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[14]) The condition (6.1) for a point to lie in the plane of a conic is due to Dr. TODD, who derived it by use of the symbolic calculus and communicated it to me. I had previously obtained the condition in the form $d_{ABCD}d^{AE}{}_{\cdot\cdot KL}X^D X^L = 0$, which, though not linear in X^A, appears to be equally correct.

It is easy to show that the theorem is still true if the conic is a pair of lines.

Corollary. *The necessary and sufficient condition that a point X^A should lie on the conic is*

$$d_{ABCD} X^A X^C = 0. \tag{6.4}$$

Theorem 5. *The necessary and sufficient condition that the point X^A should be the pole with respect to the conic of the line p^{AB} is*

$$d_{ABCD} X^A p^{CE} = 0. \tag{6.5}$$

The proof is similar to that of Theorem 4.

Theorem 6. *If p^{AB} is a line such that*

$$d_{ABCD} p^{CD} = 0, \tag{6.6}$$

then, if the conic d is non-degenerate, p lies in its plane; but if d is a pair of lines, p either lies in their plane or passes through their point of intersection, or both.

For multiplying (6.6) by h^B_a, using (3.20a) and dividing by λ_A, we get $\lambda_{aCD} p^{CD} = 0$. So p meets each of the lines λ_a. Hence if the conic is non-degenerate, $(a = 1, 2, 3)$, p lies in its plane. If the conic is a pair of lines, $(a = 1, 2)$, the line p may be either concurrent or coplanar with the lines λ_a, and hence with the lines which constitute the conic; or it may be both concurrent and coplanar.

If d is a repeated line, p intersects it.

It is easy to see by the use of (1.15) and Theorem IV, Cor. b that alternative forms of the condition are

$$d_{ABCD} p^{DE} + d_{AB}{}^{\cdot\cdot DE}{}_{\cdot\cdot} p_{CD} = 0, \tag{6.7}$$

$$d_{ABCD} p^{CE} p^{DF} = 0. \tag{6.8}$$

Theorem 7. *The necessary and sufficient condition that the line p^{AB} should lie in the plane of the conic is*

$$d^{ABCD} d^{EFGH} p_{AE} = 0. \tag{6.9}$$

For, if p^{AB} lies in the plane of the conic, the lines λ_a, p are coplanar. Hence by Theorem IV, $\lambda^{AB}_a p_{AE} \lambda^{EF}_b = 0$. Multiply by $\lambda^{CD}_a \lambda^{GH}_b$, and the necessity of (6.9) follows at once.

To prove the sufficiency, assume the truth of (6.9). Lower the suffixes CD and GH, multiply by $h^C_a h^G_b$ and use (3.20a). On dividing by $\lambda_D \lambda_H$ we get $\lambda^{AB}_a p_{AE} \lambda^{EF}_b = 0$. Since λ^{AB}_a and λ^{EF}_b intersect, p lies in their plane by Theorem IV, Cor. a. Hence p lies in the plane of the conic.

If the conic is a repeated line, p intersects it.

THEOREM 8. *If p, q are lines such that*

$$d_{ABCD}d_{EFGH}p^{AE}q^{CG}=0, \tag{6.10}$$

then one lies in the plane of the conic and the other passes through its pole with respect to the conic.

Hence, if they both lie in the plane of the conic, they are conjugate lines.

Suppose first that the conic is non-degenerate. By (6.10) and (3.16) we have

$$\lambda_{aAB}\lambda_{aCD}\lambda_{bEF}\lambda_{bGH}p^{AE}q^{CG}=0, \tag{6.11}$$

where a, b sum from 1 to 3. Multiplying by $h^B_c\lambda^D\lambda^F h^H_d$ and using (3.19), (3.20), we quickly get $\lambda_A h_{cC}h_{dE}\lambda_G p^{AE}q^{CG}=0$, whence either $\lambda_A h_{dE}p^{AE}=0$ or $\lambda_G h_{cC}q^{CG}=0$; that is, either $\lambda_{dAE}p^{AE}=0$ or $\lambda_{cGC}q^{GC}=0$. Hence either p or q meets all the lines λ_a, so that either p or q lies in the plane of the conic. Suppose that p does so. Multiplying (6.11) by $\lambda^B\lambda^D\lambda^F\lambda^H$ and using (3.19) and (3.9), we get

$$(g_{AC}-\lambda_A\lambda_C)(g_{EG}-\lambda_E\lambda_G)p^{AE}q^{CG}=0.$$

That is, the polar line of p with respect to the cone (3.31) (which line is unique since p does not pass through the vertex), meets q. So the pole of p with respect to the conic lies on q.

If the conic is a pair of lines, it is easily shown that (6.10) means that either p or q passes through their point of intersection; and if a repeated line, that either p or q meets it.

COROLLARY. *The necessary and sufficient condition that the line p should touch the conic d is*

$$d_{ABCD}d_{EFGH}p^{AE}p^{CG}=0. \tag{6.12}$$

If d is a pair of lines, this means that p passes through their point of intersection. If d is a repeated line, p meets it.

THEOREM 9. *If λ_A is the plane of the conic d, then*

$$d^{ABCD}\lambda_D=0, \tag{6.13}$$

and conversely.

The proof of this follows easily from (1.11), (3.16) and (3.20a).

THEOREM 10. *If a plane φ_A touches the conic d, then*

$$d^{ABCD}d_{BEDF}\varphi_A\varphi_C=0, \tag{6.14}$$

and conversely.

Assume first that the conic is non-degenerate. Let φ_A cut the plane of the conic in the line l_{AB}, so that l_{AB} is a tangent. Let m_{AB} be any other tangent, and let n_{AB} be the chord of contact. Then

$$d^{ABCD} = l^{AB}m^{CD} + l^{CD}m^{AB} + n^{AB}n^{CD},$$

whence
$$d^{ABCD}\varphi_A\varphi_C = n^{AB}\varphi_A n^{CD}\varphi_C$$

since φ_A contains the line l. But $n^{AB}\varphi_A$ is the point where the plane φ_A meets the line n, that is, it is the point of contact of the line l, and therefore lies on the conic. Using (6.4) we get (6.14).

If d is a line-pair, and φ_A "touches" d, that is, if φ_A passes through the point of intersection of the lines d, it is easily shown that (6.14) is satisfied. If d is a repeated line, (6.14) is an identity by (5.4).

To prove the converse, assume (6.14) to be true for a plane φ_A, and suppose for the moment that d is non-degenerate. By (3.16) and (3.17) we may write (6.14) in the form

$$\begin{aligned} 0 &= (\lambda_B h_{aE} - \lambda_E h_{aB})\lambda_b^{AB}(\lambda_D h_{aF} - \lambda_F h_{aD})\lambda_b^{CD}\varphi_A\varphi_C \\ &= \lambda_E\lambda_F h_{aB}\lambda_b^{AB}h_{aD}\lambda_b^{CD}\varphi_A\varphi_C \end{aligned}$$

by (3.21). Dividing by $\lambda_E\lambda_F$ and using (3.22), we get

$$g\varepsilon_{abc}h_c^A\varepsilon_{abd}h_d^C\varphi_A\varphi_C = 0$$

or, since $\varepsilon_{abc}\varepsilon_{abd} = 2\delta_{cd}$, we get by (3.15) the equation

$$(g^{AC} - \lambda^A\lambda^C)\varphi_A\varphi_C = 0.$$

That is, the plane φ_A touches the cone (3.32) and hence also touches the conic.

If d is a pair of distinct lines l, m, it may be expressed in the form (3.35). Substituting in (6.14) and using (1.16) and (1.15), we quickly deduce that either $\varphi_A l^{AB}m_{BE} = 0$ or $\varphi_C m^{CD}l_{DF} = 0$. The former equation means that the point $\varphi_A l^{AB}$ in which φ_A meets l lies on m, and the latter that the point in which φ_A meets m lies on l. So φ_A passes through the point of intersection of l, m.

§ 7. *Relations between two conics.*

In the present section certain invariant relations are found between two conics d and d'. For d we take the general forms (3.16), (3.33), and for d' the corresponding formulae

$$d'_{ABCD} = \lambda'_{aAB}\lambda'_{aCD}, \tag{7.1}$$

$$d'_{ABCD} = \varrho'(l'_{AB}m'_{CD} + l'_{CD}m'_{AB}) + \sigma' n'_{AB}n'_{CD}. \tag{7.2}$$

It is implicitly assumed in (7.1) that the triangle λ'_a self-polar with respect to d' is not the same as the triangle λ_a self-polar with respect to d. In dealing with coplanar conics it is of course possible to refer them to a common self-polar triangle, in which case, if (3.16) were taken for d, it would be necessary to take d' in the form $g_{ab}\lambda_{aAB}\lambda_{aCD}$, where the g_{ab} are numbers such that $g_{ab} = 0$ when $a \neq b$.

Symbols such as h'^A_a used below bear to d' the same relation as the corresponding unaccented symbols bear to d.

An identity.

If d, d' are any two conics, then

$$d_{ABCD}d'^{AEFG} + d^{AE}{}_{..CD}d'_{AB}{}^{FG}{}_{..} \equiv \tfrac{1}{2}d_{HKCD}d'^{HKFG}\delta^E_B. \tag{7.3}$$

For by (1.14),

$$\lambda_{aAB}\lambda'^{AE}_b + \lambda^{AE}_a\lambda'_{bAB} \equiv \tfrac{1}{2}\lambda_{aHK}\lambda'^{HK}_b\delta^E_B$$

for any two lines λ_a, λ'_b. Multiplying by $\lambda_{aCD}\lambda'^{FG}_b$, the identity follows at once.

THEOREM 11. *If d, d' are two conics such that*

$$d_{AB}{}^{CD}{}_{..}d'_{CDEF} = 0, \tag{7.4}$$

then either they are coplanar or they are pairs of lines meeting in the same point; and conversely.

Suppose first that neither conic is a repeated line. Multiply (7.4) by $h^B_a h'^F_b$ and use (3.20a), and we quickly obtain $\lambda^{CD}_a\lambda'_{bCD} = 0$; so each of the lines λ_a meets each of the lines λ'_a. Hence, unless both conics are degenerate, they are coplanar. If both are degenerate, the four lines λ_a, λ'_a may be coplanar or concurrent, and hence also the four lines which constitute the two conics.

If either conic is a repeated line, it follows at once from (6.1) that the conics are coplanar.

The converse easily follows from (1.10) and (3.16).

For non-degenerate conics, (7.4) is evidently a necessary and sufficient condition that they should be coplanar.

THEOREM 12. *The necessary and sufficient condition that two conics d, d' should be coplanar is*

$$d^{ABCD}d^{EFGH}d'_{AEKL} = 0 \tag{7.5}$$

or

$$d'^{ABCD}d'^{EFGH}d_{AEKL} = 0. \tag{7.5a}$$

If either of the relations (7.5), (7.5a) is true, then also is the other.

Condition (7.5) is necessary. For if d' lies in the plane of d, so does each of the lines λ'_a to which d' is referred. Hence by Theorem 7,

$$d^{ABCD} d^{EFGH} \lambda'_{aAE} = 0. \tag{7.6}$$

Multiplying by λ'_{aKL} we get (7.5).

To establish the sufficiency assume that (7.5) is true. If d' is a repeated line, it follows at once from Theorem 7 that this line lies in the plane of d. If d' is not a repeated line, multiply (7.5) by h'^L_a and use (3.20a). We get (7.6), which means that each line λ'_a lies in the plane of d. Hence the conics are coplanar.

Interchanging d, d', it is evident that the condition (7.5a) is equivalent to (7.5), and that the one relation must be deducible from the other.

THEOREM 13. *If d, d' are two conics such that* [15])

$$d^{ABCD} d'_{ABCD} = 0, \tag{7.7}$$

then either (I) *they are coplanar, or* (II) *they are non-coplanar and cut the line of intersection of their planes in harmonically conjugate points, or* (III) *one conic is a pair of lines of which one lies in the plane of the other conic.*

Special cases of (II), for which the theorem is conventionally true, are: (IV) one of the conics touching the line of intersection of their planes and the other passing through the point of contact; (V) one of the conics a pair of lines meeting on the other conic; (VI) one of them a repeated line intersecting the other conic.

Possibility (I) follows at once from (7.4) by raising the suffixes AB and changing them into EF, which gives (7.7).

Suppose then that the conics are not coplanar. Assume for the moment that neither touches the line of intersection of their planes, that neither is a repeated line, and that, if either is a pair of distinct lines, then neither of these lines lies in the plane of the other conic. Then if n_{AB} is the line of intersection of their planes, we may take d in the form

$$d^{ABCD} = l^{AB} m^{CD} + l^{CD} m^{AB} + n^{AB} n^{CD},$$

with the last term absent if d is a line-pair. We refer d' to a self-polar triangle. Take n to be one of its sides, choose a second, μ say, to pass through the intersection of m and n, and let λ be the third side. Then by (3.16),

$$d'_{ABCD} = \lambda_{AB} \lambda_{CD} + \mu_{AB} \mu_{CD} + n_{AB} n_{CD},$$

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[15]) Cf TODD, loc. cit., 190; also R. A. JOHNSON [Trans. Amer. Math. Soc. **15** (1914), 354].

with the last term absent if d' is a line-pair. Now since n is common to both planes it meets all the other lines, and by supposition μ meets m. Hence by (1.10) and (7.7),

$$0 = d^{ABCD}d'_{ABCD} = 2(\lambda l)(\lambda m)$$

in the notation of (1.13). So either $(\lambda l) = 0$ or $(\lambda m) = 0$ or both; that is, λ meets either l or m or both. If λ meets l but not m, then, since the triangle $(\lambda\mu n)$ is self-polar with respect to the conic d' (or since λ, μ separate the lines d' harmonically when d' is a line-pair), the conics cut n in harmonically conjugate points; for it will be remembered that d passes through the intersections of l, m with n. If however λ meets m, then since μ also meets it, we obtain (V); and if λ meets both l and m we get (V) again.

It is easy to prove that the only remaining possibilities are (III), (IV) and (VI), the last being indeed obvious from (2.1). It is also easy to prove the converse, namely that conics satisfying any one of the conditions (I)—(VI) satisfy (7.7).

Before proceeding further, it will be convenient to introduce an abbreviated notation for certain expressions which occur in the following theorems. If d, d' are two conics, we write

$$\Phi^{AB}_{CD} \equiv d^{ABEF}d'_{EFCD}, \tag{7.8}$$

$$\Phi \equiv \Phi^{AB}_{AB} \equiv d^{ABCD}d'_{ABCD}. \tag{7.9}$$

Since Φ^{AB}_{CD} is introduced purely as an abbreviation, its lack of symmetry as between the two conics is of no consequence. It could be made symmetrical if desired by lowering the upper pair of suffixes. The invariant defined by (7.9) is the well-known invariant Φ of two quadrics (in this case degenerate).

With this notation, the conditions (7.4), (7.7) may be written $\Phi^{AB}_{CD} = 0$, $\Phi = 0$.

THEOREM 14. *The necessary and sufficient condition that the two conics d, d' should intersect is*

$$\Phi^{AB}_{CD}\Phi^{CD}_{AB} = \tfrac{1}{2}\Phi^2. \tag{7.10}$$

The necessity of the condition is easily proved: if the conics are coplanar, (7.10) follows at once from (7.4). If they are not coplanar, but meet in a point P on the line of intersection n of their planes, then, provided that neither touches n, they may be taken in the forms

$$d^{ABEF} = \varrho(l^{AB}m^{EF} + l^{EF}m^{AB}) + \sigma n^{AB}n^{EF}, \tag{7.11}$$

$$d'_{EFCD} = \varrho'(l'_{EF}m'_{CD} + l'_{CD}m'_{EF}) + \sigma' n_{EF}n_{CD}, \tag{7.12}$$

and we may assume that l, l' meet in P. From this and the fact that n meets all the other lines, (7.10) follows from (1.10). If either conic touches n, a similar proof shows that (7.10) still holds provided that the conics intersect.

To establish the sufficiency of the condition, we assume that (7.10) is true. Then obviously d, d' may be coplanar, and if so intersect. Suppose however that they are not coplanar and that neither touches the line of intersection n of their planes. Then we may take d, d' in the forms (7.11), (7.12), where n meets l, l', m, m'. Using (1.10), we get

$$\Phi^{AB}_{CD} = \varrho\varrho'[l^{AB}m'_{CD}(l'm) + l^{AB}l'_{CD}(mm') + \\ + m^{AB}m'_{CD}(ll') + m^{AB}l'_{CD}(lm')], \quad (7.13)$$

whence $$\Phi = 2\varrho\varrho'[(ll')(mm') + (lm')(l'm)] \quad (7.14)$$

and

$$\Phi^{AB}_{CD}\,\Phi^{CD}_{AB} = 2\varrho^2\varrho'^2[(ll')^2(mm')^2 + (lm')^2(l'm)^2 + \\ + 6(ll')(mm')(lm')(l'm)], \quad (7.15)$$

so by (7.10),

$$\varrho^2\varrho'^2(ll')(mm')(lm')(l'm) = 0.$$

Hence at least one of the following equations is true: $\varrho = 0$, $\varrho' = 0$, $(ll') = 0$, $(mm') = 0$, $(lm') = 0$, $(l'm) = 0$. If it is remembered that the vanishing of the inner product of the coordinates of two lines means that the lines intersect, it is at once evident that the conics d, d' must meet in at least one point. A similar proof holds if either conic touches the line of intersection of their planes.

THEOREM 15. *The necessary and sufficient condition that the conics d, d' should meet in at least two points is*

$$\Phi^{AB}_{CD}\,\Phi^{CD}_{EF} = \tfrac{1}{2}\Phi\,\Phi^{AB}_{EF}. \quad (7.16)$$

This is easily proved by methods similar to those adopted above. The two points of intersection may of course be coincident.

THEOREM 16. *If*

$$\Phi^{AB}_{CD}\,\Phi^{CD}_{AB} = \Phi^2, \quad (7.17)$$

then either the conics are coplanar or one at least touches the line of intersection of their planes; and conversely [16]).

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[16]) That the condition (7.17) might have the meaning stated was suggested to me by Dr. Todd.

(A line-pair "touches" a given line if the three lines are concurrent.)

That the conics may be coplanar is obvious from (7.4). Suppose that they lie in different planes, and that one, say d, meets the line of intersection n of their planes in distinct points P, Q. Let l, m be the tangents to it at these points. Then

$$d^{ABEF} = l^{AB}m^{EF} + l^{EF}m^{AB} + n^{AB}n^{EF}$$

with the last term absent if d is a line-pair. For d' take

$$d'_{EFCD} = l'_{EF}m'_{CD} + l'_{CD}m'_{EF} + n'_{EF}n'_{CD},$$

where l', m' meet in P and n' is the polar of P with respect to d'. Then n meets all the other lines and l meets l', m'. Hence $\Phi = 2(ln')(mn')$ and $\Phi^{AB}_{CD}\Phi^{CD}_{AB} = 4(ln')^2[(mn')^2 + (mm')(l'm)]$, so (7.17) gives $(ln')^2(mm')(l'm) = 0$. Hence at least one of the following statements is true: (I) n' meets l, (II) m' meets m, (III) l' meets m. If (I) is true, d' is a pair of lines meeting on n. In case (II), m' coincides with n so n touches d', and similarly in case (III).

Other cases not included in the above proof are: d' a general conic passing through P; d' a line-pair meeting on n but not at P. These may be treated separately.

The proof of the converse presents no difficulty.

§ 8. *Conclusion.*

Every theorem given above has a dual, so that a similar theory of cones in 3-space is easily deducible. Both theories may be included in a more general one, namely that of quadrics in 3-space, a quadric being definable in terms of the quadratic complex of lines which touch it.

This remark explains some peculiarities which appear in the present paper. It might have been expected, for example, that any covariantive relation expressing a geometrical relationship between two non-degenerate conics would have the same meaning for degenerate conics. This however is not always the case: thus in Theorem 11 a condition that a pair of non-degenerate conics should be coplanar means, when the conics are both degenerate, that they are *either* coplanar *or* possess a common point of intersection. This is due to the fact that, from the point of view of the present paper, a pair of lines is a degenerate cone as well as a degenerate conic.

Consideration of the present theory from the more general standpoint will be deferred to a later paper. It seemed advisable to begin with the special case of conics for two reasons: first, that the generalised theory of quadrics is thereby rendered more illuminating, and secondly, that conics (and cones) have many properties not possessed by proper quadrics, so that it is desirable to consider such special properties before treating conics and cones merely as degenerate quadrics.

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