COMPOSITIO MATHEMATICA

T. Y. THOMAS

Fields of parallel vectors in the large

Compositio Mathematica, tome 3 (1936), p. 453-468
<http://www.numdam.org/item?id=CM_1936__3__453_0>

Fields of Parallel Vectors in the Large

by

T. Y. Thomas

Princeton, N.J.

1. Let S be a topological manifold homeomorphic to the real n-dimensional number space. By this homeomorphism there is defined a system of coordinates $x^1, \ldots, x^n$ in S. The manifold S becomes an *affinely connected space* by the introduction of a general affine connection L whose components $L^\alpha_{\beta\gamma}$ are functions of the coordinates of S. We shall assume in the following that the $L^\alpha_{\beta\gamma}(x)$ are analytic functions of the coordinates.

A field of contravariant vectors ξ defined over S is said to be parallel if the components $\xi\,(x)$ of the vectors are continuous and constitute a non-trivial solution of the system

$$(1) \qquad \frac{\partial \xi^\alpha}{\partial x^\beta} = - \sum_{\mu=1}^{n} L^\alpha_{\mu\beta}\, \xi^\mu, \qquad (\alpha,\ \beta = 1, \ldots, n),$$

over S. In this paper we seek the *algebraic characterization* of affinely connected spaces S and their generalization (§ 10) which admit one or more fields of parallel contravariant vectors [1]). Analogous considerations will of course apply in the case of fields of parallel covariant vectors.

2. From (1) we deduce the infinite sequence of necessary conditions

$$(2) \qquad \begin{cases} \sum_{\mu=1}^{n} \xi^\mu\, B^\alpha_{\mu\beta\gamma} = 0, \\ \sum_{\mu=1}^{n} \xi^\mu\, B^\alpha_{\mu\beta\gamma,\,\delta} = 0, \\ \sum_{\mu=1}^{n} \xi^\mu\, B^\alpha_{\mu\beta\gamma,\,\delta,\,\varepsilon} = 0, \\ \cdots\cdots\cdots \\ \cdots\cdots\cdots \end{cases}$$

[1]) Concerning the algebraic characterization of spaces admitting other properties, see T. Y. Thomas, Algebraic characterizations in complex differential geometry [Trans. Am. Math. Soc. **38** (1935), 501—514]; On the metric representations of affinely connected spaces [Bull. Am. Math. Soc. (1936), 77]. Also Riemann spaces of class one and their characterization [to appear in the Acta Mathematica].

where the $B^{\alpha}_{\mu\beta\gamma}$ are the components of the curvature tensor and the quantities $B^{\alpha}_{\mu\beta\gamma,\delta}$, $B^{\alpha}_{\mu\beta\gamma,\delta,\varepsilon}$, ... are the components of the successive covariant derivatives of the curvature tensor [2]). In fact the first set of equations (2) follows directly as the conditions of integrability of (1) and any set after the first results by covariant differentiation of the set immediately preceding it in the sequence, account being taken of the equations (1) which express the vanishing of the covariant derivative of the vector ξ.

Consider in particular the first $n+1$ sets of equations of the sequence (2), namely the equations

$$(\mathrm{E}_0) \qquad \sum_{\mu=1}^{n} \xi^{\mu} B^{\alpha}_{\mu\beta\gamma} = 0,$$

$$(\mathrm{E}_1) \qquad \sum_{\mu=1}^{n} \xi^{\mu} B^{\alpha}_{\mu\beta\gamma,\delta_1} = 0,$$

. . . .

. . . .

$$(\mathrm{E}_n) \qquad \sum_{\mu=1}^{n} \xi^{\mu} B^{\alpha}_{\mu\beta\gamma,\delta_1,\ldots,\delta_n} = 0.$$

We assume the existence of a non-trivial solution of the above equations at any point of S, this being expressible by the vanishing of the resultant system R of the equations over S. Let P be any point of S and $N(P)$ any neighborhood of P. Suppose that the matrix of the equations (E_0) has its maximum rank r_0 in $N(P)$ at a point P_0; similarly that the matrix of the system (E_0) and (E_1) has its maximum rank r_1 in $N(P)$ at P_1; ... and finally that the entire system composed of $(E_0), \ldots, (E_n)$ has its maximum rank r_n in $N(P)$ at P_n. Then in passing from $P_0 \to P_1 \to \ldots \to P_n$ the above ranks r_i can not always increase, i.e. we can not have $r_0 < r_1 < \ldots < r_n$; in fact it would follow from these inequalities that $r_\alpha = \alpha$ for $\alpha = 0, 1, \ldots, n$ and the condition $r_n = n$ is in contradiction with the above hypothesis concerning the existence of a non-trivial solution of the system $(\mathrm{E}_0), \ldots, (\mathrm{E}_n)$ at the point P_n. Hence $r_0 < r_1 < \ldots < r_i = r_{i+1}$ for some value of the integer $i \leqq n-1$. Let Δ be a determinant of order r_i formed from the matrix of the system $(\mathrm{E}_0), \ldots, (\mathrm{E}_i)$ which does not vanish at P_i. Then Δ will not vanish in a neighborhood $N(P_i) \subset N(P)$ in which the ranks of the matrices of

[2]) See T. Y. Thomas, Differential Invariants of Generalized Spaces [Cambridge University Press, 1934], § 13. This book may be consulted on questions of tensor analysis which may arise in the present paper.

the systems $(E_0), \ldots, (E_i)$ and $(E_0), \ldots, (E_{i+1})$ will be the same. Hence the equations (E_{i+1}) will be linearly dependent on the equations $(E_0), \ldots, (E_i)$ in the neighborhood $N(P_i)$, i.e. we have

$$(3) \quad B^{\alpha}_{\mu\beta\gamma,\delta_1,\ldots,\delta_{i+1}} = \Sigma\, \Phi^{\alpha bc}_{a\beta\gamma\delta_1\ldots\delta_{i+1}} B^{a}_{\mu bc} + \ldots + \\ + \Sigma\, \Phi^{\alpha bcd_1\ldots d_i}_{a\beta\gamma\delta_1\ldots\delta_{i+1}} B^{a}_{\mu bc,d_1,\ldots,d_i}$$

in $N(P_i)$ where the Φ's are either zero or rational functions of the $B^{\alpha}_{\mu\beta\gamma}, \ldots, B^{\alpha}_{\mu\beta\gamma,\delta_1,\ldots,\delta_{i+1}}$ for each of which the denominator is the above determinant Δ. On account of the above hypothesis of analyticity the Φ's are therefore analytic functions of the coordinates in the neighborhood $N(P_i)$.

Now take any point Q of S distinct from P_i and join the points P_i and Q by an analytic curve C. Let C be defined by equations of the form $x^{\alpha} = f^{\alpha}(t)$, the f^{α} being analytic functions of the variable t in the closed interval $0 \leqq t \leqq 1$ such that $f^{\alpha}(0) = x^{\alpha}_0$ and $f^{\alpha}(1) = x^{\alpha}_1$ where x^{α}_0 and x^{α}_1 are the coordinates of the points P_i and Q respectively. Thus as t varies from $t = 0$ to $t = 1$ the point whose coordinates are $f^{\alpha}(t)$ moves along C from the point P_i to the point Q. Such a curve C is given for example by the „straight line" joining P_i to Q defined by the equations

$$x^{\alpha} = x^{\alpha}_0 + (x^{\alpha}_1 - x^{\alpha}_0)t \qquad (0 \leqq t \leqq 1).$$

Now consider the equations

$$(4) \qquad \frac{d\xi^{\alpha}}{dt} = -\Sigma\, L^{\alpha}_{\mu\beta}(f(t))\xi^{\mu}\frac{df^{\beta}}{dt}$$

which define the parallel displacement along the curve C of the vector ξ_0 with components ξ^{α}_0 at the point P_i. *We choose the values* ξ^{α}_0 *so that at the point* P_i *the system* $(E_0), \ldots, (E_n)$ *is satisfied.* By the well known existence theorem the equations (4) will have a unique solution $\xi^{\alpha}(t)$ such that $\xi^{\alpha}(0) = \xi^{\alpha}_0$ where the functions $\xi^{\alpha}(t)$ are analytic in the above interval $0 \leqq t \leqq 1$. If the equations (1) admit a continuous solution $\xi^{\alpha}(x)$ in the space S which assumes at the point P_i the above values ξ^{α}_0 then at the point Q the functions $\xi^{\alpha}(x)$ must have the values at $t = 1$ of the above solution $\xi^{\alpha}(t)$ of the equations (4). Hence the equations (1) can admit at most, one such solution $\xi^{\alpha}(x)$ having the values ξ^{α}_0 at the point P_i.

3. In conformity with the above considerations let us seek to define continuous functions $\xi^\alpha(x)$ satisfying (1) by the procedure of parallel displacement of the vector ξ_0 at P_i along analytic curves C to the various points of S. This procedure will define a set of functions $\xi^\alpha(x)$ over S if, and only if, the values of the solutions $\xi^\alpha(t)$ of the equations (4) at the arbitrary point Q are independent of the analytic curve over which these equations are integrated. Let us therefore join the points P_i and Q by an analytic curve C_1 analogous to the curve C above considered, i.e. we assume that C_1 is defined by equations of the form $x^\alpha = f_1^\alpha(t)$ where the $f_1^\alpha(t)$ are analytic functions of t in the interval $0 \leqq t \leqq 1$ such that $f_1^\alpha(0) = x_0^\alpha$ and $f_1^\alpha(1) = x_1^\alpha$. Now consider an analytic surface Ω defined by equations of the form $x^\alpha = G^\alpha(t, p)$ where the functions G^α are analytic in the closed intervals $0 \leqq t \leqq 1$ and $0 \leqq p \leqq 1$ such that $G^\alpha(t, 0) = f^\alpha(t)$ and $G^\alpha(t, 1) = f_1^\alpha(t)$. Also $G^\alpha(0, p) = x_0^\alpha$ and $G^\alpha(1, p) = x_1^\alpha$ for all values of the variable p. Thus the surface Ω is generated by a family F of analytic curves joining the points P_i and Q analogous to the curves C and C_1 and in fact containing these latter two curves. That such surfaces Ω exist is seen from the fact that we may take

$$G^\alpha(t, p) = (1-p) f^\alpha(t) + p f_1^\alpha(t) \qquad \begin{pmatrix} 0 \leqq t \leqq 1 \\ 0 \leqq p \leqq 1 \end{pmatrix}.$$

Consider the system

$$\frac{d\xi^\alpha}{dt} = -\sum L^\alpha_{\mu\beta}(G(t, p))\,\xi^\mu \frac{\partial G^\beta}{\partial t} \tag{5}$$

along any curve $p = \text{const.}$ on the surface Ω. Taking $\xi^\alpha = \xi_0^\alpha$ for $t = 0$, independently of the parameter p, the equations (5) have a unique solution $\xi^\alpha(t, p)$ analytic in the above intervals of the variables t and p. Now form the equations

$$\frac{\partial \xi^\alpha}{\partial t} = -\sum L^\alpha_{\mu\beta}\,\xi^\mu \frac{\partial G^\beta}{\partial t}, \tag{6a}$$

$$\frac{\partial \xi^\alpha}{\partial p} = -\sum L^\alpha_{\mu\beta}\,\xi^\mu \frac{\partial G^\beta}{\partial p} + \sigma^\alpha, \tag{6b}$$

the functions σ^α being defined over Ω by these latter equations. By differentiation of (6a) and (6b) with respect to p and t respectively and elimination of the left members of the resulting equations we have

$$\frac{\partial \sigma^\alpha}{\partial t} = -\sum L^\alpha_{\mu\beta}\,\sigma^\mu \frac{\partial G^\beta}{\partial t} - \sum B^\alpha_{\mu\beta\gamma}\,\xi^\mu(t, p) \frac{\partial G^\beta}{\partial t}\frac{\partial G^\gamma}{\partial p} \tag{7}$$

over the surface Ω. Now the last set of terms in these equations involves the left members of the equations (E_0) which were taken to be satisfied initially, i.e. at the point P_i. We shall show that the equations (E_0) are in fact satisfied over Ω by the above functions $\xi^\alpha(t, p)$ obtained as solutions of the system (5). For this purpose we differentiate the left members of (E_0) with respect to the variable t so as to obtain

$$\frac{\partial \Sigma (\xi^\mu B^\alpha_{\mu\beta\gamma})}{\partial t} = \sum \left[\frac{\partial B^\alpha_{\mu\beta\gamma}}{\partial x^{\delta_1}} - B^\alpha_{\varepsilon\beta\gamma} L^\varepsilon_{\mu\delta_1}\right] \xi^\mu \frac{\partial G^{\delta_1}}{\partial t},$$

$$= \sum \left[\frac{\partial B^\alpha_{\mu\beta\gamma}}{\partial x^{\delta_1}} + B^\varepsilon_{\mu\beta\gamma} L^\alpha_{\varepsilon\delta_1} - B^\alpha_{\varepsilon\beta\gamma} L^\varepsilon_{\mu\delta_1} - B^\alpha_{\mu\varepsilon\gamma} L^\varepsilon_{\beta\delta_1} - B^\alpha_{\mu\beta\varepsilon} L^\varepsilon_{\gamma\delta_1}\right] \xi^\mu \frac{\partial G^{\delta_1}}{\partial t}$$

$$- \sum [B^\varepsilon_{\mu\beta\gamma} L^\alpha_{\varepsilon\delta_1} - B^\alpha_{\mu\varepsilon\gamma} L^\varepsilon_{\beta\delta_1} - B^\alpha_{\mu\beta\varepsilon} L^\varepsilon_{\gamma\delta_1}]\, \xi^\mu \frac{\partial G^{\delta_1}}{\partial t},$$

$$= \sum (\xi^\mu B^\alpha_{\mu\beta\gamma,\delta_1}) \frac{\partial G^{\delta_1}}{\partial t} - \sum (\xi^\mu B^\varepsilon_{\mu\beta\gamma}) L^\alpha_{\varepsilon\delta_1} \frac{\partial G^{\delta_1}}{\partial t}$$

$$+ \sum (\xi^\mu B^\alpha_{\mu\varepsilon\gamma}) L^\varepsilon_{\beta\delta_1} \frac{\partial G^{\delta_1}}{\partial t} + \sum (\xi^\mu B^\alpha_{\mu\beta\varepsilon}) L^\varepsilon_{\gamma\delta_1} \frac{\partial G^{\delta_1}}{\partial t}.$$

We observe that the right members of these equations consist of expressions linear and homogeneous in the quantities appearing in the left members of equations (E_0) and (E_1) with coefficients analytic over the surface Ω. Differentiating the left members of equations (E_1) with respect to the variable t we find in a similar manner that these derivatives are equal to the quantities in the left members of (E_2) multiplied by $\frac{\partial G^{\delta_2}}{\partial t}$ plus linear homogeneous expressions in the left members of (E_1) with coefficients analytic over Ω. Analogous remarks apply to the derivatives of the left members of the remaining equations (E_2), ..., (E_n). Borrowing a convenient notation from algebra we may express the set of equations at which we thus arrive in the following form

$$(E_0') \quad \frac{\partial \Sigma (\xi^\mu B^\alpha_{\mu\beta\gamma})}{\partial t} = \sum (\xi^\mu B^\alpha_{\mu\beta\gamma,\delta_1}) \frac{\partial G^{\delta_1}}{\partial t}, \quad \text{mod } (\Sigma \xi^\mu B^\alpha_{\mu\beta\gamma}),$$

$$(E_1') \quad \frac{\partial \Sigma (\xi^\mu B^\alpha_{\mu\beta\gamma,\delta_1})}{\partial t} = \sum (\xi^\mu B^\alpha_{\mu\beta\gamma,\delta_1,\delta_2}) \frac{\partial G^{\delta_2}}{\partial t}, \quad \text{mod } (\Sigma \xi^\mu B^\alpha_{\mu\beta\gamma,\delta_1}),$$

. . . .

$$(E_n') \quad \frac{\partial \Sigma (\xi^\mu B^\alpha_{\mu\beta\gamma,\delta_1,\ldots,\delta_n})}{\partial t} = \sum (\xi^\mu B^\alpha_{\mu\beta\gamma,\delta_1,\ldots,\delta_{n+1}}) \frac{\partial G^{\delta_{n+1}}}{\partial t},$$

$$\text{mod } (\Sigma \xi^\mu B^\alpha_{\mu\beta\gamma,\delta_1,\ldots,\delta_n}).$$

In consequence of the relations (3) the equations $(\mathrm{E}_0'), \ldots, (\mathrm{E}_i')$ now become

$$(\mathrm{E}_0'') \quad \frac{\partial \Sigma (\xi^\mu B^\alpha_{\mu\beta\gamma})}{\partial t} = \sum (\xi^\mu B^\alpha_{\mu\beta\gamma,\delta_1}) \frac{\partial G^{\delta_1}}{\partial t}, \qquad \text{mod } (\Sigma\, \xi^\mu B^\alpha_{\mu\beta\gamma}),$$

. . . .

$$(\mathrm{E}_{i-1}'') \quad \frac{\partial \Sigma (\xi^\mu B^\alpha_{\mu\beta\gamma,\delta_1,\ldots,\delta_{i-1}})}{\partial t} = \sum (\xi^\mu B^\alpha_{\mu\beta\gamma,\delta_1,\ldots,\delta_i}) \frac{\partial G^{\delta_i}}{\partial t},$$

$$\text{mod } (\Sigma\, \xi^\mu B^\alpha_{\mu\beta\gamma,\delta_1,\ldots,\delta_{i-1}}),$$

$$(\mathrm{E}_i'') \quad \frac{\partial \Sigma (\xi^\mu B^\alpha_{\mu\beta\gamma,\delta_1,\ldots,\delta_i})}{\partial t} = 0, \ \text{mod } (\Sigma \xi^\mu B^\alpha_{\mu\beta\gamma}, \ldots, \Sigma \xi^\mu B^\alpha_{\mu\beta\gamma,\delta_1,\ldots,\delta_i})$$

over the portion of Ω contained in the above neighborhood $N(P_i)$. It follows from the existence theorem for equations of the type $(\mathrm{E}_0''), \ldots, (\mathrm{E}_i'')$ and the fact that the equations $(\mathrm{E}_0), \ldots, (\mathrm{E}_i)$ are satisfied at the point P_i that the equations $(\mathrm{E}_0), \ldots, (\mathrm{E}_i)$ are satisfied over the portion of Ω contained in $N(P_i)$. But since the left members of $(\mathrm{E}_0), \ldots, (\mathrm{E}_i)$ are analytic functions over Ω the equations $(\mathrm{E}_0), \ldots, (\mathrm{E}_i)$ must likewise be satisfied over the entire surface Ω. In particular the last set of terms in (7) must vanish identically over Ω so that these equations become

$$(8) \qquad \frac{\partial \sigma^\alpha}{\partial t} = - \sum L^\alpha_{\mu\beta}\, \sigma^\mu \frac{\partial G^\beta}{\partial t}.$$

Since $\xi^\alpha(0, p) = \xi^\alpha_0$ and $G^\alpha(0, p) = x^\alpha_0$ independently of p it follows from (6b) that $\sigma^\alpha = 0$ for $t = 0$. Hence it follows from (8) that $\sigma^\alpha = 0$ along any curve $p = \text{const.}$ on Ω, i.e. $\sigma^\alpha = 0$ over the surface Ω so that in particular $\sigma^\alpha = 0$ at the point Q. Then from (6b) and the fact that $G^\alpha(1, p) = x^\alpha_1$ independently of p we have that the derivatives $\dfrac{\partial \xi^\alpha}{\partial p}$ vanish at the point Q. Hence the values of the quantities $\xi^\alpha(t)$ at Q are independent of the analytic curve of integration by which the point P_i is joined to the point Q. *In other words there exists a set of functions $\xi^\alpha(x)$ defined throughout the space S, these functions being uniquely determined by the assignment of their values ξ^α_0 chosen so as to satisfy the equations $(\mathrm{E}_0), \ldots, (\mathrm{E}_n)$ at the point P_i and the process of integration of the system* (4) *along analytic curve C issuing from P_i.*

4. In the above considerations the values of the functions $\xi^\alpha(x)$ at the arbitrary point Q of the space S were thought of as being determined by integration of the equations (4) along analytic curves C issuing from P_i when the parameter t of the curve was chosen so as to have the value zero at P_i and the value one at the point Q. We see immediately however that the values of the quantities ξ^α at Q are independent of the parameterization of the curve C; it follows therefore that the functions $\xi^\alpha(t)$ obtained by integration of (4) along any analytic curve C issuing from P_i and for any parameterization of this curve have the same values as the above functions $\xi^\alpha(x)$ at corresponding points of the space S.

Now consider the values $\xi^\alpha(q)$ of the above functions $\xi^\alpha(x)$ at any point Q of S. *Starting with the point Q and the values $\xi^\alpha(q)$ the integration of the equations* (4) *along analytic curves C issuing from the point Q will determine the same functions $\xi^\alpha(x)$ over the space S.* This follows from the analysis of the preceding section. Join the point Q to an arbitrary point Q' of S by any analytic curve C' defined by $x^\alpha = F^\alpha(p)$, $0 \leqq p \leqq 1$, such that $F^\alpha(0) = x_1^\alpha$ and $F^\alpha(1) = x_2^\alpha$ where x_1^α and x_2^α are the coordinates of the points Q and Q' respectively. Let Ω^* be the analytic surface generated by the straight lines joining P_i to the points of the curve C'; such a surface Ω^* will be given by equations of the form $x^\alpha = G^\alpha(t,p)$ where

$$G^\alpha(t, p) = x_0^\alpha + [F^\alpha(p) - x_0^\alpha]t, \qquad \begin{pmatrix} 0 \leqq t \leqq 1 \\ 0 \leqq p \leqq 1 \end{pmatrix},$$

the coordinates of the point P_i being x_0^α. By integration of the equations (5) along curves $p = \text{const.}$ on Ω^* we then define analytic functions $\xi^\alpha(t, p)$ over Ω^* such that $\xi^\alpha(0, p) = \xi_0^\alpha$ independently of the parameter p. Forming the equations (6a) and (6b) we proceed as before to the derivation of equations (8) from which it follows that $\sigma^\alpha = 0$ over Ω^*. Hence (6b) reduces to

$$\frac{\partial \xi^\alpha}{\partial p} = -\sum L^\alpha_{\mu\beta} \xi^\mu \frac{\partial G^\beta}{\partial p}.$$

But these latter equations for $t = 1$ define the parallel displacement of the vector $\xi(q)$ at the point Q to the point Q' along the curve C'; from this fact follows the validity of the above italicized statement.

5. We shall show that the above functions $\xi^\alpha(x)$ are analytic functions of the coordinates x^α in the space S. Taking Q to be any point of S we consider the totality of straight lines l issuing from Q, these lines being referred to a parameter t so chosen as to have the value zero at the point Q. The straight lines l are thus given by equations of the form $x^\alpha = x_1^\alpha + a^\alpha t$ where the x_1^α are the coordinates of the point Q and the a^α are constants for a particular line l. Now along a line l the functions $\xi^\alpha(x)$ satisfy the differential equations

$$\frac{d\xi^\alpha}{dt} = -\sum L^\alpha_{\mu\beta}(x_1 + at)\xi^\mu a^\beta, \tag{9}$$

in which the $L^\alpha_{\mu\beta}$ are analytic functions of the $a^1, \ldots, a^n$ and t for all values of these variables. It follows from the general existence theorem for systems of ordinary differential equations in n parameters a^α that the above equations have a solution $\xi^\alpha(t, a)$ given by convergent power series in an open region $|t - t_1| < A$ and $|a^\alpha - a_1^\alpha| < B$ where the t_1 and a_1^α may have arbitrary values, this solution being uniquely determined by the initial conditions $\xi^\alpha(0, a) = \xi_1^\alpha$ independently of the values of the parameter a^α. Selecting $t_1 = a_1^\alpha = 0$ and taking ξ_1^α to be the values of the functions $\xi^\alpha(x)$ at the point Q the above solutions $\xi^\alpha(t, a)$ will therefore give the values of the functions $\xi^\alpha(x)$ along the lines l issuing from Q; by successive differentiation of (9) with respect to t and evaluation of the derivatives of ξ^α at $t_1 = 0$ we may calculate the successive coefficients of these power series so as to obtain

$$\xi^\alpha = \xi_1^\alpha - \sum L^\alpha_{\mu\beta}(x_1)\xi_1^\mu(a^\beta t) +$$
$$+ \tfrac{1}{2}\sum\left[-\frac{\partial L^\alpha_{\mu\beta}(x_1)}{\partial x^\gamma}\xi_1^\mu + L^\alpha_{\mu\beta}(x_1)\, L^\mu_{\nu\gamma}(x_1)\,\xi_1^\nu\right](a^\beta t)(a^\gamma t) + \ldots$$

and these series will be convergent in a region $|t| < A$ and $|a^\alpha| < B$. We observe however that the variables t and a^α in the above series occur only in the combination $(a^\alpha t)$ and that $|a^\alpha t| < AB$ in consequence of the above inequalities. Conversely it is easily seen that if $|a^\alpha t| < AB$ values of a^α and t may be selected without changing the values of the combinations $(a^\alpha t)$ so that the inequalities $|t| < A$ and $|a^\alpha| < B$ are satisfied; hence the above series are convergent whenever $|a^\alpha t| < AB$ and by eliminating the quantities $(a^\alpha t)$ by the substitution $x^\alpha - x_1^\alpha = a^\alpha t$ by which the lines l issuing from Q are defined we see that the functions $\xi^\alpha(x)$ are given by the series

$$\xi^\alpha(x) = \xi_1^\alpha - \Sigma L_{\mu\beta}^\alpha(x_1)\, \xi_1^\mu (x^\beta - x_1^\beta) +$$

$$+ \tfrac{1}{2} \sum \left[- \frac{\partial L_{\mu\beta}^\alpha(x_1)}{\partial x^\nu} \xi_1^\mu + L_{\mu\beta}^\alpha(x_1)\, L_{\nu\gamma}^\mu(x_1)\, \xi_1^\nu \right] (x^\beta - x_1^\beta)(x^\nu - x_1^\nu) + \dots$$

in the neighborhood $|x^\alpha - x_1^\alpha| < AB$ of the point Q. Since Q is an arbitrary point of S we have thus established the analyticity of the functions $\xi^\alpha(x)$ in the space S.

6. It remains finally to observe that the functions $\xi^\alpha(x)$ satisfy the equations (1) in S. In fact let Q be any point of S and as the analytic curve C issuing from Q let us take the curve of parameter x^β passing through this point. Then, in view of the italicized result in § 4 the equations (4) are satisfied along this curve C by the functions $\xi^\alpha(x)$; this gives

$$\frac{\partial \xi^\alpha(x)}{\partial x^\beta} = - \Sigma L_{\mu\beta}^\alpha \xi^\mu(x)$$

at the point Q and hence at all points of the space S.

7. We have now proved the existence of a field of parallel vectors ξ with analytic components $\xi^\alpha(x)$ in S under the condition that the resultant system R of the equations $(\mathrm{E}_0), \dots, (\mathrm{E}_n)$ vanishes over S. That this condition is necessary for the existence of a field of parallel vectors in S is easily seen. In fact suppose there is a point P of S at which the equations $(\mathrm{E}_0), \dots, (\mathrm{E}_n)$ admit only the trivial solution $\xi^\alpha = 0$ and that $\xi^\alpha(x)$ are the components of a field of parallel vectors in S; then necessarily $\xi^\alpha(x_0) = 0$ where x_0^α are the coordinates of P. Since the values of the components $\xi^\alpha(x)$ at the various points of S can be obtained by integrating the equations (4) along analytic curves issuing from P and since the initial values $\xi^\alpha(x_0)$ equal zero for this integration it follows that the components $\xi^\alpha(x)$ must vanish throughout the space S. This contradicts the hypothesis that we have a field of parallel vectors in S.

Instead of the resultant system R we may, if we wish, consider the polynomial $R_1(B)$ in the quantities B appearing in the equations $(\mathrm{E}_0), \dots, (\mathrm{E}_n)$ which is constructed by taking the sum of the squares of all determinants of the nth order that can be formed from the matrix of these equations. Our result can now be stated as follows: *The affinely connected space S will admit a field of parallel contravariant vectors with compoments $\xi^{\alpha}(x)$ which are analytic functions of the coordinates of S if, and only if, the polynomial $R_1(B)$ vanishes over S.*

8. If the space S admits K fields of parallel vectors $\xi_{(1)}, \ldots, \xi_{(K)}$ where $1 \leqq K \leqq n$, we shall say that these fields are independent provided that the matrix of their components $\xi^{\alpha}_{(1)}, \ldots, \xi^{\alpha}_{(K)}$, namely

$$\left\| \begin{matrix} \xi^1_{(1)} & \xi^2_{(1)} & \cdots & \xi^n_{(1)} \\ \xi^1_{(2)} & \xi^2_{(2)} & \cdots & \xi^n_{(2)} \\ \cdot & \cdot & \cdot & \cdot \\ \xi^1_{(K)} & \xi^2_{(K)} & \cdots & \xi^n_{(K)} \end{matrix} \right\|$$

has rank K at any point P of S. A necessary condition for the existence of K independent fields of parallel vectors is therefore that at any point of S the system $(\mathrm{E}_0), \ldots, (\mathrm{E}_n)$ admit K independent solutions; if we denote by $R_K(B)$ the polynomial defined as the sum of the squares of all determinants of order $n+1-K$ which can be formed from the matrix of the system $(\mathrm{E}_0), \ldots, (\mathrm{E}_n)$ this necessary condition can be expressed by the vanishing of $R_K(B)$ over S. It follows readily that this condition is likewise sufficient. Let the quantities $\xi^{\alpha}_{(\nu)0}$ for $\nu = 1, \ldots, K$ be solutions of the system $(\mathrm{E}_0), \ldots, (\mathrm{E}_n)$ at the point P_i established in § 2. By parallel displacement of the vectors $\xi_{(\nu)0}$ at the point P_i along analytic curves issuing from P_i to the various points of S we can then define, as has been shown, K fields of parallel vectors $\xi_{(\nu)}(x)$ whose components $\xi^{\alpha}_{(\nu)}(x)$ are analytic functions of the coordinates of S. Suppose that at a point Q of S the components $\xi^{\alpha}_{(\nu)}(x)$ are such that the above matrix has rank less than K. Denoting the values of these components at Q by $\xi^{\alpha}_{(\nu)1}$ there will then exist a linear relation

$$\sum_{\nu=1}^{K} A^{\nu} \xi^{\alpha}_{(\nu)1} = 0, \tag{10}$$

in which the A's are constants not all of which are zero. Since the vector fields $\xi_{(\nu)}(x)$ can also be generated by parallel displacement of the vectors $\xi_{(\nu)}(x)$ at Q along analytic curves C issuing from this point, it follows that we must have

$$\frac{d}{dt}\left(\Sigma A^{\nu} \xi^{\alpha}_{(\nu)}\right) = -\Sigma L^{\alpha}_{\mu\beta}\left(\Sigma A^{\nu} \xi^{\mu}_{(\nu)}\right)\frac{dx^{\beta}}{dt}$$

along such curves C. But since the right members of these equations vanish at Q in consequence of (10), the relations

$$\sum_{\nu=1}^{K} A^{\nu} \xi^{\alpha}_{(\nu)}(x) = 0$$

must be satisfied throughout S which contradicts the fact that the above matrix has rank K at the point P_i. We can thus state the following general result. *The affinely connected space S will admit K (or more) independent fields of parallel contravariant vectors with components $\xi^{\alpha}_{(1)}(x), \ldots, \xi^{\alpha}_{(K)}(x)$ which are analytic functions of the coordinates of S if, and only if, the polynomial $R_K(B)$ vanishes over S.*

9. It is evident that the above considerations will apply in every detail if, instead of the n-dimensional number space, we limit ourselves to the spherical domain

$$\sum_{\alpha=1}^{n} (x^{\alpha} - a^{\alpha})^2 < r, \qquad (r > 0),$$

where the a^{α} are arbitrary constants and r is any positive number (interior of an n-dimensional Euclidean sphere). We observe in particular that any two points P and Q of this domain can be joined by an analytic curve lying entirely in the domain; for example the straight line joining P and Q will be such a curve. Also the analytic surfaces Ω and Ω^* used in the demonstrations in § 3 and § 4 will be entirely in such a spherical domain provided that the analytic curves C which enter into the construction of these surfaces possess this property. It follows that the italicized result stated at the end of the preceding section will likewise apply if the space S is identified with the above spherical domain.

10. We shall now extend our theory to more general topological spaces [3]). Let $\mathfrak{M}$ be a connected topological space with neighborhoods N homeomorphic to the interior of an n-dimensional Euclidean sphere. By this homeomorphism coordinates x^{α} can be introduced in any neighborhood of $\mathfrak{M}$. We assume that the selection of coordinates can be made for the totality of neighborhoods N of the space $\mathfrak{M}$ in such a way that the coordinate relationships between any two intersecting neighborhoods are analytic. When such a selection of coordinates is made the space

[3]) The treatment in this section is due in part to Professor W. Mayer of the Institute for Advanced Study with whom I have had the pleasure of discussing this theory.

$\mathfrak{M}$ will be called a topological manifold of class A [4]). Two such topological manifolds $\mathfrak{M}_1$ and $\mathfrak{M}_2$ derived from the same topological space $\mathfrak{M}$ will be said to be equivalent if it is possible to pass from the coordinate neighborhoods of $\mathfrak{M}_1$ to the coordinate neighborhoods of $\mathfrak{M}_2$ by analytic transformations.

A topological manifold $\mathfrak{M}$ of class A will be said to be an *affinely connected topological space* $\mathfrak{M}$ *of class* A if there exists an affine connection L with components $L^{\alpha}_{\mu\beta}(x)$ which are analytic functions of the coordinates of the neighborhoods of the manifold $\mathfrak{M}$; such an affinely connected space will usually be referred to in the following simply as the space $\mathfrak{M}$.

A necessary condition for the existence of a field of parallel contravariant vectors in the space $\mathfrak{M}$, i.e. for the equations (1) to admit a continuous non-trivial solution $\xi^{\alpha}(x)$ over $\mathfrak{M}$, is that the polynomial $R_1(B)$ vanish through $\mathfrak{M}$. We assume this condition to be satisfied. Let us then take a neighborhood $N(P)$ of any point P of $\mathfrak{M}$ and in this neighborhood select a point P_i determined as in § 2. Let ξ^{α}_0 be a non-trivial solution of the system $(\mathrm{E}_0), \ldots, (\mathrm{E}_n)$ at P_i. We now seek to define a field of parallel vectors $\xi(x)$ over $\mathfrak{M}$ by the parallel displacement of the vector ξ_0 with components ξ^{α}_0 at P_i to the various points of $\mathfrak{M}$ along *broken* analytic curves $\mathfrak{C}$. Since $\mathfrak{M}$ is connected it is possible to join P_i to any point Q of $\mathfrak{M}$ by a continuous curve T in $\mathfrak{M}$, in fact by a curve T homeomorphic to the unit interval $0 \leqq t \leqq 1$. It is possible to cover T by a finite number of neighborhoods N_1, $N_2, \ldots, N_m$ such that in the transition from P_i to Q along the curve T these neighborhoods are entered in the order indicated [5]) and where N_1 and N_m are the neighborhoods of the points P_i and Q respectively (Heine-Borel theorem). Then it is possible to choose points $V_1, \ldots, V_{m-1}$ of T where $V_j \subset N_j \cap N_{j+1}$ for $j = 1, \ldots, m-1$, i.e. V_j lies in the intersection of the neighborhoods N_j and N_{j+1}, such that the portion of the curve T between the points P_i and V_1 will lie in the neighborhood N_1; similarly the portion of T between V_1 and V_2 will lie in N_2; ...

---

[4]) The designation class A has reference to the analyticity of the coordinate relationships. If these relationships are continuous and have continuous derivatives to the order p ($\leqq 0$) inclusive we may speak of a topological manifold of class p. Cp. O. Veblen and J. H. C. Whitehead, A set of axioms for differential geometry [Proc. Nat. Acad. **17** (1931), 551—561].

[5]) No assumption is here made that these neighborhoods $N_1, \ldots, N_m$ are distinct.

and finally the portion of T between V_{m-1} and Q will lie in N_m. Now let C_1 be an analytic curve in the neighborhood N_1 joining P_i to V_1; let C_2 be an analytic curve in the neighborhood N_2 joining V_1 to V_2; . . . and finally let C_m be an analytic curve in the neighborhood N_m joining the point V_{m-1} to the point Q. Denote by $\mathfrak{C}$ the broken analytic curve composed of the analytic curves $C_1, \ldots, C_m$ by which we can pass from the point P_i to the point Q. Thus the curves of the type $\mathfrak{C}$ form a class of curves by which we can join the point P_i to any other point Q of $\mathfrak{M}$ and if we can show that the parallel displacement of the vector ξ_0 at P_i to the point Q along any two curves of this class will result in the same vector ξ_1 at Q we shall be able to define by this process of parallel displacement of the vector ξ_0 at P_i a vector field $\xi(x)$ over $\mathfrak{M}$. It will then follow from the considerations of the proceding sections that the components $\xi^\alpha(x)$ of this field are analytic functions of the coordinates of the neighborhoods N satisfying the equations (1) over $\mathfrak{M}$, i.e. $\xi(x)$ will be a field of parallel contravariant vectors in $\mathfrak{M}$.

If we replace the neighborhoods $N_1, \ldots, N_m$ by an analogous set of neighborhoods $N'_1, \ldots, N'_M$ covering T and then select points $V'_1, \ldots, V'_{M-1}$ analogous to the above points $V_1, \ldots, V_{n-1}$ the parallel displacement of the vector ξ_0 at P_i along a broken analytic curve $\mathfrak{C}'$ determined on the basis of these neighborhoods N' and points V' will likewise lead to the same vector ξ_1 at Q [6]). Denote by I the intersection

$$(N_1 + \ldots + N_m) \cap (N'_1 + \ \ldots + N'_M).$$

It is evident that we can then construct an analytic curve $\mathfrak{C}^*$ in I, joining P_i and Q, broken at the points $V_1, \ldots, V_{m-1}$ and $V'_1, \ldots, V'_{M-1}$ (and possibly other points) such that $\mathfrak{C}^*$ will pass through these points V and V' in the order in which they occur on the curve T and possessing the following property: the portion of $\mathfrak{C}^*$ between P_i and V_1 will lie in the neighborhood N_1, the portion of $\mathfrak{C}^*$ between V_1 and V_2 will lie in the neighborhood N_2; . . . and the portion of $\mathfrak{C}^*$ between V_{m-1} and Q will lie in the neighborhood N_m; similarly the portion of $\mathfrak{C}^*$ between P_i and V'_1 will lie in the neighborhood N'_1, the portion of $\mathfrak{C}^*$ between V'_1 and V'_2 will lie in the neighborhood N'_2; . . . and the portion of $\mathfrak{C}^*$ between V'_{M-1} and Q will lie in the neighborhood N'_M.

[6]) In particular the neighborhoods $N'_1, \ldots, N'_M$ may be identical with the neighborhoods $N_1, \ldots, N_m$ in which case the points $V'_1, \ldots, V'_{M-1}$ may be taken to be another selection of the points $V_1, \ldots, V_{m-1}$.

It then follows from the results of the preceding sections that the parallel displacement of the vector ξ_0 at P_i along the curves $\mathfrak{C}$ and $\mathfrak{C}^*$ considered as curves in the neighborhoods $N_1, \ldots, N_m$ will result in the same vector at the point Q; similarly the parallel displacement of the vector ξ_0 at P_i along the curves $\mathfrak{C}'$ and $\mathfrak{C}^*$ considered as curves in the neighborhoods $N'_1, \ldots, N'_M$ will give the same vector at Q. Hence the vector ξ_1 at Q resulting from the parallel displacement of the vector ξ_0 at P_i along the broken analytic curve $\mathfrak{C}$ is independent of the selection of neighborhoods N_j covering the continuous curve T and the subsequent selection of points V_j and so depends essentially on the curve T itself; *it is therefore convenient to speak of the parallel displacement of the vector ξ_0 at P_i along the continuous curve T and to say that the vector ξ_1 at Q results from this displacement.*

Now consider the broken analytic curves $\mathfrak{C}$ and $\mathfrak{C}'$ to be any two such curves joining the points P_i and Q. Assume that the space $\mathfrak{M}$ is *simply connected.* It is then possible to pass from the curve $\mathfrak{C}$ by a continuous deformation of this curve into the curve $\mathfrak{C}'$; more precisely this means that the two curves $\mathfrak{C}$ and $\mathfrak{C}'$ can be embedded in a continuous one parameter family of curves given by equations of the form $x^\alpha = \varphi^\alpha(t, p)$ where the φ's are continuous functions of the variables t and p defining for any fixed value of the parameter p, where $0 \leqq p \leqq 1$, a curve $T(p)$, joining P_i and Q, which is a continuous map of the unit interval $0 \leqq t \leqq 1$. We take $T(0)$ to be the curve $\mathfrak{C}$ and $T(1)$ to be the curve $\mathfrak{C}'$. Parallel displacement of the vector ξ_0 at P_i along the curve $T(p)$ will result in a vector $\xi(p)$ at the point Q. We shall show that $\xi(p) = \xi(0)$ for all values $0 \leqq p \leqq 1$ of the parameter p.

Consider any curve $T(p_0)$ such that $0 \leqq p_0 \leqq 1$. Let $W_1, \ldots, W_s$ be a set of neighborhoods covering $T(p_0)$ and let $Y_1, \ldots, Y_{s-1}$ be points of $T(p_0)$ such that $Y_j \subset W_j \cap W_{j+1}$, these neighborhoods W_j and points Y_j being completely analogous to the neighborhoods N_j and points V_j considered with reference to the above curve T. Also let $M(Y_j)$ be a neigborhood of the point Y_j such that $M(Y_j) \subset W_j \cap W_{j+1}$. It is then evident that for a sufficiently small positive number δ any curve $T(p_0+\varepsilon)$ such that $|\varepsilon| < \delta$ will be covered by the neighborhoods W_j which it will enter in the natural order and will furthermore have points in each of the neighborhoods $M(Y_j)$. Hence take $Z_j \subset M(Y_j)$ to be a point on $T(p_0+\varepsilon)$. Join Y_j to Z_j by the analytic curve C_j^* contained in $M(Y_j)$. Denote by $\mathfrak{C}_1$ the broken analytic curve composed of the analytic curves $C_1, \ldots, C_s$ lying respectively in the neighborhoods $W_1, \ldots, W_s$ and joining the points P_i, $Y_1, \ldots, Y_{s-1}$, Q in this order; similarly denote by $\mathfrak{C}_2$ the broken analytic curve composed of the analytic curves $C'_1, \ldots, C'_s$ joining the points P_i, $Z_1, \ldots, Z_{s-1}$, Q and lying respectively in the neighborhoods $W_1, \ldots, W_s$. Then the configuration consisting of the points P_i, Y_1, Z_1 and the curves C_1, C_1^*, C'_1 joining them lies in the neighborhood W_1; similarly the configuration consisting of the points Y_1, Y_2, Z_2, Z_1 and the curves C_2, C_2^*, C'_2, C_1^* joining them lies in the neighborhood W_2; ... and finally the configurating consisting of the points Y_{s-1}, Q, Z_{s-1} and the curves C_s, C'_s, C_{s-1}^* joining them lies in the neighborhood W_s. It follows from these facts and the results established in the preceding sections that the same vector will be obtained at the point Q by the parallel displacement of the vector ξ_0 at P_i along the broken curve $\mathfrak{C}_1$ as will be obtained by parallel displacement of this vector along the broken curve $\mathfrak{C}_2$. *In other words, parallel displacement of the vector ξ_0 at P_i along any curve $T(p_0+\varepsilon)$ for $|\varepsilon| < \delta$ will result in the same vector $\xi(p_0)$ at the point Q.*

We now define a class division of the values of the parameter p for the interval $0 \leqq p \leqq 1$. Let p belong to the class A if $\xi(p_0) = \xi(0)$ for $p_0 \leqq p$; otherwise p will belong to the class B. The class A must have a last value or the class B must have a first value of p. But $p \leqq 1$ can not be a first value of class B as follows from the above italicized statement and the fact that $p = 0$ belongs to the class A. Similarly $p < 1$ can not be a last value of the class A. Hence $p = 1$ is the last value of the class A, i.e. $\xi(p) = \xi(0)$ for all values of p in the interval $0 \leqq p \leqq 1$. One result is therefore established.

We have now proved that the vanishing of the polynomial $R_1(B)$ over the space $\mathfrak{M}$ is a necessary and sufficient condition for the existence of the field of parallel vectors $\xi(x)$ in this space. Since the extension of this result to the case of K fields of independent vectors can be made by a consideration similar to that in § 8 the following general theorem is established.

THEOREM. *An affinely connected topological space $\mathfrak{M}$ of class A will admit K (or more) independent fields of parallel contravariant vectors with components $\xi^\alpha_{(1)}(x), \ldots, \xi^\alpha_{(K)}(x)$ which are analytic functions of the coordinates of the neighborhoods of $\mathfrak{M}$ if, and only if, the polynomial $R_K(B)$ vanishes over $\mathfrak{M}$.*

11. In particular if $K = n$ all the components B in the equations $(\mathrm{E}_0), \ldots, (\mathrm{E}_n)$ must vanish in the space $\mathfrak{M}$ as a consequence of the above theorem. But since the vanishing of the curvature tensor suffices for the vanishing of the remaining coefficients in these equations we have as a corollary to the above theorem that the space $\mathfrak{M}$ *will admit n independent fields of parallel contravariant vectors if, and only if, the curvature tensor vanishes over* $\mathfrak{M}$.

If there are n independent fields of parallal contravariant vectors in $\mathfrak{M}$ the equations

$$\frac{\partial \xi^\alpha_{(i)}}{\partial x^\beta} = -\sum L^\alpha_{\mu\beta} \xi^\mu_{(i)}, \quad (i=1, \ldots, n),$$

which are satisfied by the components of these vectors can be solved at any point of $\mathfrak{M}$ for the components of the affine connection L. This gives

$$L^\alpha_{\mu\beta}(x) = -\sum_{i=1}^{n} \xi^{(i)}_\mu \frac{\partial \xi^\alpha_{(i)}}{\partial x^\beta},$$

where the quantities $\xi^{(i)}_\mu$ are the normalized cofactors of the elements of the determinant $|\xi^\mu_{(i)}|$. Hence *the space* $\mathfrak{M}$ *becomes an affine space of distance parallelism* [7]).

12. It is known that over certain types of (simply connected) topological manifolds it is not possible to have a continuous vector field without singular points. For example, Brouwer [8]) has shown that such a field can not be defined over an n-dimensional spherical surface, if n is an even integer. It follows from this fact and the above theorem that it is not possible to define an affine connection L with analytic components over a spherical surface of even dimensionality such that the polynomial $R_1(B)$ will vanish over the surface; and in particular the affine connection can not be such that the curvature tensor will vanish at all points of this surface. Speaking generally the question of the topological character of an affinely connected space over which a specified system of differential equations (such as the equations which express the vanishing of the contracted curvature tensor) is satisfied, furnishes an interesting and important class of problems in the combined field of differential geometry and topology.

(Received March 12th, 1936.)

---

[7]) Differential Invariants of Generalized spaces, loc. cit., 17.

[8]) L. E. J. Brouwer, Über eineindeutige stetige Transformationen von Flächen in sich [Math. Annalen 69 (1910)]; see also J. W. Alexander, On Transformations with Invariant Points [Trans. Am. Math. Soc. 23 (1922), 94].