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A group $\mathfrak{G}$ is said to be an extension of $\mathfrak{N}$ by $\mathfrak{G}'$ if $\mathfrak{N}$ is a self conjugate subgroup of $\mathfrak{G}$ and $\mathfrak{G}/\mathfrak{N} \cong \mathfrak{G}'$. The problem of finding the extensions of $\mathfrak{N}$ by $\mathfrak{G}'$ has been investigated by Schreier [1]) and by Baer [2]).

Let $\mathfrak{A}$ be the automorphism group of $\mathfrak{N}$ and $\mathfrak{J}$ the subgroup of inner automorphisms. Then to each coset γ of $\mathfrak{N}$ in $\mathfrak{G}$ there corresponds a coset $X(\gamma)$ of $\mathfrak{J}$ in $\mathfrak{A}$, such that if $c \in \gamma$ then the automorphism induced by c in $\mathfrak{N}$ belongs to $X(\gamma)$. $X(\gamma)$ is a homomorphism of $\mathfrak{G}'$ in $\mathfrak{A}/\mathfrak{J}$. Baer's investigations are concerned with finding all possible groups $\mathfrak{G}$ when $\mathfrak{N}$, $\mathfrak{G}'$ and the homomorphism $X(\gamma)$ are given. As a first step towards the solution of this problem the possible structures of $\mathfrak{G}/\mathfrak{Z}(\mathfrak{N})$ are found, where $\mathfrak{Z}(\mathfrak{N})$ denotes the centre of $\mathfrak{N}$; it then only remains to solve the original problem in the case where $\mathfrak{N}$ is Abelian. This case is treated entirely differently. In the present paper it is proposed to shew how Baer's method for the case when $\mathfrak{N}$ is Abelian can be used for any group $\mathfrak{N}$.

In a practical determination of all extensions with given characteristics it is necessary to find the structure of the relation group of the factor group $\mathfrak{G}'$. This is so even when $\mathfrak{N}$ is Abelian. The problem is considered in the second half of the paper.

As an illustration the theory is applied to the case of extensions of an arbitrary group by a cyclic group.

§ 1. *Extensions with given automorphisms.*

The problem of finding extensions of a group by a given group inducing given classes of automorphisms is best treated

[1]) O. Schreier, Über die Erweiterung von Gruppen [Monats. f. Math. u. Phys. **34** (1926), 165—180].

[2]) R. Baer, Erweiterung von Gruppen und ihren Isomorphismen [Math. Zeitschr. **38** (1934), 375—416].

by reducing it to another problem which is both described and solved in the following

THEOREM 1.

$\mathfrak{N}$, $\mathfrak{F}$ are given groups[3]) *and $\mathfrak{R}$ is a self conjugate subgroup of $\mathfrak{F}$; χ_a is a homomorphism of $\mathfrak{F}$ into the group of automorphisms of $\mathfrak{N}$ (i.e. $\chi_a(\mathfrak{b})$ as a function of $\mathfrak{b}$ is an automorphism of $\mathfrak{N}$ and satisfies*

$$\chi_{ab}(\mathfrak{b}) = \chi_b(\chi_a(\mathfrak{b})) \qquad (1,\ b,\ a,\ \mathfrak{b})$$

for all b, a of $\mathfrak{F}$ and $\mathfrak{b}$ of $\mathfrak{N}$) and $\mathfrak{a}(r)$ is a homomorphism of $\mathfrak{R}$ into $\mathfrak{N}$. Then there is a group[3]) *$\mathfrak{G}$ in which $\mathfrak{N}$ is a self conjugate subgroup, and a homomorphism $\mathfrak{w}(r)$ of $\mathfrak{F}$ into $\mathfrak{G}$ which satisfies:*

$$\left.\begin{aligned} a)\qquad \chi_a(\mathfrak{b}) &= (\mathfrak{w}(a))^{-1}\mathfrak{b}\mathfrak{w}(a) \quad (\mathrm{I},\ a,\ \mathfrak{b}) \\ \mathfrak{w}(r) &= \mathfrak{a}(r) \quad (\mathrm{II},\ r) \end{aligned}\right\} \qquad (2)$$

(*for all a in $\mathfrak{F}$, r in $\mathfrak{R}$ and $\mathfrak{b}$ in $\mathfrak{N}$*),

b) *every coset of $\mathfrak{N}$ in $\mathfrak{G}$ contains an element of $\mathfrak{w}(\mathfrak{F})$*,

c) $\mathfrak{w}(\mathfrak{R}) = \mathfrak{N} \cap \mathfrak{w}(\mathfrak{F})$

if and only if

$$\left.\begin{aligned} \chi_a(\mathfrak{a}(r)) &= \mathfrak{a}(a^{-1}ra) \quad (\mathrm{I},\ a,\ r) \\ \chi_r(\mathfrak{b}) &= (\mathfrak{a}(r))^{-1}\mathfrak{b}\mathfrak{a}(r) \quad (\mathrm{II},\ r,\ \mathfrak{b}) \end{aligned}\right\} \qquad (3)$$

(*for all a in $\mathfrak{F}$, r in $\mathfrak{R}$ and $\mathfrak{b}$ in $\mathfrak{N}$*).

The relevance of this theorem to the original extension problem can be seen from the

COROLLARY:

$\mathfrak{F}$ is a free[4]) *group with a self conjugate subgroup $\mathfrak{R}$ ($\mathfrak{F}/\mathfrak{R} \cong \mathfrak{G}'$, say) and $X(\gamma)$ is a homomorphism of $\mathfrak{F}/\mathfrak{R}$ into the classes of automorphisms of a given group $\mathfrak{N}$. Let χ_a be any homomorphism of $\mathfrak{F}$ into the automorphisms of $\mathfrak{N}$ for which χ_a belongs to the class $X(\alpha)$ whenever a belongs to the coset α of $\mathfrak{R}$ in $\mathfrak{F}$. Then an extension*

[3]) Elements of $\mathfrak{F}$ are denoted by italic letters, elements of $\mathfrak{N}$ and $\mathfrak{G}$ by German letters, and elements of $\mathfrak{F}/\mathfrak{R}$ by Greek letters. e, $\mathfrak{e}$, ε are the identities of these groups and E is the identity of Φ in § 2.

[4]) It is essential that $\mathfrak{F}$ should be free if the conditions are to be necessary. A trivial example shows that at least $\mathfrak{F}$ cannot be arbitrary if we require $\mathfrak{a}(r)$ to be related to a function $\mathfrak{w}(a)$ as in the theorem. Let $\mathfrak{G}$ be the cyclic group $\{\mathfrak{b}\}$ of order 4, $\mathfrak{F}$ the cyclic group $\{g\}$ of order 2, and let $\mathfrak{N}$ be $\{\mathfrak{b}^2\}$ and $\chi_g(\mathfrak{b}^2) = \mathfrak{b}^2$. Then we should have to have $\mathfrak{a}(e) = \mathfrak{e}$. But $\mathfrak{a}(e) = \mathfrak{a}(g^2) = \mathfrak{b}^2$.

$\mathfrak{G}$ of $\mathfrak{R}$ by $\mathfrak{G}'$ in which the coset α of $\mathfrak{R}$ induces the class $X(\alpha)$ can be found if and only if there is a homomorphism $\mathfrak{a}(r)$ of $\mathfrak{R}$ in $\mathfrak{R}$ satisfying (3).

Proof of the theorem. The necessity of the conditions is trivial. (3, I, a, r) follows from (2, I, a, $\mathfrak{a}(r)$), (2, II, r), (2, II, $a^{-1}ra$) and the fact that $\mathfrak{w}$ is a homomorphism: (3, II, r, $\mathfrak{b}$) follows immediately from (2, I, r, $\mathfrak{b}$) and (2, II, r).

For the sufficiency we have to construct the group $\mathfrak{G}$. The elements of this group are to be all classes of equivalent pairs $(a, \mathfrak{a})$ (a in $\mathfrak{F}$, $\mathfrak{a}$ in $\mathfrak{R}$), $(a', \mathfrak{a}')$ being equivalent to $(a, \mathfrak{a})$ if and only if $a^{-1}a'$ belongs to $\mathfrak{R}$ and $\mathfrak{a}(a^{-1}a') = \mathfrak{a}\,\mathfrak{a}'^{-1}$.
This is an equivalence relation, for

1) If $a^{-1}a' \in \mathfrak{R}$ and $\mathfrak{a}(a^{-1}a') = \mathfrak{a}\,\mathfrak{a}'^{-1}$

then

$$a'^{-1}a \in \mathfrak{R} \text{ and } \mathfrak{a}(a'^{-1}a) = \mathfrak{a}'\mathfrak{a}^{-1},$$

i.e. the relation is symmetric.

2) $a^{-1}a = e \in \mathfrak{R}, \quad \mathfrak{a}(a^{-1}a) = \mathfrak{e} = \mathfrak{a}\,\mathfrak{a}^{-1},$

i.e. the relation is reflexive.

3) If $a^{-1}a' \in \mathfrak{R}$, $a'^{-1}a'' \in \mathfrak{R}$,

$$\mathfrak{a}(a^{-1}a') = \mathfrak{a}\,\mathfrak{a}'^{-1} \text{ and } \mathfrak{a}(a'^{-1}a'') = \mathfrak{a}'\,\mathfrak{a}''^{-1},$$

then

$$a^{-1}a'' = (a^{-1}a')(a'^{-1}a'') \in \mathfrak{R}$$

$$\mathfrak{a}(a^{-1}a'') = \mathfrak{a}(a^{-1}a')\,\mathfrak{a}(a'^{-1}a'') = \mathfrak{a}\,\mathfrak{a}'^{-1}\mathfrak{a}'\,\mathfrak{a}''^{-1} = \mathfrak{a}\,\mathfrak{a}''^{-1},$$

i.e. the relation is transitive.

The product of two pairs is defined by

$$(a, \mathfrak{a})(b, \mathfrak{b}) = (ab, \chi_b(\mathfrak{a})\mathfrak{b}).$$

This will be a valid definition of a product of classes of pairs if we can shew that if $(a', \mathfrak{a}')$ is equivalent to $(a, \mathfrak{a})$ and $(b', \mathfrak{b}')$ is equivalent to $(b, \mathfrak{b})$ then $(a', \mathfrak{a}')(b', \mathfrak{b}')$ is equivalent to $(a', \mathfrak{a})(b, \mathfrak{b})$ i.e. that if

$$\left.\begin{array}{l} a^{-1}a' \in \mathfrak{R},\ b^{-1}b' \in \mathfrak{R} \\ \mathfrak{a}(a^{-1}a') = \mathfrak{a}\,\mathfrak{a}'^{-1} \text{ and } \mathfrak{a}(b^{-1}b') = \mathfrak{b}\mathfrak{b}'^{-1} \end{array}\right\} \qquad (4)$$

then

$$(ab)^{-1}(a'b') \in \mathfrak{R} \text{ and } \mathfrak{a}((ab)^{-1}(a'b')) = (\chi_{b'}(\mathfrak{a})\mathfrak{b})(\chi_h(\mathfrak{a}'), \mathfrak{b}')^{-1}.$$

Now

$$(ab)^{-1}(a'b') = b^{-1}(a^{-1}a')b \cdot b^{-1}b' \in \mathfrak{N}$$

and

$$\begin{aligned}\mathfrak{a}\big((ab)^{-1}(a'b')\big) &= \chi_b(\mathfrak{a}\,\mathfrak{a}'^{-1})\mathfrak{b}\mathfrak{b}'^{-1}\\ &= \chi_b(\mathfrak{a})\mathfrak{b}\cdot\mathfrak{b}'^{-1}\cdot(\mathfrak{b}\mathfrak{b}'^{-1})^{-1}\chi_b(\mathfrak{a}'^{-1})\mathfrak{b}\mathfrak{b}'^{-1}\\ &= \big(\chi_b(\mathfrak{a})\mathfrak{b}\big)\cdot\big(\chi_{b'}(\mathfrak{a}')\mathfrak{b}'\big)^{-1}\end{aligned}$$

by (4), $\big(3, \text{II}, b^{-1}b', \chi_b(\mathfrak{a}')\big)$ and (1).
The product is also associative, for

$$\begin{aligned}\big((a, \mathfrak{a})(b, \mathfrak{b})\big)(c, \mathfrak{c}) &= \big(ab, \chi_b(\mathfrak{a})\mathfrak{b}\big)(c, \mathfrak{c})\\ &= \big(abc, \chi_c(\chi_b(\mathfrak{a})\mathfrak{b})\mathfrak{c}\big)\\ (a, \mathfrak{a})\big((b, \mathfrak{b})(c, \mathfrak{c})\big) &= (a, \mathfrak{a})\big(bc, \chi_c(\mathfrak{b})\mathfrak{c}\big)\\ &= \big(abc, \chi_{bc}(\mathfrak{a})\chi_c(\mathfrak{b})\mathfrak{c}\big)\end{aligned}$$

and these two expressions are equal on account of (1) and the fact that χ_c is an automorphism.

$(e, \mathfrak{e})$ is an identity and $\big(a^{-1}, \chi_{a^{-1}}(\mathfrak{a}^{-1})\big)$ is an inverse to $(a, \mathfrak{a})$. Consequently with this product our classes of pairs form a group.

$\mathfrak{w}(a)$ is defined by

$$\mathfrak{w}(a) = (a, \mathfrak{e})$$

and is clearly a homomorphism. To show that (2, II) is satisfied it is only necessary to verify that $(r, \mathfrak{e})$ is equivalent to $\big(e, \mathfrak{a}(r)\big)$. As regards (2, I) we have

$$\begin{aligned}\big(\mathfrak{w}(a)\big)^{-1}\mathfrak{b}\mathfrak{w}(a) &= (a^{-1}, \mathfrak{e})\mathfrak{b}(a, \mathfrak{e})\\ &= (a^{-1}, \mathfrak{e})\big(a, \chi_a(\mathfrak{b})\big)\\ &= \big(e, \chi_a(\mathfrak{b})\big) = \chi_a(\mathfrak{b}).\end{aligned}$$

Since

$$\begin{aligned}(g, \mathfrak{b}) &= (g, \mathfrak{e})(e, \mathfrak{b})\\ &= \mathfrak{w}(g)\mathfrak{b}\end{aligned}$$

condition b is satisfied.

The condition that an element $\mathfrak{b}$ of $\mathfrak{N}$ should also lie in $\mathfrak{w}(\mathfrak{F})$ is that $(e, \mathfrak{b})$ be equivalent to some pair $(a, \mathfrak{e})$. This means that a is in $\mathfrak{R}$ and $\mathfrak{a}(a) = \mathfrak{b}$. Hence $\mathfrak{N} \cap \mathfrak{w}(\mathfrak{F}) \subseteqq \mathfrak{w}(\mathfrak{R})$. This is satisfied since $\mathfrak{w}(r) = \mathfrak{a}(r)$ so that $\mathfrak{w}(\mathfrak{R}) \subseteqq \mathfrak{N}$.

Proof of the corollary. Suppose $\mathfrak{G}$ is a group with the required properties, $e_1, e_2, \ldots, e_n$ a set of free generators of $\mathfrak{F}$ and ω the function determining the homomorphism of $\mathfrak{F}$ on $\mathfrak{G}'$. Let $\mathfrak{e}_1$,

$\mathfrak{e}_2, \ldots, \mathfrak{e}_n$ be elements of $\mathfrak{G}$ with the property that $\mathfrak{e}_i$ is in the coset $\omega(e_i)$ of $\mathfrak{N}$ and that

$$\chi_{e_i}(\mathfrak{b}) = \mathfrak{e}_i^{-1}\mathfrak{b}\mathfrak{e}_i \text{ for all } \mathfrak{b} \text{ in } \mathfrak{N}.$$

Then if $\mathfrak{w}$ be a homomorphism of $\mathfrak{F}$ into $\mathfrak{G}$ satisfying $\mathfrak{w}(e_i) = \mathfrak{e}_i$ for all i, we shall have

$$\chi_a(\mathfrak{b}) = \left(\mathfrak{w}(a)\right)^{-1}\mathfrak{b}\mathfrak{w}(a) \text{ for all } \mathfrak{b} \text{ in } \mathfrak{N}.$$

If we put $\mathfrak{w}(r) = \mathfrak{a}(r)$ for elements r of $\mathfrak{R}$ then by the first half of theorem 1 (whose proof makes no use of b, c) the conditions (3) must hold.

If on the other hand we have a function $\mathfrak{a}(r)$ satisfying (3) we can form the group $\mathfrak{G}$ of theorem 1 which is easily seen to have the required properties.

For specific applications the corollary to theorem 1 is more useful in the form of

THEOREM 2.

$\mathfrak{N}$ is a given group. $\mathfrak{F}$ is a free group with the generators $e_1, e_2, \ldots$

$r_1, r_2, \ldots, r_l$. Let us put $\mathfrak{a}(r_i) = \mathfrak{r}_i$ and $\mathfrak{z}_i = \mathfrak{r}_i^{-1}\mathfrak{r}_i^*$. $\mathfrak{z}_i$ is certainly

$\mathfrak{a}(p) = \mathfrak{a}(e) = \mathfrak{e}$. But if we make use of (3, I) and the fact that $\mathfrak{a}(r)$ is a homomorphism we obtain

§ 2. *The relations between the relations of a group.*

Suppose $\mathfrak{F}$ is a free group with the generators $e_1, \ldots, e_n$ and $\mathfrak{R}$ is the least self conjugate subgroup containing certain elements $r_1, r_2, \ldots, r_l$. The factor group will be called $\mathfrak{G}'$. As has been shewn it is important in the extension problem to be able to express the structure of $\mathfrak{G}'$ in terms of relations between the conjugates of the relations $r_1, r_2, \ldots, r_l$. This problem has been solved by Reidemeister [6]). It is necessary to repeat his conclusions to obtain another theorem on extensions (theorem 4).

Precisely the problem may be stated as follows. $\mathfrak{R}$ is generated by all elements of $\mathfrak{F}$ of form $a^{-1}r_i a$; it may therefore be regarded as the factor group $\Phi/\mathbf{P}$ of the the free group Φ with the generators $E_{i,a}$ with respect to some self conjugate subgroup $\mathbf{P}$. The problem is to find a set of elements of Φ whose conjugates generate $\mathbf{P}$. $\mathbf{P}$ contains for instance all elements of form

$$E_{i,\, ab^{-1}r_j b}\, E_{j,b}^{-1}\, E_{i,a}^{-1}\, E_{j,b}. \tag{9}$$

If our method for finding the relations $\mathbf{P}$ is to be constructive it is necessary that the structure of the original group $\mathfrak{G}'$ should be known, or what amounts to the same, that we have a constructive method for determining whether a given member of $\mathfrak{F}$ is a member of $\mathfrak{R}$. If this is the case we can find a constructive function v_a defined for all a in $\mathfrak{F}$, constant in each coset of $\mathfrak{R}$, taking its value in that coset and satisfying $v_e = e$. These elements are a set of representatives of the cosets of $\mathfrak{R}$. If we put $v_a^{-1}a = r_a$ then r_a is a relation (member of $\mathfrak{R}$) for each a.

We define r_{a,e_i} by the condition

$$v_a e_i = v_{ae_i}\, r_{a,e_i}.$$

Then $\mathfrak{R}$ is generated by the relations $b^{-1}r_{a,e_i}b$. For if $\overline{\mathfrak{R}}$ is the group generated by these and contains r_c, then

$$\left.\begin{aligned} r_{ce_i} &= r_{c,e_i}\, e_i^{-1}\, r_c\, e_i \in \overline{\mathfrak{R}} \\ \text{and} \qquad & \\ r_{ce_i^{-1}} &= e_i\, r_{ce_i^{-1},e_i}\, r_c\, e_i^{-1} \in \overline{\mathfrak{R}}. \end{aligned}\right\} \tag{10}$$

But $\overline{\mathfrak{R}}$ contains $r_e = e$; it therefore contains r_c for each c.

Now suppose that for each r_{v_c, e_i} we have chosen an element R_{v_c, e_i} of Φ corresponding to it in the homomorphism τ of Φ on $\mathfrak{R}$ and let us define automorphisms χ_a by

6) K. Reidemeister, Knoten und Gruppen [Hamb. Abhandl. 5 (1926), 8—23].

$$\chi_a(E_{i,b}) = E_{i,ba}. \tag{11}$$

Then we may define R_c recursively by the equations

$$\left.\begin{aligned} R_e &= E \\ R_{ce_i} &= R_{v_c, e_i}\,\chi_{e_i}(R_c) \\ R_{ce_i^{-1}} &= \chi_{e_i^{-1}}(R_{v_c, e_i}\, R_c) \end{aligned}\right\} \tag{12}$$

so that if either $k = e_i$ or $k = e_i^{-1}$ we shall have

$$R_{ck} = R_{v_c k}\,\chi_k(R_c). \tag{13}$$

Our definition will be valid if and only if we always have

$$R_{(ce_i)e_i^{-1}} = R_{(ce_i^{-1})e_i} = R_c.$$

It may easily be verified that this is so.

Since the equations (13) and

$$r_{ck} = r_{v_c k}\, k^{-1}\, r_c\, k \tag{14}$$

hold whenever k is a generator or its inverse, R_c must correspond to r_c in τ. Now for all b, i we have

$$r_{v_b r_i} = r_i$$

and therefore $R_{v_b r_i}$ and $E_i (= E_{i,e})$ must belong to the same coset of **P**. I.e.

$$R_{v_b r_i}\, E_i^{-1} \tag{15}$$

must belong to **P**. By operating with the automorphisms χ_a we see that all elements of form

$$\chi_a(R_{v_b r_i}\, E_i^{-1}) \tag{16}$$

belong to **P**. The structure of $\mathfrak{R}$ may now be decribed by

THEOREM 3.

The group of relations **P** *of* $\mathfrak{R}$ *is the least self conjugate subgroup of* Φ *containing all elements of form* (9) *and* (16).

Only a sketch is given for the proof of this theorem. The first step is to shew that

$$R_{ax} = R_{v_a x}\,\chi_x(R_a) \tag{17}$$

for all x, a. For this purpose we consider the set Ξ of all x such that (17) holds for all a. Then we can shew that xy belongs to Ξ if x and y belong to it. The generators e_i, e_i^{-1} belong to Ξ by (13).

Thus ϑ maps the whole of **P** on the identity and the conditions of theorem 2 are satisfied.

§ 3. *Cyclic extensions.*

When $\mathfrak{G}'$ is a cyclic group of order n we take $\mathfrak{F}$ to be the free group with the single generator a and $\mathfrak{R}$ to be the subgroup generated by $q \equiv a^n$. The representative elements v_b may be taken to be

$$e, \; a, \; a^2, \ldots, \; a^{n-1}.$$

We easily find that

$$r_{a^p, a} = e \text{ if } p \not\equiv -1 \pmod{n}$$
$$r_{a^{n-1}, a} = q.$$

If Q_{a^p} be the element of Φ corresponding to $a^{-p} q a^p$ we have

$$R_{a^p, a} = E,$$
$$R_{a^{n-1}, a} = Q$$

and from the equations (12) we obtain

$$\left.\begin{array}{l} R_{a^p} = E \\ R_{a^{n+p}} = Q_{a^p}. \end{array}\right. \quad (0 \leqq p \leqq n-1)$$

The expressions (15) are therefore

$$Q_{a^p} Q^{-1} \quad (p = 0, 1, \ldots, n-1). \tag{19}$$

If ϑ is a homomorphism of Φ and $\vartheta(Q_a^{-1} Q) = \mathfrak{e}$, then

$$\vartheta(Q_{a^p}) = \vartheta(Q_{a^{p-1}}) = \ldots = \vartheta(Q). \tag{20}$$

Now making use of (19), (20), theorem 4 for a cyclic extension becomes

THEOREM 5.

A is a class of automorphisms of a group $\mathfrak{N}$. A^w is the first power of A which is the class of inner automorphisms and ξ is an arbitrary automorphism out of A. $\mathfrak{x}^*$ is an element of $\mathfrak{N}$ which induces the inner automorphism ξ^n. Then there is an extension of $\mathfrak{N}$ by the cyclic group of order n realising the classes $A, A^2, \ldots$ of automorphisms if and only if there is an element $\mathfrak{z}$ in the centre

We have put

$$\vartheta(Q) = \mathfrak{r}^* \mathfrak{z}^{-1}$$

$$\vartheta(Q_a) = \xi(\mathfrak{r}^* \mathfrak{z}^{-1}).$$

If we put $\xi(\mathfrak{z})\mathfrak{z}^{-1} = \varphi(\mathfrak{z})$ then φ is a (possibly improper) automorphism of the centre $\mathfrak{z}$ of $\mathfrak{N}$. For some groups all of the automorphisms φ are improper. This is the case for all cyclic groups whose order is a power of 2. In these cases we shall have unrealisable classes of automorphisms if $\xi(\mathfrak{r}^*)\mathfrak{r}^{*-1}$ is not in $\varphi(\mathfrak{Z})$.

Suppose for instance that $\mathfrak{N}$ is the dihedral group D_{10} of order 20, generated by a, b with the relations

$$a^{10} = b^2 = (ab)^2 = e.$$

The centre of this group consists of e and a^5. We define the automorphism ξ by

$$\xi(a) = a^7$$

$$\xi(b) = ba^5,$$

then

$$\xi^2(a) = a^{-1} = b^{-1}ab,$$

$$\xi^2(b) = b = b^{-1}bb.$$

$\mathfrak{r}^*$ can therefore be taken to be b. The equation (21) becomes

$$\xi(\mathfrak{z})\mathfrak{z}^{-1} = \xi(b)b^{-1} = a^5,$$

but $\xi(\mathfrak{z})\mathfrak{z}^{-1} = e$ for both centre elements.

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