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The total length of the edges of a polyhedron

by

J. M. Hammersley

Fejes Tóth, in a paper ¹) to which I have not had access, has conjectured that L, the sum of the lengths of the edges of a convex polyhedron containing a sphere of unit diameter, satisfies $L \ge 12$; and he has proved that L > 10 for all such polyhedra, and L > 14 for polyhedra with triangular faces only. In this note I prove that, if no face is a polygon of more than n sides, then

$$L > \frac{10}{3} \sqrt[n]{(\pi n \tan \frac{\pi}{n})}.$$
 (1)

For triangular faces only, this is weaker (L > 13.47...) than Fejes Tóth's result; for triangular and/or quadrilateral faces it gives L > 11.82...; and for faces with any number of sides it gives

$$L \ge 10\pi/3 = 10.47 \dots,$$
 (2)

which is slightly stronger than Fejes Tóth's result.

Let S be the surface and the area of the sphere, centre 0, radius $\frac{1}{2}$. Let P be the plane containing any face of the polyhedron. Let p denote the perimeter of this face and its length. The area A' of this face cannot exceed that of a regular polygon of n sides with perimeter p; so

$$A' \leq \frac{p^2}{4n} \cot \frac{\pi}{n}.$$
 (3)

Let A denote the projection (and its area) from 0 of the interior of p upon S. Define θ by

$$A = \frac{1}{2}\pi(1 - \cos\theta), \qquad (0 \le \theta \le \frac{1}{2}\pi). \tag{4}$$

Let C be the cone of semi-vertical angle θ , with vertex at 0 and axis normal to P. Let B and B' be the areas cut off by C upon S and P respectively. Since B = A, we have

$$A' > B' \ge \frac{1}{4}\pi \tan^2 \theta. \tag{5}$$

¹) Norske. Vid. Selsk. Forh., Trondhjem (1948) 21, 32-4. See Math. Rev. (1950) 11, 386.

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Strict inequality holds in (5) because A has not got a circular boundary. From (3) and (5)

$$p > \left(n\pi \tan \frac{\pi}{n}\right)^{\frac{1}{2}} \tan \theta.$$
 (6)

Use the suffix i = 1, 2, ... for the various faces of the polyhedron. Summing we have

$$\sum_{i} A_{i} = S = \pi = \frac{1}{2}\pi\sum_{i} (1 - \cos \theta_{i}), \quad (0 \le \theta_{i} \le \frac{1}{2}\pi).$$
(7)

$$2L = \sum_{i} p_{i} > \left(n\pi \tan \frac{\pi}{n} \right)^{\frac{1}{2}} \sum_{i} \tan \theta_{i} = T.$$
 (8)

A minimum of T cannot occur unless either

$$\sec^2 \theta_i + \lambda \sin \theta_i = 0 \tag{9}$$

where λ is a Lagrangian undetermined multiplier, or θ_i is an endpoint of the interval $0 \leq \theta_i \leq \frac{1}{2}\pi$. If $\theta_i = \frac{1}{2}\pi$, *T* is infinite. Suppose that exactly *N* of the θ_i are not zero. Since these values must then satisfy (9), they are equal; whence, from (7), for these θ_i

$$\cos \theta_i = 1 - \frac{2}{N}, \ \tan \theta_i = \frac{2(N-1)^{\frac{1}{2}}}{(N-2)},$$
 (10)

the first of the relations (10) implying $N \ge 2$. Then

$$T \ge \left(n\pi \tan \frac{\pi}{n}\right)^{\frac{1}{2}} \frac{2N(N-1)^{\frac{1}{2}}}{(N-2)} = \left(n\pi \tan \frac{\pi}{n}\right)^{\frac{1}{2}} U(N). \quad (11)$$

When $N \ge 2$ ranges over the positive integers, U(N) attains its minimum for N = 5; whereupon (8) and (11) yield (1).

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