COMPOSITIO MATHEMATICA

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Order in projective and in descriptive geometry

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Introduction.

In Hilbert's system of axioms for geometry ([2]), the axioms of groups I (incidence), III (congruence), and V (continuity) are valid in hyperbolic and elliptic as well as in Euclidean geometry, while the axioms of group II, the axioms of order, are valid in Euclidean and in hyperbolic, but not in elliptic geometry. It is the aim of this paper to give an axiomatic theory of order in geometry, which serves equally well for all three geometries. As no metrical concepts are involved, we obtain a unified theory of order in projective and in descriptive geometry.

Our theory of order is based on Hilbert's axioms of incidence, and on seven axioms of order. The axioms of incidence have been modified in order to obtain axioms for a geometry of any dimension (§ 1). Separation of two pairs of lines in a pencil as a primitive relation of order. Six of our seven axioms of order are those of [1], with minor changes (§ 2). Our seventh axiom of order is needed to prove the fundamental properties of triangles (§ 4).

Half-flats, in particular half-lines and half-planes, are defined as sets of segments in § 5. This definition has the advantage that it may be used in projective as well as in descriptive geometry. Hilbert's theory of congruence then is valid, practically without modifications, for elliptic geometry as well as for Euclidean and for hyperbolic geometry.

In § 6, our axioms and definitions are compared with Hilbert's axioms and definitions.

Points are denoted by capital letters, lines by lower case letters, other flats by small greek letters, and other sets of points by small german letters. Intersections of sets of points are denoted by the sign $\cap$. The flat consisting of one point P will be identified. with the point P.

1. Axioms of incidence. Definitions.

We consider a geometry as a set, the elements of which are called *points,* and in which two classes of subsets, called the class of *lines* and the class of *planes,* are given. The following eight axioms of incidence are assumed.

AXIOM 1.1. *A line is a set of points, containing at least two points.*

AXIOM 1.2. *If A and B are two distinct points, then there is a line to which both A and B belong.*

AXIOM 1.3. *Two distinct lines have at most one point in common.*

It follows that two distinct points A and B belong to exactly one line. This line is denoted by AB.

AXIOM 1.4. *A plane is a set of points, containing three distinct points not on a line.*

AXIOM 1.5. *If A, B, C are three distinct points not on a line, then there is a plane to which A, B, and C belong.*

AXIOM 1.6. *If two distinct points of a line belong to a plane, then the line is contained in the plane.*

AXIOM 1.7. *Let a and b be two distinct lines contained in a plane π, and let P be a point not in π. If α is a plane containing a and P, and if β is a plane containing b and P, then the intersection of α and β is a line.*

AXIOM 1.8. *There are three distinct points not on a line, and four points not on a plane.*

It follows from these axioms that three points A, B, C not on a line belong to exactly one plane. This plane is denoted by ABC.

If π is a plane and P a point in π, then the set of all lines in π and on P is called the *pencil of lines P/π* with *vertex P* and *plane π.*

The second part of Axiom 1.8 is only needed in the proof of the following proposition.

PROPOSITION 1.1. *Let l be a line, and let A and B be two points not on l. If every line of the pencil A/lA intersects l in a point, then every line of the pencil B/lB intersects l in a point.*

The well-known proof is omitted.

DEFINITION 1.1. A set ϱ of points is called a *flat* if the following two conditions are satisfied:

a) If two distinct points of a line l belong to ϱ, then l is contained in ϱ.

b) If three points not on a line belong to ϱ, then a plane containing these three points is contained in ϱ.

Lines and planes are flats, and the intersection of a family

of flats is a flat. If $\mathfrak{a}$ is a set of points, then the intersection of all flats containing $\mathfrak{a}$ is the smallest flat containing $\mathfrak{a}$, and is called the flat *generated* by $\mathfrak{a}$.

If ϱ is a flat and A a point not in ϱ, then the flat generated by ϱ and A is denoted by ϱA. If ϱ contains two distinct points, and if P is a point of ϱ, then ϱA is the set-union of all planes lA, where l is a line on P and contained in ϱ. If ϱ is a flat, if A and B are two points not in ϱ, and if B is in ϱA, then $\varrho B = \varrho A$.

If ϱ is a flat, if A, B, C, ... are points not in ϱ, and if $\varrho A =$ $= \varrho B = \varrho C = \ldots$, then ϱ is called a *transversal* flat of the points A, B, C, ...

A flat is called *proper* if it is neither empty nor the set of all points.

2. Axioms of separation.

Separation of pairs of lines in a pencil is introduced as a primitive relation, characterized by six axioms. We write $ab \parallel cd$, if two lines a and b separate two lines c and d.

AXIOM 2.1. *If $ab \parallel cd$, then a, b, c, d are four distinct lines of a pencil.*

AXIOM 2.2. *If a, b, c, d are four distinct lines of a pencil, then either $ab \parallel cd$ or $ac \parallel bd$ or $bc \parallel ad$.*

AXIOM 2.3. *If $ab \parallel cd$, then $ab \parallel dc$.*

AXIOM 2.4. *If $ab \parallel cd$ and $bc \parallel de$, then $ea \parallel bc$.*

Axioms 2.5 and 2.6 will be stated later.

PROPOSITION 2.1. *If $ab \parallel cd$, then $cd \parallel ab$.*

PROOF. b, c, d, a are four distinct lines of a pencil by Axiom 2.1, hence either $bc \parallel da$ or $bd \parallel ca$ or $cd \parallel ba$ by Axiom 2.2. Now $ab \parallel cd$ with $bc \parallel da$ implies $aa \parallel bc$, and with $bd \parallel ca$ implies $aa \parallel bd$, by Axioms 2.3 and 2.4, in contradiction to Axiom 2.1. But then must we have $cd \parallel ba$, and hence $cd \parallel ab$ by Axiom 2.3.

PROPOSITION 2.2. *The three relations $ab \parallel cd$, $ac \parallel bd$, and $bc \parallel ad$ exclude each other.*

PROPOSITION 2.3. *If a, b, c are three distinct lines of a pencil, and if p and q are lines such that $ab \parallel pq$, then either $ab \parallel cp$ or $ab \parallel cq$, but not both.*

The proofs of Propp. 2.2 and 2.3 may be found e.g. in [1].

It is convenient to generalize separation as follows.

DEFINITION 2.1. We shall say that $\alpha\beta \parallel \gamma\delta$, if there are a line l, a point P not on l, and four lines a, b, c, d of the pencil P/lP such that $ab \parallel cd$, and that:

α is either the line a or a point A common to the lines a and l;

β is either the line b or a point B common to the lines b and l;

γ is either the line c or a point C common to the lines c and l;

δ is either the line d or a point D common to the lines d and l.

If two of the four flats α, β, γ, δ are lines, then they determine the pencil P/lP. If two of them are points, then they determine the line l. If at most one of the four flats α, β, γ, δ is a point, then the relation $\alpha\beta \parallel \gamma\delta$ does not depend on a particular choice of the line l. If at most one of the four flats is a line, then $\alpha\beta \parallel\!\!\mid \gamma\delta$ does not depend on a particular choice of the vertex P by Axiom 2.6 below.

Axioms 2.1—2.4 and Propositions 2.1—2.3 may also be generalized under suitable hypotheses.

AXIOM 2.5. *If a is a line, B a point not on a, and l a line of the pencil B/aB, then there are points C and D on l such that $aB \parallel CD$.*

AXIOM 2.6. *Let A and B be two distinct points, let P and P' be two points not on the line AB, let c and d be two lines of the pencil P/ABP, and let c' and d' be two lines of the pencil P'/ABP'. If $c \cap AB = c' \cap AB$, and $d \cap AB = d' \cap AB$, then $AB \parallel cd$ implies $AB \parallel c'd'$.*

Each of the intersections $c \cap AB$ and $d \cap AB$ is either empty or a point.

PROPOSITION 2.4. *If A and B are two distinct points, and if t is a transversal line of A and B, then there are points C and D on the line AB such that $AB \parallel Ct$ and $DA \parallel Bt$.*

PROOF. By Axiom 2.5 there are points P and Q on AB such that $Bt \parallel PQ$. Then $PA \parallel Bt$ or $QA \parallel Bt$ by Prop. 2.3, and there is a point D such that $DA \parallel Bt$. Similarly there is a point C such that $CD \parallel AB$, but this and $DA \parallel Bt$ imply $AB \parallel tC$ by Axiom 2.4.

COROLLARY. *If a, b, c are three distinct lines of a pencil, then there is a line d such that $ab \parallel cd$.*

PROPOSITION 2.5. *If A and B are two distincts points, and if c and d are lines such that $AB \parallel cd$, then at least one of the lines c and d intersects the line AB in a point.*

PROOF. By Prop. 2.4 there is a point P on AB such that $AB\|cP$. If neither c nor d intersect AB, we have $AB \parallel dP$ by Axiom 2.6, contrary to Prop. 2.3.

3. Sectors and segments.

DEFINITION 3.1. If α, β, γ are points or lines, then $(\alpha\beta)_\gamma$ denotes the set of all points P such that $\alpha\beta \parallel \gamma P$.

DEFINITION 3.2. A set $\mathfrak{a}$ of points is called a *sector*, if there are three distinct lines a, b, c of a pencil such that $\mathfrak{a} = (ab)_c$.

The lines a and b are called the *sides*, the vertex $a \cap b$ of the pencil is called the *vertex* of the sector $(ab)_c$. A line through the vertex and a point of a sector is called a *line* of the sector. No sector is empty (Prop. 2.4).

If at least two of the flats α, β, γ in Def. 3.1 are lines, and if the set $(\alpha\beta)_\gamma$ is not empty, then $(\alpha\beta)_\gamma$ is a sector. If at least two of the flats α, β, γ are points, and if $(\alpha\beta)_\gamma$ is not empty, then $(\alpha\beta)_\gamma$ is the intersection of a sector with the line joining these two poinst.

. PROPOSITION 3.1. *If a and b are two lines and C and D two points such that $ab \parallel CD$, then a point of the plane aC is either on a or on b or in exactly one of the two sectors $(ab)_C$ and $(ab)_D$. A point P is in $(ab)_C$ if, and only if, $(ab)_P = (ab)_D$.*

In other words, two intersecting lines a and b divide their plane into two *supplementary* sectors with sides a and b.

Prop. 3.1 follows directly from Prop. 2.3.

PROPOSITION 3.2. *If a, b, c are three distinct lines of a pencil P/π, then a point of the plane π of the pencil is either on one of the lines a, b, c, or in exactly one of the three sectors $(ab)_c$, $(ac)_b$, $(bc)_a$.*

This follows directly from Axiom 2.2 and Prop. 2.2.

PROPOSITION 3.3. *Let l be a line, P a point not on l, and let $\mathfrak{a}$ and $\mathfrak{b}$ be two supplementary sectors in the plane lP and with vertex P. If l meets one of the two sides of $\mathfrak{a}$ and $\mathfrak{b}$, then l contains points of $\mathfrak{a}$ and points of $\mathfrak{b}$. If the two sides of $\mathfrak{a}$ and $\mathfrak{b}$ do not intersect l, then l is either contained in $\mathfrak{a}$ or contained in $\mathfrak{b}$.*

This follows directly from Axiom 2.5 and Propp. 2.3 and 2.5.

DEFINITION 3.3. Let A, B, C be three distinct points of a line. The set $(AB)_C$ is called a *segment* with *endpoints* A and B if every line d, for which $AB \parallel Cd$, intersects the line AB.

A segment with endpoint A shall also be called a segment *at A*.

THEOREM 3.4. *Let A and B be two distinct points. If a transversal line t of A and B does not intersect the line AB, then $(AB)_t$ is a segment with endpoints A and B. If every transversal line of A and B intersects the line AB in a point, and if C and D are points such that $AB \parallel CD$, then $(AB)_C$ and $(AB)_D$ are supplementary segments with endpoints A and B.*

PROOF. If the transversal line t does not intersect AB, let C and D be points such that $AB \parallel Ct$ and $AB \parallel CD$. Then $(AB)_t = (AB)_D$. If u is a transversal line of A and B which does not

intersect AB, then $AB \parallel Cu$ by Axiom 2.6. It now follows from Prop. 2.3 that every line u, for which $AB \parallel Du$, intersects AB. The second part of the theorem follows directly from Prop. 3.1 and Def. 3.3.

PROPOSITION 3.5. *Let A, B. C be points and t a line such that $AB \parallel Ct$. If $(AB)_t$ is a segment, then $(AC)_B$ and $(BC)_A$ are segments contained in $(AB)_t$, and every point of $(AB)_t$, except C, is in exactly one of these segments. Conversely, if $(AC)_B$ and $(BC)_A$ are segments, then $(AB)_t$ is a segment.*

This follows directly from Propp. 3.1 and 3.2 and from Def. 3.3.

PROPOSITION 3.6. *If A and B are two distinct points, then there is a segment with endpoint A and containing B.*

PROOF. If there is a transversal line of A and B which does not meet AB, let t be such a line. Otherwise, let t be any transversal line of A and B. If C is a point such that $AC \parallel Bt$ (Prop. 2.4), then $(AC)_t$ is a segment with endpoint A and containing B by Theorem 3.4.

4. Triangles.

The exterior domain of an oval quadric in real projective space is an example of a geometry, which satisfies Axioms 1.1—1.8 and 2.1—2.6, but in which three points not on a line are not always the vertices of a triangle. This example shows that we need a further axiom in order to obtain the fundamental properties of triangles.

AXIOM 4.1. *Let A, B, C be three points not on a line, and let t be a transversal line of the points A, B, C. If $(BC)_t$ and $(CA)_t$ are segments, then $(AB)_t$ is a segment.*

This axiom is a special case of the following theorem.

THEOREM 4.1. *Let A, B, C be three points not on a line, let $\mathfrak{a}$ be a segment at B and C, and let $\mathfrak{b}$ be a segment at A and C. Then there is a uniquely determined segment $\mathfrak{c}$ at A and B, such that any transversal line of the points A, B, C intersects either none or exactly two of the three segments $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$.*

PROOF. Let P and Q be points such that $\mathfrak{a} = (BC)_P$ and $\mathfrak{b} = (AC)_Q$, and let $t = PQ$. We prove the theorem for $\mathfrak{c} = (AB)_t$. This is a segment by Axiom 4.1. If R is a point of $\mathfrak{a}$, we have $BC \parallel Rt$, and hence $BA \parallel tp$ for $p = QR$. If q is a line through R and such that $AB \parallel pq$, then $(AB)_q = (AB)_t = \mathfrak{c}$ by Prop. 3.1, and $\mathfrak{b} = (AC)_p$. If s is a transversal line of A, B, C through R, then $AC \parallel ps$ or $AB \parallel qs$, but not both (Prop. 2.3). It follows that s intersects one of the two segments $\mathfrak{b}$ and $\mathfrak{c}$, but not the

other. We may prove in the same way that a transversal line through a point of $\mathfrak{b}$ or of $\mathfrak{c}$ intersects exactly one of the other two segments, completing the proof.

In order to generalize Theorem 4.1 we introduce the following definition.

DEFINITION 4.1. Let A, B, C be three distinct points, and let $\mathfrak{a}$ be a segment at B and C, $\mathfrak{b}$ a segment at A and C, and $\mathfrak{c}$ a segment at A and B. We shall say that $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$ are the *sides of a generalized triangle* with *vertices* A, B, C, in symbols: $\Delta(\mathfrak{a}, \mathfrak{b}, \mathfrak{c})$, if any transversal flat of the points A, B, C intersects either none or exactly two of the segments $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$.

THEOREM 4.2. *Let A, B, C be three distinct points, let $\mathfrak{a}$ be a segment at B and C, and let $\mathfrak{b}$ be a segment at A and C. There is a uniquely determined segment $\mathfrak{c}$ at A and B such that $\Delta(\mathfrak{a}, \mathfrak{b}, \mathfrak{c})$.*

PROOF. If A, B, C are not on a line, and if ϱ is a transversal flat, then $\varrho \cap ABC$ is either empty or a transversal line of A, B, C, and the contention follows from Theorem 4.1.

If A, B, C are on a line, let T be a point such that $AB \parallel CT$. If T is in $\mathfrak{a}$, but not in $\mathfrak{b}$, then A is in $\mathfrak{a}$, B is not in $\mathfrak{b}$, $\mathfrak{b} = (AC)_B$ is contained in $\mathfrak{a}$, and $\Delta(\mathfrak{a}, \mathfrak{b}, \mathfrak{c})$ for $\mathfrak{c} = (AB)_C$ by Prop. 3.5. Similarly, if T is in $\mathfrak{b}$, but not in $\mathfrak{a}$, then $\mathfrak{a}$ is contained in $\mathfrak{b}$, and $\Delta(\mathfrak{a}, \mathfrak{b}, \mathfrak{c})$ for $\mathfrak{c} = (AB)_C$. If T is neither in $\mathfrak{a}$ nor in $\mathfrak{b}$, then $\mathfrak{a} = (BC)_A$ and $\mathfrak{b} = (AC)_B$, so that $\Delta(\mathfrak{a}, \mathfrak{b}, \mathfrak{c})$ for $\mathfrak{c} = (AB)_T$, containing $\mathfrak{a}$ and $\mathfrak{b}$ (Prop. 3.5.). If T is in $\mathfrak{a}$ and in $\mathfrak{b}$, then $\mathfrak{a} \cap \mathfrak{b} = (AB)_C$, $\mathfrak{a} \cap (AB)_T = (AC)_B$. and $\mathfrak{b} \cap (AB)_T = (BC)_A$; hence $\Delta(\mathfrak{a}, \mathfrak{b}, \mathfrak{c})$ for $\mathfrak{c} = (AB)_T$.

COROLLARY. *We have $\Delta(\mathfrak{a}, \mathfrak{b}, \mathfrak{c})$ for $\mathfrak{c} = (AB)_C$ if, and only if, one of the two segments $\mathfrak{a}$, $\mathfrak{b}$ is contained in the other.*

THEOREM 4.3. *Let A, B, C be three distinct points, let $\mathfrak{a}$ be a segment at B and C, $\mathfrak{b}$ a segment at A and C, and $\mathfrak{c}$ a segment at A and B, and let ϱ be a transversal flat of A, B, C. If ϱ intersects either none or exactly two of the segments $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, then $\Delta(\mathfrak{a}, \mathfrak{b}, \mathfrak{c})$.*

PROOF. We either have $\Delta(\mathfrak{a}, \mathfrak{b}, \mathfrak{c})$ or $\Delta(\mathfrak{a}, \mathfrak{b}, \mathfrak{d})$ with $\mathfrak{d}$ supplementary to $\mathfrak{c}$. In the second case, ϱ intersects either none or two of the segments $\mathfrak{a}$, $\mathfrak{b}$, δ, and $AB \cap \varrho$ is a point. It follows that ϱ intersects either all three segments $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, or exactly one of them, contrary to our assumption.

PROPOSITION 4.4. *Let A, B, C, D be four distinct points, let $\mathfrak{a}$ be a segment at A and D, let $\mathfrak{b}$ be a segment at B and D, and let $\mathfrak{c}$ be a segment at C and D. If $\mathfrak{p}$, $\mathfrak{q}$, and $\mathfrak{r}$ are segments such that $\Delta(\mathfrak{b}, \mathfrak{c}, \mathfrak{p})$, $\Delta(\mathfrak{a}, \mathfrak{c}, \mathfrak{q})$, and $\Delta(\mathfrak{a}, \mathfrak{b}, \mathfrak{r})$, then $\Delta(\mathfrak{p}, \mathfrak{q}, \mathfrak{r})$.*

PROOF. If ϱ is a transversal flat of A, B, C, D which intersects

$\mathfrak{b}$ and $\mathfrak{p}$, then ϱ does not meet $\mathfrak{c}$, and ϱ intersects either $\mathfrak{r}$, but neither $\mathfrak{a}$ nor $\mathfrak{q}$, or $\mathfrak{a}$ and $\mathfrak{q}$, but not $\mathfrak{r}$. In both cases $\varDelta(\mathfrak{p}, \mathfrak{q}, \mathfrak{r})$ by Theorem 4.3.

THEOREM 4.5. *Let* $\mathfrak{a}$, $\mathfrak{b}$, *and* $\mathfrak{c}$ *be three segments such that* $\varDelta(\mathfrak{a}, \mathfrak{b}, \mathfrak{c})$. *If there is a segment* $\mathfrak{p}$ *supplementary to* $\mathfrak{a}$, *then there are segments* $\mathfrak{q}$ *supplementary to* $\mathfrak{b}$, *and* $\mathfrak{r}$ *supplementary to* $\mathfrak{c}$, *and we have* $\varDelta(\mathfrak{a}, \mathfrak{q}, \mathfrak{r})$, $\varDelta(\mathfrak{q}, \mathfrak{b}, \mathfrak{r})$, *and* $\varDelta(\mathfrak{p}, \mathfrak{q}, \mathfrak{c})$.

PROOF. If ϱ is a transversal flat of the endpoints of $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, which intersects $\mathfrak{a}$, then ϱ intersects either $\mathfrak{b}$ or $\mathfrak{c}$, but not both, and we cannot have $\varDelta(\mathfrak{p}, \mathfrak{b}, \mathfrak{c})$. It follows from Theorem 4.2 that $\varDelta(\mathfrak{q}, \mathfrak{b}, \mathfrak{r})$ for a segment $\mathfrak{r}$ supplementary to $\mathfrak{c}$. Similarly $\varDelta(\mathfrak{p}, \mathfrak{q}, \mathfrak{c})$ for $\mathfrak{q}$ supplementary to $\mathfrak{b}$, and $\varDelta(\mathfrak{a}, \mathfrak{q}, \mathfrak{r})$.

5. Half-flats.

DEFINITION 5.1. Let ϱ be a proper flat and P a point of ϱ. If $\mathfrak{a}$ is a segment at P, but not in ϱ, then $\varrho \mid \mathfrak{a}$ denotes the set of all segments $\mathfrak{p}$ at P such that $\mathfrak{p}$ is either supplementary to $\mathfrak{a}$, or that there is a segment $\mathfrak{q}$ intersecting ϱ in a point, and for which $\varDelta(\mathfrak{a}, \mathfrak{p}, \mathfrak{q})$. A set $\mathfrak{H}$ of segments at P is called a *half-flat* with *boundary* (ϱ, P), if there is a segment $\mathfrak{a}$ at P, but not in ϱ, such that $\mathfrak{H} = \varrho \mid \mathfrak{a}$.

If A is the other endpoint of $\mathfrak{a}$, then $\varrho \mid \mathfrak{a}$ consists of segments at P and in ϱA, but not in ϱ. We shall say that $\varrho \mid \mathfrak{a}$ is a half-flat *on* ϱA. A half-flat on a line is called a *half-line*; a half-flat on a plane is called a *half-plane*.

PROPOSITION 5.1. *If* ϱ *is a proper flat,* P *a point of* ϱ, A *a point not in* ϱ, *and* $\mathfrak{a}$ *a segment at* P *and* A, *then* $\varrho \mid \mathfrak{a}$ *is a non-empty set.*

PROOF. There is a segment $\mathfrak{q}$ at A containing P (Prop. 3.6), and a segment $\mathfrak{p}$ such that $\varDelta(\mathfrak{a}, \mathfrak{p}, \mathfrak{q})$. $\mathfrak{p}$ belongs to $\varrho \mid \mathfrak{a}$.

Let ϱ be a proper flat, let P be a point of ϱ, let $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$ be segments at P, and let A, B, C be the other endpoints of $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, in this order.

LEMMA 5.2. *If* $\mathfrak{b}$ *is in* $\varrho \mid \mathfrak{a}$, *then* $\mathfrak{a}$ *is in* $\varrho \mid \mathfrak{b}$.

This follows directly from the definition.

LEMMA 5.3. *If* $\mathfrak{b}$ *and* $\mathfrak{c}$ *are in* $\varrho \mid \mathfrak{a}$, *then* $\mathfrak{c}$ *is not in* $\varrho \mid \mathfrak{b}$.

PROOF. If $B = C$, then $\mathfrak{a}$ is either supplementary to $\mathfrak{b}$ and to $\mathfrak{c}$, or there is a segment $\mathfrak{r}$ intersecting ϱ, for which $\varDelta(\mathfrak{a}, \mathfrak{b}, \mathfrak{r})$ and $\varDelta(\mathfrak{a}, \mathfrak{c}, \mathfrak{r})$. In both cases, $\mathfrak{b} = \mathfrak{c}$. If $B \neq C$, let $\varDelta(\mathfrak{b}, \mathfrak{c}, \mathfrak{p})$. If $A = B$, and if $\varDelta(\mathfrak{a}, \mathfrak{c}, \mathfrak{q})$, then $\mathfrak{a}$ and $\mathfrak{b}$ are supplementary segments. But then $\mathfrak{p}$ and $\mathfrak{q}$ are supplementary, and ϱ intersects $\mathfrak{q}$, but not $\mathfrak{p}$. If A, B, C are three distinct points, and if $\varDelta(\mathfrak{a}, \mathfrak{c}, \mathfrak{q})$ and $\varDelta(\mathfrak{a}, \mathfrak{b}, \mathfrak{r})$,

then ϱ intersects q and r. Since $\varDelta(\mathfrak{p}, q, r)$ by Prop. 4.4, ϱ does not intersect $\mathfrak{p}$.

LEMMA 5.4. *If ϱ is a transversal flat of A and B, and if $\mathfrak{b}$ is not in $\varrho \mid \mathfrak{a}$, then $\varrho \mid \mathfrak{a} = \varrho \mid \mathfrak{b}$.*

PROOF. If $A = B$, then $\mathfrak{a} = \mathfrak{b}$. Otherwise, let $\varDelta(\mathfrak{a}, \mathfrak{b}, r)$. r does not meet ϱ. If $C = A$, and if $\varDelta(\mathfrak{b}, \mathfrak{c}, \mathfrak{p})$, then c is supplementary to $\mathfrak{a}$ if, and only if, $\mathfrak{p}$ is supplementary to r, that is if, and only if, ϱ intersects $\mathfrak{p}$. If A, B, C are three distinct points, and if $\varDelta(\mathfrak{b}, \mathfrak{c}, \mathfrak{p})$ and $\varDelta(\mathfrak{a}, \mathfrak{c}, q)$, then $\varDelta(\mathfrak{p}, q, r)$ by Prop. 4.4, so that ϱ intersects $\mathfrak{p}$ if, and only if, ϱ intersects q. In each case, c is in $\varrho \mid \mathfrak{a}$, if, and only if, c is in $\varrho \mid \mathfrak{b}$.

THEOREM 5.5. *If ϱ is a proper flat, P a point of ϱ, A a point not in ϱ, $\mathfrak{a}$ a segment at P and A, and $\mathfrak{b}$ a segment of $\varrho \mid \mathfrak{a}$, then a segment c at P and in ϱA is either contained in ϱ or belongs to exactly one of the two half-flats $\varrho \mid \mathfrak{a}$ and $\varrho \mid \mathfrak{b}$ with boundary (ϱ, P). If c is in $\varrho \mid \mathfrak{a}$, then $\varrho \mid \mathfrak{c} = \varrho \mid \mathfrak{b}$, and if c is in $\varrho \mid \mathfrak{b}$, then $\varrho \mid \mathfrak{c} = \varrho \mid \mathfrak{a}$.*

In other words, there are exactly two half-flats on ϱA and with boundary (ϱ, P). Two half-flats in this position are called *complementary* half-flats.

PROOF. Let c be a sement at P and in ϱA, but not in ϱ. If c does not belong to $\varrho \mid \mathfrak{a}$, then $\varrho \mid \mathfrak{c} = \varrho \mid \mathfrak{a}$ by Lemma 5.4.; hence c is in $\varrho \mid \mathfrak{b}$ by Lemma 5.2. If c is in $\varrho \mid \mathfrak{a}$, then c is not in $\varrho \mid \mathfrak{b}$ by Lemma 5.3, and $\varrho \mid \mathfrak{c} = \varrho \mid \mathfrak{b}$ by Lemma 5.4.

COROLLARY. *If c and $\mathfrak{b}$ are two supplementary segments at P and in ϱA, but not in ϱ, then one of the two segments c, $\mathfrak{b}$ is in $\varrho \mid \mathfrak{a}$, the other in $\varrho \mid \mathfrak{b}$.*

PROPOSITION 5.6. *Let ϱ be a proper flat, P a point of ϱ, A a point not in ϱ, and $\mathfrak{a}$ a segment at P and A. If σ is a flat contained in ϱ and containing P, then $\sigma \mid \mathfrak{a}$ is the set of all segments belonging to $\varrho \mid \mathfrak{a}$ and contained in σA.*

This follows directly from Def. 5.1.

PROPOSITION 5.7. *Two segments at a point P are in the same half-line with boundary (P, P) if, and only if, one of the two segments is contained in the other.*

This follows directly from the Corollary of Theorem 4.2.

PROPOSITION 5.8. *Let ϱ be a proper flat containing two distinct points P and Q, let $\mathfrak{p}$ be a segment at P and Q, let $\mathfrak{a}$ and $\mathfrak{b}$ be two segments at P, but not in ϱ, and let c and $\mathfrak{d}$ be segments at Q such that $\varDelta(\mathfrak{a}, \mathfrak{p}, \mathfrak{c})$ and $\varDelta(\mathfrak{b}, \mathfrak{p}, \mathfrak{d})$. If $\mathfrak{b}$ is in $\varrho \mid \mathfrak{a}$, then $\mathfrak{d}$ is in $\varrho \mid \mathfrak{c}$.*

PROOF. If $\mathfrak{b}$ is supplementary to $\mathfrak{a}$, then $\mathfrak{d}$ is supplementary to c. Otherwise there is a segment r intersecting ϱ and such that $\varDelta(\mathfrak{a}, \mathfrak{b}, r)$. But then $\varDelta(\mathfrak{c}, \mathfrak{d}, r)$ by Prop. 4.4, and $\mathfrak{d}$ is in $\varrho \mid \mathfrak{c}$.

Prop. 5.8 establishes a one-one correspondence between segments at P and segments at Q, which maps half-flats with boundary (ϱ, P) on half-flats with boundary (ϱ, Q).

PROPOSITION 5.9. *Let a, b, c, d be four distinct lines of a pencil P/π, let c be a segment at P and on c, and let b be a segment at P and on d. $ab \parallel cd$ if, and only if, b is in one of the two half-planes $a \mid c$ and $b \mid c$, but not in the other.*

PROOF. Let $\varDelta(c, b, p)$, and let a be the sector with sides c and d containing p. $ab \parallel cd$ if, and only if, exactly one of the lines a, b is a line of a, hence if, and only if, exactly one of the lines a, b intersects the segment p, and thus if, and only if, b is in one of the two half-planes $a \mid c$ and $b \mid c$, but not in the other.

If a and b are two distinct lines through a point P, if a is a segment on a and at P, and if b is a segment on b and at P, then the intersection of the half-planes $b \mid a$ and $a \mid b$ is called an *angle* with *sides* $P \mid a$ and $P \mid b$ and with *vertex* P. It follows from Prop. 5.9 that the segments of an angle with sides on a and on b are contained in a sector with sides a and b. The segments of supplementary angles are contained in supplementary sectors; the segments of opposite angles are contained in the same sector.

6. Descriptive geometry.

THEOREM 6.1. *If there is a line l and a point P not on l such that every line of the pencil P/lP intersects the line l in a point, then two lines in a plane always intersect in a point.*

PROOF. Let A and B be two distinct points, and let C and D be two points on l and different from A and from B. It follows from Prop. 1.1 that every transversal line of C and D intersects l, and hence from Theorem 3.4 that there are two supplementary segments c and b at C and D. Let p be a segment at A and C and q a segment at B and D. By Theorem 4.5, we have $\varDelta(c, p, r)$ and $\varDelta(b, p, \hat{s})$ for supplementary segments r and $\hat{s}$, and hence $\varDelta(q, r \, a)$ and $\varDelta(p, \hat{s}, b)$ for supplementary segments a and b at A and B. But then every transversal line of A and B intersects AB, proving the theorem.

An incidence geometry may be called a *descriptive* geometry if, for any two distinct points A and B, there is a transversal line of A and B which does not intersect the line AB. It follows from Theorem 6.1 that a geometry satisfying our axioms is either a projective geometry or a descriptive geometry in this sense.

DEFINITION 6.1. Let A, B, C be three distinct points. We say that C is *between* A and B, in symbols: (ACB), if there is a line t not intersecting the line AB and such that $AB \parallel Ct$.

If there are points A, B, C such that (ACB), then our geometry is descriptive. There is a unique segment with endpoints A and B, and Theorem 3.4 implies the following proposition.

PROPOSITION 6.2. *Let A and B be two distinct points. If there is a point C such that (ACB), then the segment with endpoints A and B is the set of all points P such that (APB).*

In other words, Def. 3.3 is equivalent in a descriptive geometry to the usual definition of segments in terms of betweenness.

Hilbert's axioms of order are easily verified for a descriptive geometry satisfying our axioms. If (ABC), then A, B, C are three points on a line by definition, and (ABC) implies (CBA) by Axiom 2.3 and Prop. 2.1. This proves Hilbert's Axiom II.1. If A and B are two distinct points, then there is a point C such that (ABC) by Prop. 2.4. The relations (BAC), (ABC), and (ACB) exclude each other by Prop. 2.2. This proves Axioms II.2 and II.3. Axiom II.4, the Axiom of Pasch, is an immediate consequence of our Theorem 4.1.

Conversely, separation of lines in a pencil in descriptive geometry may be defined in terms of betweenness as follows: $ab \parallel cd$ if, and only if, a, b, c, d are four distinct lines of a pencil P/π, such that there are points A and E on a, B on b, C on c, and D on d, for which (ACB), (BDE), and (APE).

With this definition, our Axioms 2.1—2.6 become consequences of Hilbert's axioms of order, and so does Axiom 4.1 for segments in the sense of Prop. 6.2. The first part of Theorem 3.4 remains valid for segments in this sense, but our Def. 3.3 is equivalent to the usual definition of segments (by Prop. 6.2) only if, for any two distinct points A and B, there is a transversal line of A and B which does not meet the line AB. This is not a consequence of Hilbert's axioms of order, as is well known.

If ϱ is a proper flat in descriptive geometry and A a point not in ϱ, then the *half-flat* ϱ/A is usually defined as the set of all points B such that the segment AB intersects the flat ϱ in a point. If P is a point of ϱ and $\mathfrak{a}$ the segment at P and A, then ϱ/A is the set of all second endpoints of the segments of $\varrho \mid \mathfrak{a}$, as defined in § 5.

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