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by

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Introduction.

It is known that certain positive (regular) transforms of a function $s(t)$, or of a sequence s_n, possess the property that the boundedness of the transform implies the boundedness of $s(t)$, or of s_n, when an appropriate Tauberian condition is imposed on $s(t)$, or on s_n. Two transforms with the property stated above, covering many particular cases of summability processes, have been considered by Karamata ([3], Théorème IV) and Hardy ([2], Theorem 238). Theorems A and B of this note show that, in the case of these two transforms, the oscillation of the transform is equal to the oscillation of $s(t)$, or s_n, provided that certain additional conditions are assumed, one of which consists in a refinement of the Tauberian hypothesis on $s(t)$, or on s_n.

1. The first theorem.

The following theorem generalizes the essentials of a Tauberian oscillation theorem of V. Ramaswami ([6], Theorem I.2), somewhat like a theorem of H. Delange ([1], Théorème 11), but more completely, since it includes an analogue of Ramaswami's theorem for the Borel transform. It is a refinement of an oscillation theorem of which one version is Lemma 3 A of this note proved by Karamata ([4], Satz IV); and it is on the same lines as a convergence theorem of Karamata ([4], Satz VI; [3], pp. 36—7, § 18).[1])

THEOREM A. *Let the following assumptions be made.*

(i) $\psi(x, t) \geqq 0$ *for all large* x *and every* $t \geqq 0$;

(ii) $\varphi(x, y) \equiv \int_y^\infty \psi(x, t)dt \to 1$ *as* $x \to \infty$, *for every* $y \geqq 0$; $\varphi(x, 0) = 1$.

[1]) Basing ourselves on Karamata's ideas, we can extend also a theorem of Delange on absolute Tauberian constants ([1]), Théorème 3) so that it covers the case of the Borel transform. Such an extension appears elsewhere [5].

(iii) $y' = y'(x), y'' = y''(x)$ *are single-valued, steadily increasing, unbounded functions of* x, *and conversely, such that, given any small* $\varepsilon > 0$, *we have, for all large enough* x, y', y'',

$$\text{(1)} \qquad \varphi(x, y') < \varepsilon, \ \varphi(x, y'') > 1 - \varepsilon,$$

with the implication, on account of (i), *that* $y'' < y'$.

(iv) $A(t)$ *is a continuous, strictly increasing, unbounded function satisfying the condition*

$$\text{(2)} \qquad \int_0^\infty \psi(x, t) \left| \log \frac{A(t)}{A(y)} \right| dt < K \ \ (\textit{a constant}),$$

where $y =$ *either* y' *or* y''.

(v) $s(t)$ *is a function of bounded variation in every finite interval of* $(0, \infty)$ *subject to the conditions*:

$$\text{(3)} \qquad \underset{t \leqq t' \leqq T}{\text{bound}} \{s(t') - s(t)\} = o_L(1) \log \lambda \ \ \textit{as} \ t \to \infty,$$

for $A(T) = \lambda A(t)$, $\lambda > 1$;

$$\text{(4)} \qquad \Psi(x) \equiv \int_0^\infty \psi(x, t) s(t) dt = O(1) \ \ \textit{as} \ x \to \infty.$$

Then

$$\text{(5)} \qquad \underline{\lim}_{t \to \infty} s(t) = \underline{\lim}_{x \to \infty} \Psi(x), \ \overline{\lim}_{t \to \infty} s(t) = \overline{\lim}_{x \to \infty} \Psi(x).$$

2. Lemmas.

To prove Theorem A we require the following lemmas.

Lemma 1A. *Hypotheses* (i) *and* (ii) *of Theorem* A *imply*

$$\underline{\lim}_{t \to \infty} s(t) \leqq \underline{\lim}_{x \to \infty} \Psi(x) \leqq \overline{\lim}_{t \to \infty} s(t), \ \textit{where} \ \Psi(x) = \int_0^\infty \psi(x, t) s(t) dt,$$

$s(t)$ *being a function of bounded variation in every finite interval of* $(0, \infty)$.

This is a simple Abelian result proved by an argument of the usual type (cf. [2], proof of Theorem 9).

Lemma 2A. *The condition*

$$\text{(3')} \qquad \underset{t \leqq t' \leqq T}{\text{bound}} \{s(t') - s(t)\} > - \omega < 0, \ A(T) = \lambda A(t), \ \lambda > 1,$$

involves

$$\text{(6)} \qquad s(u) - s(t) > - \frac{\omega}{\log \lambda} \log \left\{ \lambda \frac{A(u)}{A(t)} \right\} \ \textit{for every} \ u \geqq t > 0.$$

This lemma is due to Karamata ([4], Hilfsatz 1) and may be readily proved in the form

$$s(V(u)) - s(V(t)) > -\frac{\omega}{\log \lambda}\log\left(\lambda\frac{u}{t}\right),\ u \geqq t > 0,$$

obtained by replacing u, t in (6) by $V(u)$, $V(t)$ respectively, $V(x)$ being the function which is the inverse of $\Lambda(x)$. For, assuming that $\lambda^{r-1}t \leqq u < \lambda^r t\ (r \geqq 1)$, we have

$$s(V(u)) - s(V(t)) = \{s(V(u)) - s(V(\lambda^{r-1}t))\}$$
$$+ \{s(V(\lambda^{r-1}t)) - s(V(\lambda^{r-2}t))\} + \ldots + \{s(V(\lambda t)) - s(V(t))\},$$

and thus

$$(7) \qquad s(V(u)) - s(V(t)) > -\omega r,$$

by virtue of (3′) in the form

$$s(V(t')) - s(V(t)) > -\omega,\ t \leqq t' \leqq \lambda t.$$

Since

$$r - 1 \leqq \frac{\log (u/t)}{\log \lambda},$$

it follows from (7) that

$$s(V(u)) - s(V(t)) > -\omega\left\{1 + \frac{\log (u/t)}{\log \lambda}\right\}$$

which, as already explained, is equivalent to (6).

Lemma 3A. *If, in the hypotheses of Theorem* A, (1) *is replaced by*

$$(1') \qquad \varphi(x, y') < c' < c'' < \varphi(x, y'')$$

and (3) *is replaced by* (3′), *the conclusion of the theorem will assume the form*

$\Psi(x) = O(1)$ *as* $x \to \infty$ *involves* $s(t) = O(1)$ *as* $t \to \infty$.

In particular, since (1) *implies* (1′) *and* (3) *implies* (3′), *conditions* (1)—(4) *together imply* $|s(t)| < \varkappa$ *for* $t \geqq 0$.

This lemma again is due to Karamata ([4], Satz IV; cf. [3], p. 35, Théorème IV).

3. Proof of Theorem A.

In the identity

$$(8) \qquad \Psi(x) = \int_0^{y''} \psi(x, t)s(t)\,dt + \int_{y''}^{y'} \ldots + \int_{y'}^{\infty} \ldots,$$

suppose that $x \geqq X$ and $\psi(x, t) \geqq 0$. First let y'' assume an ascending, divergent sequence of values for which

$$s(y'') \to \bar{s} \equiv \overline{\lim_{t\to\infty}}\, s(t),$$

this limit being finite as a result of Lemma 3A. By hypothesis (iii), the corresponding sequence of values of $x = x(y')$ and the sequence of values of $y = y'(x)$ are both ascending, divergent and such that, for all large y', x, y'',

$$(9)\quad \int_0^{y''} \psi(x, t)dt \equiv \int_0^{\infty} \ldots - \varphi(x, y'') < \varepsilon,\ \int_{y'}^{\infty} \psi(x, t)dt \equiv \varphi(x, y') < \varepsilon.$$

Also, in consequence of (3), we can choose t_0 so that, for $t \geqq t_0$,

$$\underline{\operatorname{bound}}_{t \leqq t' \leqq T} \{s(t') - s(t)\} > -\varepsilon \log \lambda,\ A(T) = \lambda A(t),$$

and hence, by Lemma 2A,

$$(10)\qquad s(u) - s(t) > -\varepsilon \log\left\{\lambda \frac{A(u)}{A(t)}\right\},\ u \geqq t \geqq t_0.$$

In (8) we can confine ourselves to values of $y'' \geqq t_0$ for which $x \geqq X$, and use (10), obtaining

$$\begin{aligned}\Psi(x) &> \int_0^{y''} \psi(x, t)\, s(t)\, dt + s(y'') \int_{y''}^{y'} \psi(x, t)\, dt \\ &\quad - \varepsilon \int_{y''}^{y'} \psi(x, t) \log\left\{\lambda \frac{A(t)}{A(y'')}\right\} dt + \int_{y'}^{\infty} \psi(x, t)\, s(t)\, dt \\ &> -\varkappa \int_0^{y''} \psi(x, t)dt + s(y'')\Big[\int_0^{\infty} \psi(x, t)\, dt - \int_0^{y''} \ldots - \int_{y'}^{\infty} \ldots\Big] \\ &\quad - \varepsilon \int_{y''}^{y'} \psi(x, t) \log\left|\lambda \frac{A(t)}{A(y'')}\right| dt - \varkappa \int_{y'}^{\infty} \psi(x, t)\, dt\end{aligned}$$

since $s(t) > -\varkappa$ by Lemma 3A. From the last step, letting x, y', $y'' \to \infty$, and using (9) and (2), we get

$$(11)\qquad \overline{\Psi} \equiv \overline{\lim_{x\to\infty}}\, \Psi(x) \geqq \bar{s} - 2(\bar{s} + \varkappa)\varepsilon - K\varepsilon.$$

Hence, ε being arbitrary,

$$\overline{\Psi} \geqq \bar{s}.$$

Next we let y' assume, an ascending divergent sequence of values such that

$$s(y') \to \underline{s} \equiv \underline{\lim}_{t\to\infty}\, s(t)$$

where $\underline{s}$ is finite, by Lemma 3A. Hypothesis (iii) shows that the corresponding sequences of values of $x = x(y')$, $y'' = y''(x)$ are also ascending, divergent and conditioned by (9). If we restrict ourselves to values of $x \geqq X$ and of $y'' \geqq t_0$, (8) and (10) together give

$$\begin{aligned} \Psi(x) &< \int_0^{y''} \psi(x, t)\, s(t)\, dt + s(y') \int_{y''}^{y'} \psi(x, t)\, dt \\ &+ \varepsilon \int_{y''}^{y'} \psi(x, t) \log\left\{ \lambda \frac{A(y')}{A(t)} \right\} dt + \int_{y'}^{\infty} \psi(x, t)\, s(t)\, dt \\ &< \varkappa \int_0^{y''} \psi(x, t) dt + s(y') \left[\int_0^{\infty} \psi(x, t)\, dt - \int_0^{y''} \dots - \int_{y'}^{\infty} \dots \right] \\ &+ \varepsilon \int_{y''}^{y'} \psi(x, t) \log \left| \lambda \frac{A(t)}{A(y')} \right| dt + \varkappa \int_{y'}^{\infty} \psi(x, t)\, dt \end{aligned}$$

since $s(t) < \varkappa$ by Lemma 3A. Letting x, y', $y'' \to \infty$ in the last step, appealing to (9) and (2), and remembering that ε can be chosen arbitrarily small, we conclude that

$$\underline{\Psi} \equiv \varliminf_{x \to \infty} \Psi(x) \leqq \underline{s}.$$

From the results proved above, we have

$$\underline{\Psi} \leqq \underline{\bar{s}} \leqq \overline{\Psi},$$

whence the desired conclusion follows by an appeal to Lemma 1A.

4. Deductions from Theorem A.

(i) Following Karamata ([4], p. 6), we shall define the Borel transform of a sequence s_n as

$$B(x) = \int_0^{\infty} \varphi(x, t)\, d\{s(t)\},$$

where

$$s(t) = s_n \text{ for } n \leqq t < n + 1,\ n = 0, 1, 2, \dots,$$

$$\varphi(x, y) = \frac{1}{\Gamma(y)} \int_0^x e^{-t} t^{y-1}\, dt,$$

and $\psi(x, t) \equiv -\partial\varphi(x, t)/\partial t$ satisfies the initial conditions in hypotheses (i), (ii), of Theorem A.

Now, in the case of the Borel transform, it is known ([4], p. 7) that

$$\varphi(x, y) \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} e^{-t^2/2}\, dt \text{ when } x = y - 1 + a\sqrt{y - 1},\ y \to \infty.$$

Consequently we can choose $a'' > 0$ and $a' < 0$ so that

$$\varphi(x, y'') \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a''} e^{-t^2/2} dt > 1 - \varepsilon/2,\ x = y'' - 1 + a''\sqrt{y'' - 1},\ y'' \to \infty,$$

$$\varphi(x, y') \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a'} e^{-t^2/2} dt < \varepsilon/2,\ x = y' - 1 + a'\sqrt{y' - 1},\ y' \to \infty,$$

where the sign before each square root is positive. Therefore

$$\varphi(x, y'') > 1 - \varepsilon,\ \varphi(x, y') < \varepsilon,\ \text{for } x > x_0,\ y' > y_0',\ y'' > y_0'';$$

i.e. (1) is satisfied.

In the case of the Borel transform, it is also known ([4], pp. 7—8) that, with $A(t) = e^{\sqrt{t}}$ and any given real a,

$$(12)\quad \int_0^\infty \psi(x, t) \left| \log \frac{A(t)}{A(y)} \right| dt < 2\frac{x^n \sqrt{n}}{e^x n!} + \frac{|x - n|}{\sqrt{n}} + o(1) < K(a)$$

where

$$n = [y],\quad x = y - 1 + a\sqrt{y - 1} \to \infty.$$

Hence, taking successively $a = a'$, $y = y'$ and $a = a''$, $y = y''$ in (12), we see that (2) holds. The fact that K depends on a' in the first case and on a'' in the second case does not vitiate the conclusion drawn from a step like (11) since a', a'' are kept fixed when x, y' $y'' \to \infty$.

Lastly, with our choice of $A(t) = e^{\sqrt{t}}$, we find that T in (3) is given by

$$T = V\{\lambda A(t)\} = (\sqrt{t} + \log \lambda)^2,$$

or

$$T/t = 1 + O(\log \lambda/\sqrt{t}),\ t \to \infty.$$

Combining the results of the last three paragraphs, we see that Theorem A contains the following as a particular case.

COROLLARY 1A. *The conditions*

$$B(x) = e^{-x} \sum_{n=0}^{\infty} s_n \frac{x^n}{n!} = O(1),\ \textit{as}\ x \to \infty,$$

$$(\mathrm{T}_1)\qquad \min_{n \leqq n' \leqq n + \delta\sqrt{n}} \{s_{n'} - s_n\} = o_L(1)\delta,\ \textit{as}\ n \to \infty,$$

together imply

$$\operatorname*{osc}_{n\to\infty} s_n = \operatorname*{osc}_{x\to\infty} B(x).$$

Condition (T_1) *can be replaced by* (T_1^*) *below, by an argument as in* § 5.

$$(\mathrm{T}_1^*)\qquad \varliminf_{n\to\infty} \min_{n \leqq n' \leqq n + \delta\sqrt{n}} \{s_{n'} - s_n\} = o_L(\delta),\ \textit{as}\ \delta \to 0.$$

(ii) A version of Ramaswami's oscillation theorem referred to at the outset is as follows.

COROLLARY 2A. *The condition*

$$\Psi^*(x) = \frac{1}{x}\int_0^\infty \psi^*\left(\frac{t}{x}\right)s(t)\,dt = O(1), \text{ as } x \to \infty,$$

where

$$\psi^*(t) \geqq 0, \quad \int_0^\infty \psi^*(t)\,dt = 1, \qquad \int_0^\infty \psi^*(t)\left|\log t\right| dt \text{ exists},$$

in conjunction with the condition

$$(\mathrm{T}_2) \qquad \operatorname*{bound}_{t \leqq t' \leqq \lambda t} \{s(t') - s(t)\} = o_L(1)\log\lambda, \text{ as } t \to \infty,$$

implies $$\operatorname*{osc}_{t\to\infty} s(t) = \operatorname*{osc}_{x\to\infty} \Psi^*(x).$$

We can replace condition (T_2) *by* (T_2^*) *below, arguing as in* § 5.

$$(\mathrm{T}_2^*) \qquad \lim_{t\to\infty} \operatorname*{bound}_{t \leqq t' \leqq \lambda t} \{s(t') - s(t)\} = o_L(\log\lambda), \text{ as } \lambda \to 1.$$

To prove Corollary 2A, we take

$$\psi(x, t) = \frac{1}{x}\psi^*\left(\frac{t}{x}\right)$$

so that

$$\varphi(x, y) = \int_y^\infty \frac{1}{x}\psi^*\left(\frac{t}{x}\right)dt = \int_a^\infty \psi^*(t)\,dt, \quad a = \frac{y}{x};$$

and we choose a', a'' so as to satisfy condition (1):

$$\varphi(x, y') \equiv \varphi(x, a'x) = \int_{a'}^\infty \psi^*(t)dt < \varepsilon,$$

$$\varphi(x, y'') \equiv \varphi(x, a''x) = \int_{a''}^\infty \psi^*(t)dt > 1 - \varepsilon.$$

Also, with $\psi(x, t) = \frac{1}{x}\psi^*\left(\frac{t}{x}\right)$, $\Lambda(t) = t$, $y' = a'x$, $y'' = a''x$, we find that condition (2) reduces to

$$\int_0^\infty \psi^*(t)\log\left|\frac{t}{a}\right| dt < K(a), \quad a = \text{either } a' \text{ or } a'',$$

and is ensured by the restrictions on ψ^* in Corollary 2A. The proof of Corollary 2A is thus complete.

The well-known particular cases of Corallary 2A, as of Ramaswami's theorem, are given by:

$$(a)\quad \psi^*(u) = -\frac{d}{du}e^{-u},\qquad (b)\quad \psi^*(u) = -\frac{d}{du}(1+u)^{-\varrho},\quad \varrho > 0,$$

$$(c)\quad \psi^*(u) = -\frac{d}{du}\left(\frac{u}{e^u - 1}\right).$$

5. Remark on hypothesis (3) of Theorem A.

This hypothesis may be replaced by the apparently milder one that there exists a sequence $\{\lambda_p\}$ *such that* $1 < \lambda_p \to 1$ *as* $p \to \infty$ *and*

$$(3^*)\qquad \varliminf_{t\to\infty} \underline{\operatorname{bound}}_{t \leqq t' \leqq T} \{s(t') - s(t)\} = o_L(\log \lambda_p) \text{ as } p \to \infty,$$
$$\text{for } \Lambda(T) = \lambda_p \Lambda(t).$$

To justify the replacement of (3) by (3*) we take (3), (3*) in the forms

$$(13)\qquad \underline{\operatorname{bound}}_{t \leqq t' \leqq \lambda t} \{s(V(t')) - s(V(t))\} = o_L(1)\log\lambda, \text{ as } t \to \infty,$$

$$(13^*)\quad \varliminf_{t\to\infty} \underline{\operatorname{bound}}_{t \leqq t' \leqq \lambda_p t} \{s(V(t')) - s(V(t))\} = o_L(\log \lambda_p), \text{ as } p \to \infty,$$

respectively, and argue, as in the proof of Lemma 2A, that (13*) implies (13). The actual argument is as follows.

(13*) shows that, corresponding to any $\lambda > 1$, we can find $\lambda_p < \lambda$ and such that, for all large t,

$$(14)\qquad s(V(t')) - s(V(t)) > -\frac{\varepsilon}{2}\log\lambda_p,\quad t \leqq t' \leqq \lambda_p t.$$

There is evidently a positive integer $r \geqq 2$ such that $\lambda_p^{r-1} < \lambda \leqq \lambda_p^r$. Hence (14) gives, for all large t and $t \leqq t' \leqq \lambda t$,

$$s(V(t')) - s(V(t)) = \{s(V(t') - s(V(\lambda_p^{r-1}t))\}$$
$$+ \{s(V(\lambda_p{}^{r-1}t)) - s(V(\lambda_p{}^{r-2}t))\} + \ldots + \{s(V(\lambda_p t)) - s(V(t))\}$$
$$> -\frac{\varepsilon}{2} r\log\lambda_p > -\frac{\varepsilon}{2}\ \frac{r}{r-1}\log\lambda \geqq -\varepsilon\log\lambda.$$

The conclusion reached above leads at once to (13) and shows that (3*) implies (3) and so may replace (3) in the enunciation of Theorem A.

6. A supplementary theorem.

THEOREM B. *Let the following assumptions be made.*

(i) $\qquad c_n(x) \geqq 0$ *for* $n = 0, 1, 2, \ldots$ *and* $x > 0$;

(ii) $c_n(x) \to 0$ *as* $x \to \infty$, $\Sigma\, c_n(x) = 1$.

(iii) $f(u)$ *is positive and differentiable for* $u \geqq 1$;

$$f \to \infty,\ 0 < f' < k = a \text{ constant},$$

$$F(u) = \int_1^u \frac{dt}{f(t)} \quad (\text{so that } F \to \infty \text{ with } u).$$

(iv) x *and positive integers* M, N *are defined by the relations*

$$F(x) - F(M) = \mu,\ F(N) - F(x) = \nu,$$

with the condition that, given any small $\varepsilon > 0$, *we have, for all sufficiently large* x, M, N, μ, ν,

$$(15)\quad \sum_{n=0}^{M} c_n(x) < \varepsilon,\ \sum_{n=N}^{\infty} c_n(x) < \varepsilon,\ \sum_{n=N}^{\infty} c_n(x)\{F(n) - F(N)\} < \varepsilon,$$

while, for large enough fixed μ, ν *and all sufficiently large* x, M, N, *we have, in addition to* (15),

$$(16)\quad \sum_{n=M}^{\infty} c_n(x)\,|\,F(n) - F(p)\,| < K(\mu, \nu), \text{ where } p = \text{either } M \text{ or } N.$$

(v) $s(t) = s_n$ *for* $n \leqq t < n + 1$ *and satisfies the conditions*:

$$(17)\quad \underset{t \leqq t' \leqq T}{\underline{\text{bound}}}\, \{s(t') - s(t)\} = o_L(1)\delta, \text{ as } t \to \infty, \text{ for } T = t + \delta f(t),\ \delta > 0;$$

$$(18)\quad \tau(x) \equiv \Sigma\, c_n(x) s_n = O(1) \text{ as } x \to \infty.$$

Then

$$(19)\quad \varliminf_{n \to \infty} s_n = \varliminf_{x \to \infty} \tau(x),\ \varlimsup_{n \to \infty} s_n = \varlimsup_{x \to \infty} \tau(x).$$

7. Further Lemmas.

In the proof of Theorem B we require the following lemmas which are similar to Lemmas 1A, 2A, 3A.

LEMMA 1B. *Hypotheses* (i), (ii) *of Theorem* B *make*

$$\varliminf_{n \to \infty} s_n \leqq \varlimsup_{x \to \infty} \tau(x) \leqq \varlimsup_{n \to \infty} s_n, \text{ where } \tau(x) = \Sigma\, c_n(x) s_n.$$

This lemma is established like Lemma 1A.

LEMMA 2B. *The condition*

$$(17')\quad \underset{t \leqq t' \leqq T}{\underline{\text{bound}}}\, \{s(t') - s(t)\} > -\omega < 0,\ T = t + \delta f(t),\ \delta > 0,$$

where $s(t)$ *is defined as in hypothesis* (v) *of Theorem* B *and* $f(u)$ *is as in hypothesis* (iii) *of Theorem* B, *implies*

$$(20)\quad s(u) - s(t) > -\omega\left(\frac{1}{\delta} + k\right)\{F(u) - F(t)\} - \omega \text{ for } u \geqq t \geqq 1.$$

This is a known result ([2], Theorem 239).

LEMMA 3B. *If, in the hypotheses of Theorem* B, (17) *is replaced by condition* (17′) *of Lemma* 2B, *and* (16) *is dropped, then the conclusion of Theorem* B *will assume the form*

$$\tau(x) = O(1) \text{ as } x \to \infty \text{ involves } s_n = O(1) \text{ as } n \to \infty.$$

Since (17) *implies* (17′), *conditions* (15), (17), (18) *together imply* $|s_n| < \varkappa$ *for* $n \geqq 0$.

This is a theorem of Vijayaraghavan and Hardy ([2], Theorem 238).

8. Proof of Theorem B.

The proof may be modelled on that of Theorem A and divided into two parts which separately lead us to infer that

$$(21) \qquad \bar{\tau} \equiv \overline{\lim_{x\to\infty}}\, \tau(x) \geqq \overline{\lim_{n\to\infty}}\, s_n \equiv \bar{s}, \quad \underline{\tau} \equiv \underline{\lim_{x\to\infty}}\, \tau(x) \leqq \underline{\lim_{n\to\infty}}\, s_n \equiv \underline{s}.$$

To justify the first inference of (21), we begin by fixing μ, ν, x_0, M_0, N_0 so that, for $x \geqq x_0$, $M \geqq M_0$, $N \geqq N_0$, (15) and (16) hold. We then find $M_1 \geqq M_0$ (and correspondingly $x_1 \geqq x_0$, $N_1 \geqq N_0$) so that

$$\min_{M \leqq n \leqq M+\delta\varphi(M)} (s_n - s_M) > -\varepsilon\delta \text{ for } M \geqq M_1.$$

This is possible by hypothesis (17) of Theorem B, and it ensures, as a result of Lemma 2B,

$$(22) \quad s_n - s_M > -\varepsilon(1+k\delta)\{F(n) - F(M)\} - \varepsilon\delta \text{ for } n \geqq M \geqq M_1.$$

Next we write

$$(23) \quad \tau(x) = \sum_{n=0}^{M-1} c_n(x)s_n + \sum_{n=M}^{N} \ldots + \sum_{n=N+1}^{\infty} \ldots = \tau_1(x) + \tau_2(x) + \tau_3(x)$$

and choose M to be one of an ascending, divergent sequence of integers such that

$$s_M \to \bar{s}$$

where $\bar{s}$ is finite since $|s_n| < \varkappa$ by Lemma 3B. Then, using (15) in $\tau_1(x)$ and $\tau_3(x)$, and using (22) in $\tau_2(x)$, we obtain from (23):

$$\tau(x) > -\varkappa \sum_{n=0}^{M-1} c_n + \sum_{n=M}^{N} c_n(s_M - \varepsilon\delta)$$
$$-\sum_{n=M}^{N} c_n \varepsilon(1 + k\delta)\{F(n) - F(M)\} - \varkappa \sum_{n=N+1}^{\infty} c_n$$

$$= -(s_M - \varepsilon\delta + \varkappa)\left(\sum_{n=0}^{M-1} c_n + \sum_{n=N+1}^{\infty} c_n\right) + s_M - \varepsilon\delta$$
$$- \varepsilon(1 + k\delta)\sum_{n=M}^{N} c_n\{F(n) - F(M)\}$$
$$> -2(s_M - \varepsilon\delta + \varkappa)\varepsilon + s_M - \varepsilon\delta - \varepsilon(1 + k\delta)K, \tag{24}$$

if we suppose (as we may) that $s_M > -\varkappa + \varepsilon\delta$ and use (16). Letting $M \to \infty$ in (24), and remembering that ε is arbitrary and K fixed (on account of μ, ν being fixed), we obtain

$$\bar{\tau} \geqq \bar{s}.$$

The second inference of (21) is justified in the same way as the first, and the two inferences taken along with Lemma 1B yield conclusion (19).

9. Deductions from Theorem B.

We can deduce Corollary 1A *from Theorem* B, *taking*

$$c_n(x) = e^{-x}x^n/n!, \quad f(u) = 2\sqrt{u},$$

and using well-known properties of the Borel transform ([2], p. 313, § 12.15; [4], pp. 7—8).

We may also take

$$c_n(x) = \frac{1}{x}\, g\left(\frac{n}{x}\right), \quad g(t) = \left(\frac{\sin t}{t}\right)^2, \quad f(u) = u,$$

and deduce

COROLLARY 1B. *The condition*

$$\tau(x) = \frac{2x}{\pi}\sum \frac{\sin^2 (n/x)}{n^2}\, s_n = O(1) \ \textit{as}\ x \to \infty,$$

along with either (T_2) *or* (T_2^*) *of Corollary* 2A, *ensures*

$$\operatorname*{osc}_{n\to\infty} s_n = \operatorname*{osc}_{x\to\infty} \tau(x).$$

The deduction of Corollary 1B from Theorem B requires us to verify that conditions (15) and (16) of the theorem are fulfilled for the particular choices of c_n and f in the corollary. That (15) is fulfilled is known ([2], proof of Theorem 240). That (16) is fulfilled follows from the facts:

$$x\sum_{M}^{\infty}\frac{\sin^2(n/x)}{n^2}\log\frac{n}{M} = O\left(\frac{x}{M}\int_1^{\infty}\frac{\log u}{u^2}du\right) < K$$

when x, $M \to \infty$, $\log x - \log M = \mu$ (fixed); and

$$x\sum_{M}^{\infty}\frac{\sin^2(n/x)}{n^2}\left|\log\frac{n}{N}\right| = x\sum_{M}^{N}\frac{\sin^2(n/x)}{n^2}\log\frac{N}{n} + x\sum_{N+1}^{\infty}\frac{\sin^2(n/x)}{n^2}\log\frac{n}{N}$$

$$< O\left(x\int_{M-1}^{N}\frac{\log(N/u)}{u^2}du\right) + K'$$

$$= O\left(\frac{x}{N}\int_1^{N/(M-1)}\log u\,du\right) + K' < K'' + K'$$

when $x, M, N \to \infty$, $\log N - \log x = \nu$ (fixed), $\log N - \log M = \mu + \nu$ (fixed).

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