J. P. JAISWAL

On Meijer transform III

Compositio Mathematica, tome 12 (1954-1956), p. 284-297
<http://www.numdam.org/item?id=CM_1954-1956__12__284_0>

On Meijer Transform III [1)]

by

J. P. Jaiswal

1. Meijer [1] introduced the integral equation

$$\text{(A)} \qquad \varphi(s) = s\int_0^\infty e^{-\frac{1}{2}st}(st)^{-k-\frac{1}{2}}\, W_{k+\frac{1}{2},m}(st)f(t)dt$$

where $W_{k,m}(z)$ is Whittaker's confluent hypergeometric function. In this equation $f(t)$ is known as the *original* of $\varphi(s)$, $\varphi(s)$ the image of $f(t)$ and (A) is symbolically denoted by [2]

$$f(t) \xrightarrow[m]{k+\frac{1}{2}} \varphi(s).$$

Particular cases of Meijer transform are:—

(a) when $k = -\frac{1}{2}$, (A) reduces to [2)]

$$\varphi(s) = \frac{s}{\sqrt{\pi}}\int_0^\infty e^{-\frac{1}{2}st}(st)^{\frac{1}{2}}K_m(\tfrac{1}{2}st)f(t)dt$$

and will be known as K_m-transform, and symbolically denoted as

$$f(t) \xrightarrow[m]{0} \varphi(s);$$

(b) when $k = \frac{1}{2}n + \frac{1}{4}$, $m = \pm\frac{1}{4}$, (A) reduces to [2)]

$$\varphi(s) = 2^{-n/2}s\int_0^\infty e^{-\frac{1}{2}st}(st)^{-n/2}D_n(\sqrt{2st})\,f(t)dt$$

and will be known as D_n-transform, and symbolically denoted as

$$f(t) \xrightarrow[\pm\frac{1}{4}]{n/2+\frac{1}{4}} \varphi(s);$$

(c) when $k = \pm m$, (A) reduces to the Laplace Integral,

$$\varphi(s) = s\int_0^\infty e^{-st}f(t)dt$$

---

1) This paper is in continuation of my earlier papers [2] and [3].

2) $K_m(z)$ and $D_n(z)$ are the Bessel function and the parabolic cylinder function respectively.

which will be denoted symbolically as

$$\varphi(s) \doteqdot f(t).$$

In this paper we have established some more properties of the Meijer transform by utilising integral representations of the Whittaker's function, the parabolic cylinder function, and the Bessel function, and also the integrals involving these functions. The analogues of these properties are not known in the case of the Laplace transform. The conditions imposed on the theorems may in some cases be relaxed by analytic continuation. We have illustrated this fact in example to Theorem 1.

2. THEOREM 1. [3] *If*

$$t^{\lambda} f(t) \xrightarrow[m]{k+\frac{1}{2}} \varphi_{\lambda,\, k+\frac{1}{2},\, m}(s),$$

then

$$(1)\quad \varphi_{\lambda,\, k+\frac{1}{2},\, m}(s) = \frac{s^{2\lambda}}{\Gamma(2\lambda)} \int_1^{\infty} (u-1)^{2\lambda-1} u^{k-m-1} \varphi_{3\lambda,\, k+\lambda+\frac{1}{2},\, m+\lambda}(su)\, du,$$

provided that

(i) $R(\mu_1 + 2\lambda - k + 1 \pm \overline{m+\lambda}) > 0$, *and* $R(\mu_1 + \lambda - k + m + 1) > 0$, *where* $f(t) = 0(t^{\mu_1})$ *for small* t,

(ii) $e^{-\frac{1}{2}sut}\, t^{2\lambda-k-\frac{1}{2}}\, W_{k+\lambda+\frac{1}{2},\, m+\lambda}(sut) f(t) \to 0$ *as* $t \to \infty$ *for* $R(s) \geqq s_0 > 0$ and $u > 0$,

(iii) $f(t)$ *is continuous for* $t \geqq 0$,

(iv) $R(\lambda) > 0$, *and*

(v) *the integral in* (1) *is absolutely convergent.*

PROOF: We have ([4], p. 441)

$$z^{\lambda} e^{-\frac{1}{2}z} W_{k+\frac{1}{2},\, m}(z) = \frac{z^{2\lambda}}{\Gamma(2\lambda)} \int_1^{\infty} e^{-\frac{1}{2}zu} W_{k+\lambda+\frac{1}{2},\, m+\lambda}(zu)(u-1)^{2\lambda-1} u^{-m-\lambda-\frac{1}{2}} du$$

$$R(\lambda) > 0,\ |\arg z| > \pi/2,\ z \neq 0.$$

Replacing z by st and multiplying by $(st)^{-k-\frac{1}{2}} f(t)$ and integrating with respect to t, from $t = 0$ to ∞, we get

$$\int_0^{\infty} e^{-\frac{1}{2}st} (st)^{-k-\frac{1}{2}} (st)^{\lambda} W_{k+\frac{1}{2},\, m}(st) f(t)\, dt$$

$$(2)\quad = \frac{1}{\Gamma(2\lambda)} \int_0^{\infty} (st)^{2\lambda-k-\frac{1}{2}} f(t)\, dt \int_1^{\infty} e^{-\frac{1}{2}stu} W_{k+\lambda+\frac{1}{2},\, m+\lambda}(stu)(u-1)^{2\lambda-1} u^{-\lambda-m-\frac{1}{2}} du$$

$$= \frac{1}{\Gamma(2\lambda)} \int_1^{\infty} (u-1)^{2\lambda-1} u^{-\lambda-m-\frac{1}{2}} du \int_0^{\infty} e^{-\frac{1}{2}stu} W_{k+\lambda+\frac{1}{2},\, m+\lambda}(stu)(st)^{2\lambda-k-\frac{1}{2}} f(t)\, dt,$$

[3]) In all the theorems $\varphi(s)$ comes out to be an analytic function regular in a certain region which depends on the particular function $f(t)$ chosen; and the method employed to evaluate the integral (A).

whence

$$\varphi_{\lambda, k+\frac{1}{2}, m}(s) = \frac{s^{2\lambda}}{\Gamma(2\lambda)} \int_1^\infty (u-1)^{2\lambda-1} u^{k-m-1} \varphi_{3\lambda, k+\lambda+\frac{1}{2}, m+\lambda}(su)du.$$

Regarding the change of order of integration in (2), we see that the t-integral is absolutely convergent if $R(\mu_1 + 2\lambda - k + 1 \pm \overline{m + \lambda}) > 0$, where $f(t) = 0(t^{\mu_1})$ for small t, $e^{-\frac{1}{2}sut} t^{2\lambda-k-\frac{1}{2}} W_{k+\lambda+\frac{1}{2}, m+\lambda}(sut) f(t) \to 0$ as $t \to \infty$ for $R(s) \geqq s_0 > 0$, $u > 0$; and $f(t)$ is continuous for $t \geqq 0$. Also the u-integral is absolutely convergent if $R(\lambda) > 0$; and the repeated integral (2) is absolutely convergent due to (1). Hence by de la Vallée Poussin's theorem the change of order of integration is justified. We may also note that the integral on the left hand side of equation (2) also converges under the conditions already imposed together with $R(\mu_1 + \lambda - k + m + 1) > 0$.

Example: Let

$$t^\lambda f(t) = t^{\lambda+\mu} H_\nu(2t),$$

then ([3], p. 132.) [4])

$$\varphi_{\lambda, k+\frac{1}{2}, m}(s) = \frac{\Gamma_\times(\lambda+\mu+\nu+2-k \pm m) s^{-\lambda-\mu-\nu-1}}{\Gamma(\frac{3}{2})\Gamma(\nu+\frac{3}{2})\Gamma(\lambda+\mu+\nu-2k+2)} \times$$

$${}_5F_4\left[\begin{matrix}\frac{\lambda+\mu+\nu-k\pm m+2}{2}, \frac{\lambda+\mu+\nu-k\pm m+3}{2}, 1 \\ \frac{3}{2}, \nu+\frac{3}{2}, \frac{\lambda+\mu+\nu-2k+2}{2}, \frac{\lambda+\mu+\nu-2k+3}{2}\end{matrix}; -\frac{4}{s^2}\right]$$

$$R(\lambda+\mu+\nu-k+2 \pm m) > 0, \quad R(s) > 0 \text{ and } |s| > 2.$$

Since $f(t)$ satisfies all the conditions of the above theorem, we therefore get

$$\text{(B)}\quad \int_1^\infty (u-1)^{2\lambda-1} u^{k-m-3\lambda-\mu-\nu-2} {}_5F_4\left[\begin{matrix}\frac{2\lambda+\mu+\nu-k+2\pm\overline{m+\lambda}}{2}, \frac{2\lambda+\mu+\nu-k+3\pm\overline{m+\lambda}}{2}, 1 \\ \frac{3}{2}, \nu+\frac{3}{2}, \frac{\lambda+\mu+\nu-2k+2}{2}, \frac{\lambda+\mu+\nu-2k+3}{2}\end{matrix}; -\frac{4}{s}\right]$$

$$= \frac{\Gamma(\lambda+\mu+\nu-k+m+2)\Gamma(2\lambda)}{\Gamma(3\lambda+\mu+\nu-k+m+2)} {}_5F_4\left[\begin{matrix}\frac{\lambda+\mu+\nu-k+2\pm m}{2}, \frac{\lambda+\mu+\nu-k+3\pm m}{2}, 1 \\ \frac{3}{2}, \nu+\frac{3}{2}, \frac{\lambda+\mu+\nu-2k+2}{2}, \frac{\lambda+\mu+\nu-2k+3}{2}\end{matrix}; -\frac{}{s}\right]$$

[4]) The symbol $\Gamma_\times(a \pm b)$ denotes $\Gamma(a+b)\,\Gamma(a-b)$ and ${}_2F_1\left[\begin{matrix}\alpha \pm \beta \\ \gamma\end{matrix}; x\right]$ denotes ${}_2F_1\left[\begin{matrix}\alpha+\beta, \alpha-\beta \\ \gamma\end{matrix}; x\right]$.

$$R(\lambda+\mu+\nu-k+2\pm m)>0,\quad R(3\lambda+\mu+\nu-k+m+2)>0,$$
$$R(\lambda)>0,\quad R(s)>0,\quad \text{and}\quad |s|>2.$$

As regards the absolute convergence of the integral, it easily follows under the conditions already imposed, if we note that:

(i) the integrand is continuous for $u \geqq 1$, and $R(s) \geqq s_0 > 0$,

(ii) the integrand $= O(u^{-(\lambda+\mu+\nu-k+m+3)})$ for large u, and

(iii) the integrand $= O(u^{2\lambda-1})$ for $u = 1+\epsilon$, $\epsilon > 0$.

We may also note that the conditions $|s| > 2$, and $R(\lambda+\mu+\nu-k-m+2) > 0$ may be waived out by the help of analytic continuation, if we make a cut in the s-plane from 0 to $-\infty$. For in (B) the integral converges absolutely and the function on the right is regular in the cut s-plane without these conditions.

3. THEOREM 2. *If*

$$f(t) \xrightarrow[m]{k+\frac{1}{2}} \varphi(s),$$

and

$$\overline{\psi\left(\frac{x}{2}+\frac{1}{2}\cdot s\right)} \doteqdot t^{\frac{1}{2}-k} f(t),$$

then

$$(3)\quad \int_1^\infty (1+x)^{-1}\left(\frac{x+1}{x-1}\right)^{\frac{1}{2}k+\frac{1}{4}} P^{k+\frac{1}{2}}_{m+\frac{1}{2}}(x)\, \overline{\psi\left(\frac{1}{2}+\frac{x}{2}\cdot s\right)}dx = s^{k-\frac{1}{2}}\varphi(s),$$

provided that

(i) $R(\mu-k+\frac{3}{2})>0$, $R(\mu-k\pm m+1)>0$, *where* $f(t) = O(t^\mu)$ *for small* t,

(ii) $\left|e^{-s(\frac{x}{2}+\frac{1}{2})t}\, t^{\frac{1}{2}-k} f(t)\right| \to 0$ *as* $t\to\infty$ *for* $x > -\frac{1}{2}$, $R(s)\geqq s_0>0$,

(iii) $f(t)$ *is continuous for* $t \geqq 0$, *and*

(iv) $R(k) < \frac{1}{2}$.

PROOF: We have

$$(\alpha)\qquad \frac{1}{s}\varphi(s) = \int_0^\infty e^{-\frac{1}{2}st}(st)^{-k-\frac{1}{2}}\, W_{k+\frac{1}{2},m}(st) f(t)\,dt.$$

Now, using the result ([5], p. 480)

$$W_{k,m}(z) = \tfrac{1}{2}z\int_1^\infty e^{-\frac{zt}{2}}\left(\frac{t+1}{t-1}\right)^{\frac{1}{2}k} P^k_{m-\frac{1}{2}}(t)\,dt$$

$$R(k)<1,\ \text{and}\ |\arg z| < \pi/2,$$

we get

$$(4)\quad \frac{1}{s}\varphi(s) = \int_0^\infty f(t)e^{-\frac{1}{2}st}(st)^{-k-\frac{1}{2}}(\tfrac{1}{2}st)\left[\int_1^\infty e^{-\frac{stx}{2}}\left(\frac{x+1}{x-1}\right)^{\frac{1}{2}k+\frac{1}{4}} P^{k+\frac{1}{2}}_{m-\frac{1}{2}}(x)\,dx\right]dt$$

$$= \frac{s^{-\frac{1}{2}-k}}{2}\int_1^\infty \left(\frac{x+1}{x-1}\right)^{\frac{1}{2}k+\frac{1}{4}} P^{k+\frac{1}{2}}_{m-\frac{1}{2}}(x)\left[s\int_0^\infty e^{-s\left(\frac{1}{2}+\frac{x}{2}\right)t}t^{\frac{1}{2}-k}f(t)\,dt\right]dx$$

$$= s^{-\frac{1}{2}-k}\int_1^\infty (1+x)^{-1}\left(\frac{x+1}{x-1}\right)^{\frac{1}{2}k+\frac{1}{4}} P^{k+\frac{1}{2}}_{m-\frac{1}{2}}(x)\,\psi\left(\overline{\frac{1}{2}+\frac{x}{2}\cdot s}\right)dx,$$

where

$$\psi\left(\overline{\frac{1}{2}+\frac{x}{2}\cdot s}\right) \doteqdot t^{\frac{1}{2}-k}f(t).$$

The change of order of integration in (4) can be justified as follows:

The x-integral is absolutely convergent if $R(k) < \frac{1}{2}$, $t > 0$, and $R(s) \geqq s_0 > 0$; the t-integral converges absolutely if $f(t)$ is continuous for $t \geqq 0$; $R(\mu - k + \frac{3}{2}) > 0$, where $f(t) = O(t^{\mu})$ for small t, and $\left| e^{-s(\frac{x}{2}+\frac{1}{2})t}t^{-k+\frac{1}{2}}f(t)\right| \to 0$ as $t \to \infty$ for $R(s) \geqq s_0 > 0$ and $x > -\frac{1}{2}$. The repeated integral (α) converges absolutely if $R(\mu - k + 1 \pm m) > 0$, and under the conditions already imposed. Hence the change of order of integration is justified by de la Vallée Poussin's theorem.

Example: Let

$$f(t) = t^{\frac{n}{2}}J_n(2\sqrt{t}),$$

then ([2], p. 389)

$$\varphi(s) = \frac{\Gamma_{\times}(n-k+1\pm m)}{n!\Gamma(n-2k+1)s^n}\,{}_2F_2\left[\begin{matrix} n-k+1\pm m \\ n+1,\, n-2k+1\end{matrix};\, -\frac{1}{s}\right]$$
$$R(n-k+1\pm m) > 0, \quad \text{and} \quad R(s) > 0,$$

and ([6], p. 29)

$$\psi\left\{\frac{x}{2}+\frac{1}{2}\cdot s\right\} = \frac{\Gamma(n-k+\frac{3}{2})}{\Gamma(n+1)}\left[\left(\frac{x}{2}+\frac{1}{2}\right)s\right]^{-n+k-\frac{1}{2}} {}_1F_1\left[\begin{matrix} n-k+\frac{3}{2} \\ n+1\end{matrix};\, -\frac{2}{(x+1)s}\right]$$
$$R(n-k+\tfrac{3}{2}) > 0.$$

Now applying the above theorem we get

$$(5)\quad \int_1^\infty (x+1)^{-n+k-\frac{3}{2}}\left(\frac{x+1}{x-1}\right)^{\frac{1}{2}k+\frac{1}{4}} P^{k+\frac{1}{2}}_{m-\frac{1}{2}}(x)\,{}_1F_1\left[\begin{matrix} n-k+\frac{3}{2} \\ n+1\end{matrix};\, -\frac{2}{(x+1)s}\right]dx$$

$$= \frac{\Gamma_{\times}(n-k+1\pm m)}{\Gamma(n-2k+1)\Gamma(n-k+\frac{3}{2})}\,2^{-n+k-\frac{1}{2}}\,{}_2F_2\left[\begin{matrix} n-k+1\pm m \\ n+1,\, n-2k+1\end{matrix};\, -\frac{1}{s}\right]$$

$R(n-k+\frac{3}{2}) > 0$, $R(n-k+1\pm m) > 0$, $R(k) < \frac{1}{2}$ and $R(s) > 0$.

4. **Theorem 3.** *If*

$$f(t) \xrightarrow[m]{k+\frac{1}{2}} \varphi(s),$$

and

$$\chi(s, u) \doteqdot \left(1 + \frac{u}{t}\right)^{k+m} f(t),$$

then

$$\int_0^\infty e^{-su} u^{-k+m-1} \chi(s, u)\, du = \frac{\Gamma(m-k)}{s^{m-k}} \varphi(s), \tag{6}$$

provided that

(i) $R(m-k) > 0$, *and* $(k-m)$ *is not an integer,*

(ii) $R(\mu_1 + 1) > 0$, *and* $R(\mu_1 - k - m + 1) > 0$, *where* $f(t) = O(t^{\mu_1})$ *for small* t,

(iii) $\left| e^{-st}\left(1 + \frac{u}{t}\right)^{k+m} f(t)\right| \to 0$ *as* $t \to \infty$ *for* $R(s) \geqq s_0 > 0$, $u \geqq 0$, *and*

(iv) $f(t)$ *is continuous for* $t \geqq 0$.

Proof: We have

$$\varphi(s) = s\int_0^\infty e^{-\frac{1}{2}st}(st)^{-k-\frac{1}{2}} W_{k+\frac{1}{2}, m}(st) f(t)\, dt.$$

Now, using the integral representation of $W_{k,m}(z)$ ([7], p. 340)

$$W_{k,m}(z) = \frac{e^{-\frac{1}{2}z} z^k}{\Gamma(\frac{1}{2} - k + m)} \int_0^\infty t^{-k-\frac{1}{2}+m}\left(1 + \frac{t}{z}\right)^{k-\frac{1}{2}+m} e^{-t}\, dt$$

$R(\frac{1}{2} - k + m) > 0$, and $(k - \frac{1}{2} - m)$ not an integer, we get

$$\begin{aligned} \varphi(s) &= s\int_0^\infty e^{-\frac{1}{2}st} f(t)\left[\frac{e^{-\frac{1}{2}st}}{\Gamma(m-k)}\int_0^\infty u^{-k+m-1}\left(1 + \frac{u}{st}\right)^{k+m} e^{-u}\, du\right] dt \\ &= \frac{s^{-k+m}}{\Gamma(m-k)}\int_0^\infty e^{-su} u^{m-k-1}\left[s\int_0^\infty e^{-st}\left(1 + \frac{u}{t}\right)^{k+m} f(t)\, dt\right] du \\ &= \frac{s^{m-k}}{(\Gamma m - k)}\int_0^\infty e^{-su} u^{m-k-1} \chi(s, u)\, du, \end{aligned} \tag{7}$$

where

$$\chi(s, u) \doteqdot \left(1 + \frac{u}{t}\right)^{k+m} f(t).$$

The change of order of integration in (7) can be justified by de la Vallée Poussin's theorem.

5. **Theorem 4.** *If*

$$f(t) \xrightarrow[m]{0} \varphi(s),$$

then

(8) $$\varphi(s) = \frac{1}{\sqrt{\pi}\, 2^{2m+1}} \int_0^\infty e^{-x} x^{-m-1} \psi(x, s) dx,$$

where

$$\psi(x, s) = s \int_0^\infty e^{-\frac{1}{2}st - \frac{s^2 t^2}{16x}} (st)^{m+\frac{1}{2}} f(t)\, dt,$$

provided that

(i) $R(\mu_1 + m + \frac{3}{2}) > 0$, *where* $f(t) = O(t^{\mu_1})$ *for small* t,
(ii) $f(t)$ *is continuous in* $t \geqq 0$,
(iii) $R(s) \geqq s_0 > 0$, *and*
(iv) *the* K_m*-transform of* $|f(t)|$ *exists.*

Proof: We have

$$\varphi(s) = \frac{s}{\sqrt{\pi}} \int_0^\infty e^{-\frac{1}{2}st} (st)^{\frac{1}{2}} K_m(\tfrac{1}{2}st) f(t)\, dt.$$

Now, using the integral representation of $K_m(z)$ ([8] p. 193)

$$K_m(z) = \tfrac{1}{2}(\tfrac{1}{2}z)^m \int_0^\infty e^{-x - \frac{z^2}{4x}} \frac{dx}{x^{m+1}}, \quad R(z^2) > 0,$$

we get

(9) $$\varphi(s) = \frac{s}{\sqrt{\pi}} \int_0^\infty e^{-\frac{1}{2}st} (st)^{\frac{1}{2}} \left[\tfrac{1}{2} (\tfrac{1}{4}st)^m \int_0^\infty e^{-x - \frac{s^2 t^2}{16x}} \frac{dx}{x^{m+1}} \right] f(t)\, dt$$

$$= \frac{1}{\sqrt{\pi}\, 2^{2m+1}} \int_0^\infty e^{-x} x^{-m-1} \left[s \int_0^\infty e^{-\frac{1}{2}st - \frac{s^2 t^2}{16x}} (st)^{m+\frac{1}{2}} f(t)\, dt \right] dx$$

$$= \frac{1}{\sqrt{\pi}\, 2^{2m+1}} \int_0^\infty e^{-x} x^{-m-1} \psi(x, s)\, dx,$$

where

$$\psi(x, s) = s \int_0^\infty e^{-\frac{1}{2}st - \frac{s^2 t^2}{16x}} (st)^{m+\frac{1}{2}} f(t)\, dt.$$

The change of order of integration in (9) can be easily justified by de la Vallée Poussin's theorem.

Example: Let

$$f(t) = t^\nu,$$

then ([2], p. 387)

$$\varphi(s) = \frac{\Gamma_{\times}(\nu + \frac{3}{2} \pm m)}{\Gamma(\nu + 2)s^{\nu}}, \quad R(\nu + \tfrac{3}{2} \pm m) > 0, \quad R(s) > 0$$

and

$$\psi(x, s) = s \int_0^{\infty} e^{-\frac{1}{2}st - \frac{s^2t^2}{16x}} (st)^{m+\frac{1}{2}} t^{\nu}\, dt.$$

On simplifying and using the result [9]

$$D_{-n}(z) = \frac{e^{-\frac{1}{4}z^2}}{\Gamma(n)} \int_0^{\infty} e^{-tz-\frac{1}{2}t^2} t^{n-1}\, dt, \quad R(n) > 0,$$

we get

$$\psi(x, s) = \frac{(2\sqrt{2x})^{\nu+m+\frac{3}{2}}\, \Gamma(\nu + m + \frac{3}{2}) e^{\frac{x}{2}}}{s^{\nu}} D_{-(\nu+m+\frac{3}{2})}(\sqrt{2x})$$

Hence applying the above theorem, we get

$$\int_0^{\infty} e^{-\frac{x}{2}} x^{\frac{1}{2}\nu-m-\frac{1}{4}} D_{-(\nu+m+\frac{3}{2})}(\sqrt{2x})\, dx = \frac{\sqrt{\pi}\,.\, 2^{2m+1}\, \Gamma(\nu - m + \frac{3}{2})}{(2\sqrt{2})^{\nu+m+\frac{3}{2}}\, \Gamma(\nu + 2)}$$

$$R(\nu \pm m + \tfrac{3}{2}) > 0.$$

6. Theorem 5. *If*

$$f(t) \xrightarrow[m]{0} \varphi(s),$$

then

$$\varphi(s) = \sqrt{\frac{s}{\pi}} \int_0^{\infty} \frac{\cosh mu}{\cosh^2 \frac{u}{2}}\, \psi\left(s \cosh^2 \frac{u}{2}\right) du, \tag{10}$$

where

$$\psi\left(s \cosh^2 \frac{u}{2}\right) \doteqdot t^{\frac{1}{2}} f(t),$$

provided that

(i) $R(\mu_1 + \frac{3}{2} \pm m) > 0$, *where* $f(t) = O(t^{\mu_1})$ *for small* t,

(ii) $|e^{-st} t^{\frac{1}{2}} f(t)| \to 0$ *as* $t \to \infty$ *for* $R(s) \geqq s_0 > 0$, *and*

(iii) $f(t)$ *is continuous for* $t \geqq 0$.

Proof: We have

$$\varphi(s) = \frac{s}{\sqrt{\pi}} \int_0^{\infty} e^{-\frac{1}{2}st} (st)^{\frac{1}{2}} K_m(\tfrac{1}{2}st) f(t)\, dt.$$

Now, using the result ([10], p. 599)

$$K_m(z) = \int_0^\infty e^{-z\cosh u} \cosh mu\, du,$$

we get

$$\text{(11)}\quad \varphi(s) = \frac{s}{\sqrt{\pi}} \int_0^\infty e^{-\frac{1}{2}st} (st)^{\frac{1}{2}} f(t) \left[\int_0^\infty e^{-\frac{1}{2}st\cosh u} \cosh mu\, du \right] dt$$

$$= \sqrt{\frac{s}{\pi}} \int_0^\infty \cosh mu \left[s \int_0^\infty e^{-st\cosh^2\frac{u}{2}} t^{\frac{1}{2}} f(t)\, dt \right] du$$

$$= \sqrt{\frac{s}{\pi}} \int_0^\infty \frac{\cosh mu}{\cosh^2 \frac{u}{2}} \psi\left(s\cosh^2\frac{u}{2}\right) du,$$

where

$$\psi\left(s\cosh^2\frac{u}{2}\right) \doteqdot t^{\frac{1}{2}} f(t).$$

The change of order of integration in (11) can easily be justified as before.

Further, we can find out the behaviour of $\psi(s\cosh^2 u/2)$ for large values of u in the following way:

$$\psi\left(s\cosh^2\frac{u}{2}\right) = s\cosh^2\frac{u}{2} \int_0^\infty e^{-s\cosh^2\frac{u}{2}\cdot t} t^{\frac{1}{2}} f(t)\, dt$$

If the function $t^{\frac{1}{2}}f(t)$ satisfies the conditions of Watson's lemma, regarding the asymptotic expansion of functions representable by Laplace's integral, and is $O(t^{\mu_1+\frac{1}{2}})$ for small t, we have from Watson's lemma

$$\frac{\psi(p)}{p} = O(p^{-\mu_1-\frac{3}{2}}) \text{ for large } p,$$

where $p = s\cosh^2\dfrac{u}{2}$ and $\mu_1 + \frac{3}{2}$ is not an integer. If we fix s, and note that

$$\cosh^2\frac{u}{2} = O(e^u) \text{ for large } u,$$

then

$$\psi\left(s\cosh^2\frac{u}{2}\right) = O(e^{-\overline{\mu_1+\frac{1}{2}}\cdot\mu}) \text{ for large } u.$$

Example: Let

$$f(t) = t^{\nu-\frac{3}{2}} e^{-qt},$$

then ([2], p. 388).

$$\varphi(s) = \frac{\Gamma_{\times}(\nu \pm m)}{s^{\nu-\frac{3}{2}}\Gamma(\nu+\frac{1}{2})}\, {}_2F_1\left[\begin{matrix}\nu \pm m\\ \nu+\frac{1}{2}\end{matrix};\, -\frac{q}{s}\right]$$

$$R(\nu \pm m) > 0,\ R(s+q) > 0 \text{ and } |s| > |q|.$$

and ([6], p. 16)

$$t^{\nu-1} e^{-qt} \doteqdot \frac{\Gamma(\nu)\, s\cosh^2\frac{u}{2}}{\left(s\cosh^2\frac{u}{2}+q\right)^{\nu}};\quad R(\nu) > 0,\quad R(s+q) > 0$$

$$= \psi\left(s\cosh^2\frac{u}{2}\right).$$

Applying the above theorem, we get

$$\int_0^{\infty}\frac{\cosh mu}{\left(s\cosh^2\frac{u}{2}+q\right)^{\nu}}\,du = \frac{\sqrt{\pi}\ \Gamma_{x}(\nu \pm m)}{s^{\nu}\Gamma(\nu+\frac{1}{2})\Gamma(\nu)} \times {}_2F_1\left[\begin{matrix}\nu \pm m\\ \nu+\frac{1}{2}\end{matrix},\, -\frac{q}{s}\right]$$

$$R(\nu) > 0,\ R(\nu \pm m) > 0,\ R(s) > 0, \text{ and } |s| > |q|.$$

7. **Theorem 6.** *If*

$$f(t) \xrightarrow[\pm\frac{1}{4}]{\frac{1}{2}n+\frac{1}{4}} \varphi(s),$$

then

$$\varphi(s) = 2\sqrt{\frac{s}{\pi}}\int_0^{\frac{\pi}{2}} \cos^{-n}\varphi \cos n\varphi\ \psi\{s\sec^2\varphi\}\,d\varphi, \tag{12}$$

where

$$\psi\{s\sec^2\varphi\} \doteqdot x^{\frac{1}{2}} f(x),$$

provided that

(i) $R(\mu_1+\frac{3}{2}) > 0$, $R(\mu_1-\frac{1}{2}n+1) > 0$, *where* $f(x) = O(x^{\mu_1})$ *for small* x,

(ii) $|e^{-sx}x^{-\frac{1}{2}n}f(x)| \to 0$ *as* $x \to \infty$ *for* $R(s) \geqq s_0 > 0$, *and*

(iii) $f(t)$ *is continuous for* $t \geqq 0$.

Proof: We have

$$\varphi(s) = 2^{-\frac{1}{2}n}s\int_0^{\infty} e^{-\frac{1}{2}sx}(sx)^{-\frac{1}{2}n}D_n(\sqrt{2sx})\,f(x)\,dx.$$

Using the result ([10] p. 600)

$$D_n(z) = \frac{2^{\frac{1}{2}} z^{n+1} e^{-\frac{1}{4}z^2}}{\sqrt{\pi}} \int_0^{\frac{\pi}{2}} e^{-\frac{1}{2}z^2 \tan^2 \varphi} \cos^{-n-2} \varphi \cos n\varphi \, d\varphi$$

we get

$$(13) \quad \varphi(s) = \frac{2s}{\sqrt{\pi}} \int_0^{\infty} e^{-sx} f(x)(sx)^{\frac{1}{2}} \left[\int_0^{\frac{\pi}{2}} e^{-sx \tan^2 \varphi} \cos^{-n-2} \varphi \cos n\varphi \, d\varphi \right] dx$$

$$= 2\sqrt{\frac{s}{\pi}} \int_0^{\frac{\pi}{2}} \cos^{-n-2} \varphi \cos n\varphi \left[s \int_0^{\infty} e^{-(s \sec^2 \varphi) x} x^{\frac{1}{2}} f(x) \, dx \right] d\varphi$$

$$= 2\sqrt{\frac{s}{\pi}} \int_0^{\pi} \cos^{-n} \varphi \cos n\varphi \, \psi [s \sec^2 \varphi] \, d\varphi,$$

where

$$\psi[s \sec^2 \varphi] \doteqdot x^{\frac{1}{2}} f(x).$$

The change of order of integration in (13) can easily be justified.

Example: Let

$$f(t) = t^{-\frac{1}{2}} e^{-bt} J_\nu(at),$$

then ([2], p. 390)

$$f(t) \xrightarrow[\pm \frac{1}{4}]{\frac{1}{2}n + \frac{1}{4}} \sum_{r=0}^{\infty} \frac{(-)^r \cdot \left(\dfrac{a}{2}\right)^{\nu+2r} s^{-(\nu+2r-\frac{1}{2})} \quad \Gamma_{\times}(\nu + 2r - \frac{1}{2}n + \frac{3}{4} \pm \frac{1}{4})}{r! \Gamma(\nu + r + 1) \qquad \Gamma(\nu - n + 2r + 1)} \times$$

$$ {}_2F_1 \left[\begin{matrix} \nu + 2r - \frac{1}{2}n + \frac{3}{4} \pm \frac{1}{4} \\ \nu + 2r - n + 1 \end{matrix} ; \; -\frac{b}{s} \right] = \varphi(s)$$

$$R(\nu - \tfrac{1}{2}n + \tfrac{1}{2}) > 0, \; R(s+b) > 0, \text{ and } |s+b| > |a|,$$

and ([6], p. 28)

$$e^{-bt} J_\nu(at) \doteqdot \frac{s \,.\, a^\nu \sec^2 \varphi}{\sqrt{(s \sec^2 \varphi + b)^2 + a^2} [s \sec^2 \varphi + b + \sqrt{(s \sec^2 \varphi + b)^2 + a^2}]^\nu}$$

$$= \psi(s \sec^2 \varphi)$$

$$R(\nu) > -1, \; R(s+b) > 0.$$

Now, using the above theorem, we get

$$\int_0^{\frac{\pi}{2}} \frac{\cos^{-n-2} \varphi \cos n\varphi \, d\varphi}{\sqrt{(s \sec^2 \varphi + b)^2 + a^2} [s \sec^2 \varphi + b + \sqrt{(s \sec^2 \varphi + b)^2 + a^2}]^\nu}$$

$$= \frac{\sqrt{\pi}}{2^{\nu+1}} \sum_{r=0}^{\infty} \frac{(-)^r \cdot \left(\dfrac{a}{2}\right)^{2r} s^{-(\nu+2r+1)} \, \Gamma_{\times}(\nu - \frac{1}{2}n + 2r + \frac{3}{4} \pm \frac{1}{4})}{r! \, \Gamma(\nu + r + 1) \Gamma(\nu - n + 2r + 1)} \times$$

$$ {}_2F_1 \left[\begin{matrix} \nu - \frac{1}{2}n + 2r + \frac{3}{4} \pm \frac{1}{4} \\ \nu - n + 2r + 1 \end{matrix} ; \; -\frac{b}{s} \right]$$

$$R(\nu) > -1, \; R(\nu - \tfrac{1}{2}n + \tfrac{1}{2}) > 0, \; R(s+b) > 0 \text{ and } |s+b| > |a|.$$

8. THEOREM 7. *If*

$$f(t)\xrightarrow[\pm\frac{1}{4}]{\frac{1}{2}n+\frac{1}{4}}\varphi(s),$$

then

$$\varphi(s)=\frac{\sqrt{\pi}\,.\,2^{\frac{1}{4}}}{\Gamma(-n)}\int_0^\infty u^{-n-\frac{1}{2}}e^{-\frac{1}{4}u^2}D_{-n}(u)\chi(s,u)\,du, \tag{14}$$

where

$$\chi(s,u)=s\int_0^\infty (st)^{-\frac{1}{4}}e^{-st-\sqrt{\frac{st}{2}}u}I_{-n-\frac{1}{2}}\left(u\sqrt{\frac{st}{2}}\right)f(t)\,dt,$$

provided that

(i) $R(\mu_1-\frac{1}{2}n+1)>0$, *where* $f(t)=O(t^{\mu_1})$ *for small* t,

(ii) $|e^{-st}t^{\frac{1}{4}}f(t)|\to 0$ *as* $t\to\infty$ *for* $R(s)\geqq s_0>0$, *and* $u>0$,

(iii) $R(n)<0$, *and*

(iv) $f(t)$ *is continuous for* $t\geqq 0$.

PROOF: We have

$$\varphi(s)=2^{-\frac{1}{2}n}s\int_0^\infty e^{-\frac{1}{2}st}(st)^{-\frac{n}{2}}D_n(\sqrt{2st})f(t)\,dt$$

Now, using ([11], p. 442)

$$D_n(z)=\frac{\sqrt{\pi}\,z^{n+\frac{1}{2}}}{\Gamma(-n)}\int_0^\infty u^{-n-\frac{1}{2}}e^{-\frac{1}{4}(z+u)^2}D_{-n}(u)I_{-n-\frac{1}{2}}(\tfrac{1}{2}zu)\,du$$

$R(n)<0,\ z\neq 0,$

we get

$$\varphi(s)=2^{-\frac{1}{2}n}s\int_0^\infty e^{-\frac{1}{2}st}(st)^{-\frac{1}{2}n}f(t)\left[\frac{\sqrt{\pi}}{\sqrt{(-n)}}(2st)^{\frac{1}{2}n+\frac{1}{4}}\int_0^\infty u^{-n-\frac{1}{2}}\times e^{-\frac{1}{4}(\sqrt{2st}+u)^2}D_{-n}(u)I_{-n-\frac{1}{2}}\left(u\sqrt{\frac{st}{2}}\right)du\right]dt$$

$$=\frac{\sqrt{\pi}}{\Gamma(-n)}2^{\frac{1}{4}}\int_0^\infty u^{-n-\frac{1}{2}}e^{-\frac{1}{4}u^2}D_{-n}(u)\left[s\int_0^\infty e^{-st-u\sqrt{\frac{st}{2}}}(st)^{\frac{1}{4}}\times I_{-n-\frac{1}{2}}\left(u\sqrt{\frac{st}{2}}\right)f(t)dt\right]du. \tag{15}$$

Hence

$$\varphi(s)=\frac{\sqrt{\pi}}{\Gamma(-n)}2^{\frac{1}{4}}\int_0^\infty u^{-n-\frac{1}{2}}e^{-\frac{1}{4}u^2}D_{-n}(u)\chi(s,u)\,du$$

where

$$\chi(s, u) = s\int_0^\infty e^{-st-u\sqrt{\frac{st}{2}}}(st)^{\frac{1}{4}} I_{-n-\frac{1}{2}}\left(u\sqrt{\frac{st}{2}}\right) f(t)\,dt.$$

The change of order of integration in (15) can easily be justified.

9. **Theorem 8.** *If*

$$f(t) \xrightarrow[m]{k+\frac{1}{2}} \varphi(s),$$

then

$$(16)\quad \varphi(s) = \frac{4\sqrt{\pi s}}{\Gamma_\times(-k\pm m)}\int_0^\infty u^{-2k-1} e^{-\frac{1}{2}u^2} W_{-k,m}(u^2)\,\psi(u, s)\,du,$$

where

$$\psi(u, s) = s\int_0^\infty e^{-st} I_{-2k-1}(u\sqrt{st})\,K_{2m}(u\sqrt{st})t^{\frac{1}{2}} f(t)\,dt,$$

provided that

(i) $R(\mu_1 - k + 1 \pm m) > 0$, *where* $f(t) = O(t^{\mu_1})$ *for small* t,

(ii) $|e^{-st} I_{-2k-1}(u\sqrt{st})K_{2m}(u\sqrt{st})t^{\frac{1}{2}}f(t)| \to 0$ *as* $t \to \infty$ *for* $R(s) \geqq s_0 > 0$, *and* $u > 0$;

(iii) $R(-k \pm m) > 0$,

(iv) $f(t)$ *is continuous for* $t \geqq 0$, *and*

(v) *the integral in* (16) *is absolutely convergent.*

Proof: We have ([11], p. 442)

$$z^{-2k-2} W_{k+\frac{1}{2},m}(z^2) = \frac{4\sqrt{\pi}\, e^{-\frac{1}{2}z^2}}{\Gamma_\times(-k\pm m)}\int_0^\infty u^{-2k-1} e^{-\frac{1}{2}u^2} W_{-k,m}(u^2) I_{-2k-1}(zu)K_{2m}(zu)\,du$$

$$z \neq 0, \text{ and } R(-k\pm m) > 0.$$

Replacing z^2 by st and integrating between $t = 0$ and $t = \infty$, after multiplying by $e^{-\frac{1}{2}st} f(t)(st)^{\frac{1}{2}}$, we get

$$s\int_0^\infty e^{-\frac{1}{2}st}(st)^{-k-\frac{1}{2}} W_{k+\frac{1}{2},m}(st) f(t)\,dt$$

$$= \frac{4\sqrt{\pi}\, s}{\Gamma_\times(-k\pm m)}\int_0^\infty e^{-st}(st)^{\frac{1}{2}} f(t)\left[\int_0^\infty u^{-2k-1} e^{-\frac{1}{2}u^2} W_{-k,m}(u^2)\times I_{-2k-1}(u\sqrt{st})K_{2m}(u\sqrt{st})\,du\right]dt$$

or

$$(17) \quad \varphi(s) = \frac{4\sqrt{\pi s}}{\Gamma_\times(-k \pm m)} \int_0^\infty u^{-2u+1} e^{-\frac{1}{2}u^2} W_{-k,m}(u^2) \Big[s \int_0^\infty e^{-st} \times I_{-2k-1}(u\sqrt{st}) K_{2m}(u\sqrt{st})\, t^{\frac{1}{2}} f(t)\, dt \Big] du$$

$$= \frac{4\sqrt{\pi s}}{\Gamma_\times(-k \pm m)} \int_0^\infty u^{-2k-1} e^{-\frac{1}{2}u^2} W_{-k,m}(u^2)\, \psi(u, s)\, du,$$

where

$$\psi(u, s) = s \int_0^\infty e^{-st} I_{-2k-1}(u\sqrt{st}) K_{2m}(u\sqrt{st})\, t^{\frac{1}{2}} f(t) dt.$$

The change of order of integration in (17) can easily be justified.

I am thankful to Dr. S. K. Bose of Lucknow University (India) for his help in the preparation of this paper.

REFERENCES.

J. P. JAISWAL.

[2] Math. Zeits. 55 (1952), p. 385—98.

[3] Annales de la Soc. Scientifique de Bruxelles, 66 (1952), p. 131—151.

C. S. MEIJER.

[1] Proc. Neder. Akad. v. Wetensch., Amsterdam, 44 (1941), p. 727—737.

[4] Proc. Neder. Akad. v. Wetensch., Amsterdam, 44 (1941), p. 435—441.

[5] Math. Annalen, 112 (1936).

[10] Proc. Neder. Akad. v. Wetensch., Amsterdam, 44 (1941), p. 599—605.

[11] Proc. Neder. Akad. v. Wetensch., Amsterdam, 44 (1941), p. 442—451.

N. W. MCLACHLAN and P. HUMBERT.

[6] Formulaire pour le Calcul Symbolique (1941).

E. T. WHITTAKER and G. N. WATSON.

[7] Modern Analysis (1946).

G. N. WATSON.

[8] Theory of Bessel Functions (1944).

E. T. WHITTAKER.

[9] Proc. Lond. Math. Soc. (1) 35 (1903), p. 417—427.

(Oblatum 8-1-54).