COMPOSITIO MATHEMATICA

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Compositio Mathematica, tome 13 (1956-1958), p. 219-228
<http://www.numdam.org/item?id=CM_1956-1958__13__219_0>

On zeros, poles and mean value of meromorphic functions

by

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In this paper, I have studied some of the properties of meromorphic functions. The results obtained have been divided into three sections. Section One contains some of the properties of Nevanlinna's characteristic function $T(r)$ and the function $N(r; a)$. The notation adopted is the same as that of Nevanlinna. Section Two contains some of the results on the zeros and poles of meromorphic functions which have been given in the form of theorems. Section Three contains some of the properties of mean value of a meromorphic function, defined as in the case of integral functions ([1], p. 31).

Section I

Let $f(z)$ be a meromorphic function of finite and positive order ρ and $n(r; a)$ the number of zeros of $f(z) - a$, $f(0) \neq a$, for $|z| \leqq r$. Let

$$\lim_{r\to\infty} N(r)/r^\rho = \left.\begin{matrix} T \\ t \end{matrix}\right\}, \quad \lim_{r\to\infty} n(r)/r^\rho = \left.\begin{matrix} c \\ d \end{matrix}\right\},$$

where

$$N(r; a) = \int_0^r n(x; a)/x \, dx.$$

We prove the following:

THEOREM 1

$$d \leqq \frac{c}{e} e^{d/c} \leqq \rho T \leqq c; \tag{1.1}$$

$$d \leqq \rho t \leqq d\left(1 + \log \frac{c}{d}\right) \leqq c; \tag{1.2}$$

and

$$c + d \leqq e\rho T. \tag{1.3}$$

THEOREM 2

$$\text{(1.4)} \qquad e\rho t \leqq \rho T + ed,$$

$$\text{(1.5)} \qquad c + \rho t \leqq e\rho T.$$

PROOF OF THEOREM 1

$$N(rk^{1/\rho}) = N(r_0) + \int_{r_0}^{r} \frac{n(x)}{x} dx + \int_{r}^{rk^{1/\rho}} \frac{n(x)}{x} dx$$

where $k \geqq 1$.

Since $n(r)$ is a nondecreasing, positive function of r, therefore,

$$k \cdot \frac{N(rk^{1/\rho})}{k \cdot r^\rho} \leqq \frac{N(r_0)}{r^\rho} + \frac{c+\varepsilon}{\rho} + \frac{k \cdot n(rk^{1/\rho}) \log k}{\rho \cdot kr^\rho}$$

and taking limits, we get

$$\text{(1.6)} \qquad kT \leqq \frac{c + k \log k \cdot c}{\rho},$$

$$\text{(1.7)} \qquad kt \leqq \frac{c + k \log k \cdot d}{\rho}.$$

Also

$$k \cdot \frac{N(rk^{1/\rho})}{k \cdot r^\rho} \geqq \frac{N(r_0)}{r^\rho} + \frac{d-\varepsilon}{\rho} + \frac{n(r)}{r^\rho} \cdot \frac{\log k}{\rho}$$

and therefore taking limits, we get

$$\text{(1.8)} \qquad kT \geqq \frac{d + c \log k}{\rho},$$

$$\text{(1.9)} \qquad kt \geqq \frac{d + d \log k}{\rho}.$$

Putting $k = 1$ in (1.6) and (1.9) we get

$$\text{(2.1)} \qquad \rho T \leqq c, \quad d \leqq \rho t.$$

Further, putting $k = c/d$ in (1.7), we obtain

$$\text{(2.2)} \qquad \rho t \leqq d(1 + \log c/d) \leqq c.$$

Next we put $k = \exp(c - d/c)$ in (1.8). This gives

$$\text{(2.3)} \qquad \rho T \geqq \frac{d + c \log e^{(c-d)/c}}{e \cdot e^{-d/c}},$$

i.e.

$$\rho T \geqq \frac{c}{e} e^{d/c}.$$

Combining the inequalities (2.1), (2.2) and (2.3) the results

(1.1) and (1.2) follow. (1.3) is easily obtained on putting $k = e$ in (1.8).

PROOF OF THEOREM 2:

We know that

$$n(e^{1/\rho}\cdot r) \geqq \rho\int_r^{e^{1/\rho}\cdot r}\frac{n(x)}{x}\,dx. \tag{2.4}$$

Adding $\rho\cdot N(r)$ on both the sides, we obtain

$$\rho N(r) + n(e^{1/\rho}\cdot r) \geqq \rho\cdot N(e^{1/\rho}\cdot r). \tag{2.5}$$

Similarly, we have

$$n(r) + \rho N(r) \leqq \rho\cdot N(e^{1/\rho}\cdot r). \tag{2.6}$$

Dividing by r^ρ and taking the limits, the results follow.

THEOREM 3. *If $f(z)$ be a meromorphic function, $f(0)\neq 0$; $n(r;0)$ be the number of zeros of $f(z)$ for $|z|\leqq r$, then for any two values of r, say r_1 and r_2, for which $|f(re^{i\theta})|$ be greater than* 1,

$$n(r_1)\log\frac{r_2}{r_1}\leqq T(r_2)-T(r_1)\leqq n(r_2)\log\frac{r_2}{r_1}.$$

PROOF:

We have

$$T(r)=\frac{1}{2\pi}\int_0^{2\pi}\log^+|1/f(re^{i\theta})|\,d\theta+\int_0^r\frac{n(x)}{x}\,dx$$

and so,

$$T(r_2)=\int_0^{r_2}\frac{n(x)}{x}\,dx$$

and

$$T(r_1)=\int_0^{r_1}\frac{n(x)}{x}\,dx$$

i.e.

$$T(r_2)-T(r_1)=\int_{r_1}^{r_2}\frac{n(x)}{x}\,dx.$$

Hence

$$T(r_2)-T(r_1)\leqq n(r_2)\log r_2/r_1$$

and

$$T(r_2)-T(r_1)\geqq n(r_1)\log r_2/r_1.$$

COR. 1. If $\alpha > 1$ and $f(z)$ satisfies the condition of Theorem 3, then

$$\lim_{r\to\infty} \frac{1}{T(\alpha r) - T(r)} = 0.$$

$$T(\alpha r) - T(r) = n(r) \log \alpha + \theta[n(\alpha r) - n(r)] \log \alpha,$$

where $0 < \theta < 1$.
Taking the reciprocals and proceeding to limits, the result follows.

COR. 2:

$$\lim_{r\to\infty} \frac{T(kr) - T(r)}{n(kr)} \leqq \log k \leqq \lim_{r\to\infty} \frac{T(kr) - T(r)}{n(r)}.$$

THEOREM 4. *Let $f(z)$ be a meromorphic function, $f(0) \neq \infty$, $n(r; \infty)$ the number of poles of $f(z)$ for $|z| \leqq r$, then, for any two values r_1 and r_2 for which $|f(re^{i\theta})|$ be less than* 1,

$$n(r_1) \log \frac{r_2}{r_1} \leqq T(r_2) - T(r_1) \leqq n(r_2) \log \frac{r_2}{r_1}.$$

The method of proof is the same as that of Theorem 3. Corollary 1 and 2 hold in this case also.

Section II

THEOREM 1. *Let $f(z)$ be a meromorphic function of order $\rho < 1$, with all its zeros and poles real and negative and $\Delta(t)$ be the excess of zeros of $f(z)$ over its poles, for $|z| \leqq t$. If*

$$\Delta(t) \sim \lambda \cdot t^\rho,$$

then

$$\frac{f^1(x)}{f(x)} \sim \lambda \cdot x^{\rho-1} \cdot \pi\rho \operatorname{cosec} \pi\rho.$$

PROOF:

Let

$$f(z) = \frac{\prod_{n=1}^{\infty} (1 + z/a_n)}{\prod_{n=1}^{\infty} (1 + z/b_n)}.$$

Then

$$\log f(z) = \sum_{n=1}^{\infty} \log (1 + z/a_n) - \sum_{n=1}^{\infty} \log (1 + z/b_n)$$

and therefore, for real values of z,

$$\begin{aligned}\frac{f^1(z)}{f(z)} &= \sum_{n=1}^{\infty} 1/z + a_n - \sum_{n=1}^{\infty} 1/z + b_n \\ &= \sum_{n=1}^{\infty} n[1/z + a_n - 1/z + a_{n+1}] - \sum_{n=1}^{\infty} n\left[\frac{1}{z + b_n} - \frac{1}{z + b_{n+1}}\right] \\ &= \sum_{n=1}^{\infty} n \int_{a_n}^{a_{n+1}} \frac{dt}{(z+t)^2} - \sum_{n=1}^{\infty} n \int_{b_n}^{b_{n+1}} \frac{dt}{(z+t)^2} \\ &= \int_0^{\infty} \frac{\Delta(t)}{(z+t)^2}\, dt.\end{aligned}$$

Further, we have, for $t > t_0 = t_0(\varepsilon)$,

$$(\lambda - \varepsilon)t^{\rho} < \Delta(t) < (\lambda + \varepsilon)t^{\rho}.$$

Hence

$$\begin{aligned}\frac{f^1(x)}{f(x)} &< \int_0^{t_0} \frac{\Delta(t)}{(x+t)^2}\, dt + \int_{t_0}^{\infty} \frac{(\lambda + \varepsilon)t^{\rho}}{(x+t)^2}\, dt \\ &< \int_0^{t_0} \frac{\Delta(t) - (\lambda + \varepsilon)t^{\rho}}{(x+t)^2}\, dt + \int_0^{\infty} \frac{(\lambda + \varepsilon)t^{\rho}}{(x+t)^2}\, dt.\end{aligned}$$

The first term on R.H.S. is obviously bounded. In the second term we put $t = px$, and therefore, the second integral becomes

$$\begin{aligned}&(\lambda + \varepsilon)x^{\rho-1} \int_0^{\infty} \frac{p^{(\rho+1)-1}}{(1+p)^{(\rho+1)+(1-\rho)}}\, dp \\ &= (\lambda + \varepsilon)x^{\rho-1} \cdot \pi\rho \operatorname{cosec} \pi\rho.\end{aligned}$$

Similarly, we can show that

$$\frac{f^1(x)}{f(x)} > (\lambda - \varepsilon)x^{\rho-1} \cdot \pi\rho \operatorname{cosec} \pi\rho;$$

hence the result.

Making an appeal to analytic continuation, this result can be extended for complex values of z, by taking a suitable determination of $\log f(z)$; thus we have for $|\arg z| < \pi$, if

$$\Delta(t) \sim \lambda t^{\rho},$$

then

$$\frac{f^1(re^{i\theta})}{f(re^{i\theta})} \sim \lambda e^{i(\rho-1)\theta} r^{\rho-1} \cdot \pi\rho \operatorname{cosec} \pi\rho.$$

THEOREM 2. *If $f(z)[f(0) \neq \infty]$ is a meromorphic function, real for real z, of order less than 2, with positive and real zeros and poles, a_n and b_n arranged according to increasing moduli, then the zeros of $f^1(z)$ in the half plane $R1\, z < (a_1 + b_1)/2$ are all real provided*

PROOF:

Let

$$f(z) = cz^k e^{az} \frac{\prod_{n=1}^{\infty} (1 - z/a_n)e^{z/a_n}}{\prod_{n=1}^{\infty} (1 - z/b_n)e^{z/b_n}}$$

where k is a positive integer or zero and c, a, a_n b_n, are all real and a_n and b_n are positive for all values of n. Logarithmic differentiation leads to

$$\frac{f^1(z)}{f(z)} = \frac{k}{z} + a + \sum_{n=1}^{\infty} \left\{\frac{1}{z - a_n} + \frac{1}{a_n}\right\} - \sum_{n=1}^{\infty} \left\{\frac{1}{z - b_n} + \frac{1}{b_n}\right\}.$$

If $z = x + iy$,

$$Im \left\{\frac{f^1(z)}{f(z)}\right\} = -\, iy \left\{\frac{k}{x^2 + y^2} + \sum_{n=1}^{\infty} \frac{2(a_n - b_n)x + b_n^2 - a_n^2}{[(x-a_n)^2 + y^2][(x-b_n)^2 + y^2]}\right\},$$

which under the condition stated can vanish only if $y = 0$, for, the quantity within the brackets is positive.

THEOREM 3. *Let $f(z)$ be a cubic,*

$$f(z) = z^3 + 3Hz + G,$$

all of whose zeros are real; and let $\phi(w)$ be a meromorphic function of genus zero or 1, *which is real for real w and has all the zeros and poles* $-\alpha_n$ and $-\beta_n$ real and negative. Then the cubic

$$g(z) = \phi(3)\, z^3 + 3H.\ \phi(1)\, z + G.\ \phi(0)$$

also has its zeros real, if $\alpha_n < \beta_n$ *for* $n = 1, 2, \ldots$

PROOF:

Let

$$\phi(\omega) = ae^{k\omega} \frac{\prod_{n=1}^{\infty} \left(1 + \frac{\omega}{\alpha_n}\right) e^{-\omega/\alpha_n}}{\prod_{n=1}^{\infty} \left(1 + \frac{\omega}{\beta_n}\right) e^{-\omega/\beta_n}}.$$

Condition that all the zeros of $g(z)$ be real is

$$G^2 + 4H^3 \frac{[\phi(1)]^3}{\phi(3) \cdot [\phi(0)]^2} < 0.$$

We know that since all the roots of $f(z) = 0$ are real

$$G^2 + 4H^3 < 0.$$

Hence, if

$$[\phi(1)]^3 > \phi(3)[\phi(0)]^2,$$

all the zeros of $g(z)$ are real.

Now, since

$$0 < \alpha_n < \beta_n,$$

$$\left(\frac{1}{\alpha_n^3} - \frac{1}{\beta_n^3}\right) + 3\left(\frac{1}{\alpha_n^2} - \frac{1}{\beta_n^2}\right) + \frac{9}{\alpha_n\beta_n}\left(\frac{1}{\alpha_n} - \frac{1}{\beta_n}\right) + \frac{3}{\alpha_n\beta_n}\left(\frac{1}{\alpha_n^2} - \frac{1}{\beta_n^2}\right) > 0$$

and therefore,

$$[\phi(1)]^3 > \phi(3)[\phi(0)]^2.$$

Hence the result.

Section III

DEFINITION: Let $f(z)$ be an analytic function and

$$\log^- \alpha = \min(\log \alpha, 0), \ \alpha > 0.$$

We define

$$\begin{aligned} C(r, f) &= m(r; \infty) + m(r; 0) \\ &= \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(re^{i\theta})| \, d\theta - \frac{1}{2\pi}\int_0^{2\pi} \log^- |f(re^{i\theta})| \, d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} |\log |f(re^{i\theta})|| \, d\theta. \end{aligned}$$

Unless otherwise stated, we write $C(r, f) = C(r)$, in general. The following are the direct consequences of the definition:

(i) $$C(r, f) = C(r, f - a) + 0(1),$$

where 'a' is any finite number.

(ii) If 'a' is finite and non-zero, we have

$$C(r, f) = C(r, af) + 0(1).$$

(iii) Let $f(z)$ be a meromorphic function and $\alpha, \beta, \gamma, \delta$ finite complex numbers, independent of z, of the type that

$$\alpha\delta - \beta\gamma \neq 0;$$

then

$$C\left(r, \frac{\alpha f + \beta}{\gamma f + \delta}\right) = C(r, f) + 0(1).$$

If $\gamma = 0$, $\alpha \neq 0$, $\delta \neq 0$, (iii) follows directly from (i) and (ii). Otherwise we write

$$\frac{\alpha f + \beta}{\gamma f + \delta} = \frac{\alpha}{r} - \frac{\alpha\delta - \beta\gamma}{\gamma} \cdot \frac{1}{\gamma f + \delta}$$

and observe that $C(r, f) = C(r, 1/f)$

(iv) If $f(z)$ is an integral function, for $0 < r < R$,

$$C(r) \leqq \log M(r) \leqq \frac{R+r}{R-r} C(R).$$

First part of the inequality follows readily from the definition. Using the Poisson-Jensen formula,

$$\log | f(re^{i\theta}) | = \frac{1}{2\pi} \int_0^{2\pi} \frac{\log | f(Re^{i\phi}) | \cdot R^2 - r^2 \, d\phi}{R^2 - 2Rr \cos (\theta - \phi) + r^2}$$
$$- \sum_{|a_\nu| < R} \log \left| \frac{R^2 - \bar{a}_\nu z}{R(z - a_\nu)} \right|$$

where a_ν's are the zeros of $f(z)$. Choosing θ such that $\log | f(re^{i\theta}) | = \log M(r)$, we obtain

$$\log M(r) \leqq \frac{R+r}{R-r} \cdot C(R).$$

(v) In view of (iv), the order of an integral function $f(z)$ can be defined as

$$\overline{\lim_{r\to\infty}} \frac{\log C(r)}{\log r} = \rho.$$

(vi) $$\int^{\infty} \frac{C(r)}{r^{q+1}} dr \text{ and } \int^{\infty} \frac{\log M(r)}{r^{q+1}} dr$$

converge and diverge simultaneously. Also

$$\overline{\lim_{r\to\infty}} \frac{\log M(r)}{r^{q+1}} \text{ and } \overline{\lim_{r\to\infty}} \frac{C(r)}{r^{q+1}}$$

tend to the same, zero, finite or infinite limits.

Definition: Let $f(z)$ be a meromorphic function of order ρ, where

(A) $$f(z) = \frac{f_1(z)}{P(z)},$$

$f_1(z)$ being of order ρ and $P(z)$ of order $\rho^1 (< \rho)$. Then

$$\overline{\lim_{r\to\infty}} \frac{\log C(r, f)}{\log r} = \rho.$$

That the above definition is equivalent to that of Borel follows easily from the following lemma.

LEMMA: Let $f_1(z)$ and $f_2(z)$ be any two given integral functions and let

$$\phi(z) = f_1(z) \cdot [f_2(z)]^a.$$

Then

$$C(r, \phi) \leqq C(r, f_1) + C(r, f_2)$$

where $a = \pm 1$.

The proof follows directly from the definition.

From (A) we have

$$C(r, f) \leqq C(r, f_1) + C(r, P).$$

Here, either $C(r, f_1)$ is greater than $C(r, P)$ for all values of $r > r_0$ or else there is a sequence of values of r tending to infinity for which $C(r, f_1)$ is less than $C(r, P)$.

Obviously $C(r, f)$ cannot be of order greater than ρ. Since

$$C(r, f_1) \leqq C(r, f) + C(r, P)$$

and

$$C(r, P) \leqq C(r, f_1) + C(r, f),$$

$C(r, f)$ is always greater than or equal to the higher of the two, viz. $C(r, f_1)$ and $C(r, P)$ and hence the result.

We know ([**3**], p. 82) that the order of the derivative of a meromorphic function is the same as that of the function itself. Here we give an alternative proof of this.

THEOREM: *The order of the derivative of a meromorphic function is the same as that of the function.*

PROOF: Let $f(z)$ be a meromorphic function of order ρ and $f'(z)$ its derivative*. Then

$$\begin{aligned} C(r, f) \sim C(r, f^1) &= \frac{1}{2\pi} \int_0^{2\pi} |\log|f(re^{i\theta})|\, d\theta \\ &\sim \frac{1}{2\pi} \int_0^{2\pi} |\log|f'(re^{i\theta})||\, d\theta \\ &\leqq \frac{1}{2\pi} \int_0^{2\pi} \left|\log\left|\frac{f'(re^{i\theta})}{f(re^{i\theta})}\right|\right| d\theta \\ &< A \cdot \log r \end{aligned}$$

Thus $f'(z)$ is also of order ρ.

I wish to thank Dr. S. K. Bose for his suggestion of this problem and guidance in the preparation of this paper.

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(Oblatum 2-1-57).

[1]) The order of $f'(z)/f(z)$ has been obtained by the author [**2**] for integral functions and the same can easily be extended for meromorphic functions.

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