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Absolute Neighborhood Retracts and Local Connectedness in Arbitrary Metric Spaces

by

J. Dugundji

1. Introduction.

For separable metric spaces, the following implications and equivalences are well-known:

I. $ANR \Rightarrow$ local contractibility $\Rightarrow LC^{\infty}$. These concepts are equivalent only in the finite-dimensional spaces.

II. $AR \Rightarrow$ contractibility and local contractibility $\Rightarrow C^{\infty}$ and LC^{∞}. These notions coincide only in the finite-dimensional spaces.

III. n-dimensional and LC^n (n finite) $\equiv ANR$.

IV. $AR \equiv$ contractible $ANR \equiv C^{\infty}$ ANR.

A unified account is given in [9; Chap. VII] [1]. The proofs of the equivalences in I—III are based on embedding into Euclidean spaces; those in IV on embedding into the Hilbert cube.

Upon attempting to determine the interrelations of the above concepts in non-separable metric spaces, one meets (1): the question of what definition of dimension to adopt, and (2): that the embeddings considered above are never possible, since the image spaces mentioned are separable. The theory, therefore, does not give any information about I—IV in non-separable metric spaces.

The main object of this paper is to show that the complete statements I—IV are valid in *all* metric spaces, if the covering definition of dimension be used. In the course of this, various alternative characterizations of LC^n and ANR are derived, so well as other subsidiary results. Only the point-set aspects are given here; homology in such spaces will be considered elsewhere.

2. Preliminaries.

E^n denotes Euclidean n-space; $H^n \subset E^n$ is $\{(x_1, \ldots, x_n) \in E^n \mid \sum_1^n x_i^2 \leqq 1\}$ and the $(n-1)$-sphere $S^{n-1} =$ boundary H^n. I represents $\{x \in E^1 \mid 0 \leqq x \leqq 1\}$.

[1]) Numbers in square brackets are references to the bibliography.

"An open covering of a space X" will always mean "A covering of X by open sets". $\{U\}$ is a *nbd*-finite open covering of X if each $x \in X$ has a *nbd* intersecting at most finitely many sets U; X is paracompact if each open covering has a *nbd*-finite refinement.

2.1 Every metric space is paracompact. [15; 972]

2.2 In paracompact spaces, each open covering $\{U\}$ has a star refinement $\{V\}$, i.e. for each V, $\bigcup_\alpha \{V_\alpha \mid V_\alpha \cap V \neq 0\} \subset$ some set U. [16; 45]

"A *continuous* map f of a space X into a space Y" is written "$f: X \to Y$". The homotopy of $f_0, f_1 : X \to Y$ is symbolized "$f_0 \simeq f_1$"; if the homotopy Φ satisfies $\Phi(X \times I) \subset B$ where $B \subset Y$, f_0 and f_1 are called "homotopic in B". f is nullhomotopic (notation: $f \simeq 0$) if it is homotopic to a constant map.

2.3 $f: S^n \to Y$ is nullhomotopic in B if and only if f has an extension $F: H^{n+1} \to Y$ with $F(H^{n+1}) \subset B$. [1; 501]

A polytope is a point set composed of an arbitrary collection of closed Euclidean cells (higher-dimensional analogs of a tetrahedron) satisfying: 1. Every face of a cell of the collection is itself a cell of the collection and 2. The intersection of any two cells is a face of both of them. The dimensions of the cells need not have a finite upper bound, and the set of cells incident with any given one may have any finite or transfinite cardinal. The CW topology [17; 316] is always used: $U \subset P$ is open if and only if for each closed cell $\bar{\sigma}$, $U \cap \bar{\sigma}$ is open in the Euclidean topology of $\bar{\sigma}$.

2.4 The open sets of P are invariant under subdivision. Stars of vertices are open sets. $f: P \to Y$, Y any space, if and only if $f \mid \bar{\sigma}$ is continuous for each cell $\bar{\sigma}$. If $\{U\}$ is an open covering of P, there is a subdivision P' of P having each closed vertex-star contained in some set of $\{U\}$. [17].

For a metric space X, define dim $X \leqq n$ if each open covering $\{U\}$ has a refinement $\{V\}$ in which no more than $n+1$ sets V have a non-vacuous intersection. (i.e. $\{V\}$ is of order $\leqq n+1$).

2.5 $\dim X \times I \leqq \dim X + 1$. For any set $E \subset X$, $\dim E \leqq \dim X$. [13]

In a space X with metric d, $S(x_0, \varepsilon)$ denotes $\{x \in X \mid d(x, x_0) < \varepsilon\}$. Let $A \subset X$ be closed, cover $X - A$ by $\{S(x, \frac{1}{2}d(x, A)) \mid x \in X - A\}$ and take a *nbd*-finite refinement $\{U\}$. Then [4; 354]

2.6 $\{U\}$ has the properties: 1. Each *nbd* of $a \in [A -$ interior $A]$ contains infinitely many sets U; 2. For each *nbd* $W \supset a$ there is a *nbd* W', $a \in W' \subset W$, such that whenever $U \cap W' \neq 0$ then $U \subset W$; 3. For each U, $d(U, A) > 0$.

Open coverings of $X-A$ with the properties 1—3 above are called canonical. One can always assume order$\{U\} \leqq \dim(X-A)+1$.

If $N(U)$ is the polytope nerve of a canonical cover of $X-A$, the set $\tilde{X}=A \cup N(U)$ is given a Hausdorff topology by taking: (a). $N(U)$ with the CW topology and (b). A subbasis for the *nbds* of $a \in A \subset \tilde{X}$ to consist of all sets with form $\{W \cap A\} \cup \{$all vertex stars in $N(U)$ corresponding to sets $U \subset W\}$ where W is a *nbd* of *a in* X.

2.61 Let $K : X-A \to N(U)$ be the Alexandroff map.

The (canonical) map

$$\mu(x) = \begin{cases} x & x \in A \\ K(x) & x \in X-A \end{cases}$$

of X into $\tilde{X}$ is continuous, and $\mu \mid A$ a homeomorphism. [4; 356]

2.62 If $N(U)_0$ is the zero-skeleton of $N(U)$, then $A \subset \tilde{X}$ is a retract of $A \cup N(U)_0$.

A retraction r is obtained [4; 357] by first choosing a point $x_U \in U$; for each x_U select $a_U \in A$ with $d(a_U, x_U) < 2d(x_U, A)$ and set

$$\begin{cases} r(p_U) = a_U, & p_U \text{ the vertex of } N(U) \text{ corresponding to} \\ r(a) \;\; = a, & a \in A. \end{cases}$$

2.63 An $f : A \to E$ has an extension $F : A \cup N(U) \to E$ if either

(a). E is a convex subset of a locally convex linear space [4; 357]

or

(b). E is a convex subset of a real vector space L having the finite topology: $G \subset L$ is open if and only if for each finite-dimensional linear subspace K, $G \cap K$ is open in the Euclidean topology of K. [3; 9]

$B(Z)$ denotes the Banach space of all bounded continuous real-valued functions on Z.

2.7 Each metric X can be embedded in $B(X)$ [11; 543] as a closed subset of its convex hull $H(X)$ [18; 186].

LEMMA 2.8. Let X, Y be metric, $A \subset X$ closed and $f : A \to Y$. Then there exists a metric $Y_1 \supset Y$ and an extension $F : X \to Y_1$ such that Y is closed in Y_1 and $F \mid X-A$ is a homeomorphism of $X-A$ with $Y_1 - Y$.

PROOF. Embed Y in $H(Y)$; 2.61 land 2.63 yield and extension $F^+ : X \to H(Y)$. Regard $X \subset B(X)$ and form the cartesian product $H(Y) \times E^1 \times B(X)$; with Kuratowski [8; 139], $F(x) = [F^+(x), d(x, A), x \cdot d(x, A)]$ is the desired map and $Y_1 = F(X)$.

3. The property LC^n.

DEFINITION 3.1. A space is n-locally connected (symbol: $n-LC$)

at $y \in Y$ if for each *nbd* $U \supset y$ there is a *nbd* V, $y \in V \subset U$, such that every $f : S^n \to V$ has an extension $F : H^{n+1} \to U$, or, equivalently (2.3) every $f : S^n \to V$ is nullhomotopic in U.

Y is LC^n if it is $i-LC$ at each point for all $0 \leqq i \leqq n$

Y is LC^∞ if it is LC^n for every n.

The following is the generalization of Kuratowski's theorem [10; 273] to the non-separable metric case; the proof is similar.

THEOREM 3.2. Let Y be metric and n finite. The following four properties are equivalent:

3.21 Y is LC^n, n finite.

3.22 If X is metric, $A \subset X$ closed, and $\dim (X-A) \leqq n+1$, then every $f : A \to Y$ can be extended over a *nbd* $W \supset A$ in X.

3.23 For each $y \in Y$ and *nbd* $U \supset y$ there is a *nbd* V, $y \in V \subset U$ such that: if X is metric, $A \subset X$ closed, and $\dim (X-A) \leqq n+1$, then every $f : A \to V$ has an extension $F : X \to U$.

3.24 For each $y \in Y$ and *nbd* $U \supset y$ there is a *nbd* V, $y \in V \subset U$ such that: if X is metric, $\dim X \leqq n$, every $f : X \to V$ is nullhomotopic in U.

PROOF.

$1 \Rightarrow 2$: Form $\tilde{X} = A \cup N(U)$ of 2.61 with $\dim N(U) \leqq n+1$. Regarding f defined on $A \subset \tilde{X}$ one need only construct an open W, $A \subset W \subset \tilde{X}$ and an extension $F : W \to Y$; $F\mu \mid \mu^{-1}(W)$ will then be the required extension.

Let $N(U)_k$ be the k-skeleton of $N(U)$; for $k = 0, 1, \ldots, n+1$, open W_k, $A \subset W_k \subset \tilde{X}$, and extensions $f_k : [A \cup N(U)_k] \cap W_k \to Y$ will be defined inductively; the result follows by taking $W = W_{n+1}$ and $f = f_{n+1}$.

Take $W_0 = \tilde{X}$ by 2.62, $f_0 = fr$ is the extension.

Suppose $k-1 \leqq n$ and W_{k-1}, f_{k-1} constructed. For each $a \in A$ let W_a be a *nbd* of $f_{k-1}(a) = f(a)$ such that any $f : S^{k-1} \to W_a$ extends to a continuous map of H^k into Y. From 2.6 and continuity of f_{k-1} there is a *nbd* $\tilde{V}_a \supset a$ such that each closed k-cell $\bar{\sigma}$ in the vertex-stars that form $\tilde{V}_a$ satisfies (1) $\bar{\sigma} \subset W_{k-1}$ and (2) $f_{k-1}(bdry\, \bar{\sigma}) \subset W_a$.

Set $W_k = \bigcup_a \tilde{V}_a$.

To define f_k, let $\bar{\sigma}$ be any closed k-cell in a vertex-star forming W_k; by (2) $f_{k-1} \mid bdry\, \bar{\sigma}$ has an extension over $\bar{\sigma}$. Let $\eta(\bar{\sigma})$ be the infimum of the diameters of the images of $\bar{\sigma}$ taken over all possible extensions, and define $f_k \mid \bar{\sigma}$ to be an extension having image diameter $< 2\,\eta(\bar{\sigma})$. Proceeding in this way yields and extension f_k of f_{k-1} over $(A \cup N(U)_k) \cap W_k$. Continuity need be proved only at A, and follows at once by observing that Y is $(k-1)-LC$ and f_{k-1} is continuous.

2 ⇒ 3. Assume 3 not true at y_0. Then there is an $S(y_0, \alpha) = U$, a sequence X_i of metric spaces, a sequence $A_i \subset X_i$ of closed subsets with $\dim (X_i - A_i) \leqq n+1$, and a sequence of $f_i : A_i \to \to S(y_0, \alpha/i)$ none of which is extendable over X_i with values in U.

The points and metric of X_i will carry the subscript i. One can assume all the metrics bounded: $d_i(x_i, x'_i) < 1$. Construct a metric space $\hat{X}$ consisting of a point $\hat{x}$ and the pairwise disjoint union of the X_i by setting

$$\hat{d}(x_i, x'_i) = 2^{-i} d_i(x_i, x'_i)$$
$$\hat{d}(x_i, x_j) = 2^{-\min(i,j)} \qquad i \neq j$$
$$\hat{d}(\hat{x}, x_i) = 2^{-i}$$

$\hat{A} = \hat{x} \cup \bigcup_1^\infty A_i$ is closed $\hat{X}$ and $\dim (\hat{X} - \hat{A}) \leqq n+1$ [13]. Defining $\hat{f} : \hat{A} \to Y$ by $\hat{f}(a_i) = f(a_i)$, $\hat{f}(\hat{x}) = y_0$, $\hat{f}$ is continuous, so by 2 there is an open $\hat{W} \supset \hat{A}$ and an extension $\hat{F} : \hat{W} \to Y$. Because $\hat{x} \in \hat{W}$ and diam $(\hat{x} \cup X_i) < 2^{-i}$, $\hat{W}$ contains all X_i for i large. Using the continuity of $\hat{F}$ at $\hat{x}$ one easily finds diam $(y_0 \cup \hat{F}(X_i) < \frac{\alpha}{2}$ for large i, so that then $\hat{F}(X_i) \subset U$. Since $\hat{F} \mid X_i$ is an extension of f_i this contradiction proves the assertion.

3 ⇒ 4. This is the homotopy theorem corresponding to 3: set $\hat{X} = X \times I$ and $A = X \times 0 \cup X \times 1$ then 3 applies since $\dim (X \times I) \leqq n+1$.

4 ⇒ 1. Let $X = S^r$, $0 \leqq r \leqq n$.

Remark 3.3. In case $n = \infty$, then 3.2 holds is one restricts $X - A$ to having finite dimension.

Theorem 3.4. Let Y be metric. The following two properties are equivalent.,

3.41 Y is LC^n n finite.

3.42 If Z is metric, $Y \subset Z$ is closed, and $\dim (Z-Y) \leqq n+1$, then Y is a *nbd* retract in Z.

Proof. 1 ⇒ 2 is obvious from 3.22.

2 ⇒ 1. One shows 3.22 valid. Given $f : A \to Y$ as in 3.22, use 2.8 to obtain a metric $Y_1 \supset Y$, Y closed in Y_1, and an extension $F : X \to Y_1$.

Since $X - A$ is homeomorphic to $Y_1 - Y$, $\dim (Y_1 - Y) \leqq n+1$ so that there is an open $U \supset Y$ in Y_1 and a retraction $r : U \to Y$. $rF \mid F^{-1}(U)$ is the desired extension of f.

4. Characterization of LC^n by partial realization.

DEFINITION 4.1. Let $\{U\}$ be a covering of a space Y. Let P be a polytope and Q a subpolytope of P containing the zero-skeleton of P. An $f : Q \to Y$ is called a partial realization of P relative to $\{U\}$ if for each closed cell $\bar{\sigma}$ of P, $f(Q \cap \bar{\sigma}) \subset$ some set U.

THEOREM 4.2. Let Y be metric. The following two properties are equivalent:

4.21 Y is LC^n, n finite.

4.22 Each open covering $\{U\}$ of Y has a *nbd*-finite refinement $\{V\}$ with the property: Every partial realization of any polytope P dim $P \leqq n+1$, relative to $\{V\}$ extends to a full realization of P relative to $\{U\}$.

PROOF. $1 \Rightarrow 2$. For each $y \in Y$ select a $U \supset y$; choose an open V, $y \in V \subset U$ satisfying the definition of n-LC and let $\{V(n)\}$ be a star-refinement. Repeat, using $\{V(n)\}$ and $(n-1)-LC$ to obtain $\{V(n-1)\}$. A *nbd*-finite refinement of $\{V(0)\}$ satisfies the requirements.

$2 \Rightarrow 1$. Let $y \in Y$. For any $\varepsilon > 0$ take the open covering: $S(y, \varepsilon)$, $Y - \overline{S(y, \varepsilon/2)}$. Let $\{V\}$ be the refinement satisfying (2), and choose $V \supset y$. Then $V \subset S(y, \varepsilon)$ and, S^k being a subpolytope of H^{k+1}, any $f : S^k \to V$ extends to $F : H^{k+1} \to S(y, \varepsilon)$, $k \leqq n$. This suffices to prove the assertion.

REMARK 4.3. 4.2 is valid even if the polytopes of 4.22 are restricted to be finite.

5. Characterization of LC^n by homotopy.

THEOREM 5.1. Let Y be metric. The following two properties are equivalent:

5.11 Y is LC^n, n finite.

5.12 Each open covering $\{U\}$ has a *nbd*-finite refinement $\{W\}$ with the property: For any metric X, dim $X \leqq n$, and any $f_0, f_1 : X \to Y$ satisfying

(a). $f_0(x)$ and $f_1(x)$ belong to a common W for each $x \in X$.

Then $f_0 \simeq f_1$ and the homotopy Φ can be chosen so that $\Phi(x, I)$ lies in a set U for each x.

PROOF. $1 \Rightarrow 2$. Let $\{U(0)\}$ be given. Construct successive star refinements $\{U(i)\}$, $i = 0, 1, \ldots, n$ with $\{U(i+1)\}$ having property 4.22 relative to $\{U(i)\}$. Let $\{W\}$ be a star refinement of $\{U(n)\}$; by 2.1 $\{W\}$ may be assumed *nbd*-finite. This is the desired open covering.

Let f_0, f_1 satisfy (a) and let $\{G\}$ be a common refinement of $\{f_0^{-1}(W)\}$ and $\{f_i^{-1}(W)\}$. Take a common refinement of a canonical cover of $X \times I - [X \times 0 \cup X \times 1]$ and $\{G \times I\}$, and let $\{V\}$ be a star refinement of this resulting cover. $\{V\}$ can be assumed *nbd*-finite and of order $\leqq n+1$; it is clearly also a canonical cover.

Form $\widetilde{X \times I} = X \times 0 \cup X \times 1 \cup N(V)$ and regard f_0, f_1 defined on the subsets $X \times 0$, $X \times 1$, respectively, of $\widetilde{X \times I}$. Construct an extension $F_0 : X \times 0 \cup X \times 1 \cup N(V)_0 \to Y$ as follows: given the vertex p_V select $x_V \times i_V \in V$; if $i_V \leqq \frac{1}{2}$ set $F_0(p_V) = f_0(x_V)$, otherwise $F_0(p_V) = f_1(x_V)$. F_0 is continuous.

Let $(p_0 \ldots, p_r)$ be any r-cell of $N(V)$; all these vertices will be shown to map into a single $\{U(n)\}$. Since $V_0 \cap \ldots \cap V_r \neq 0$, $\bigcup_0^r V_i \subset G \times I$ for some G so that $\bigcup_0^r x_i \subset G$. Since $G \subset f_0^{-1}(W)$ for some W, $\bigcup_0^r f_0(x_i) \subset W$. By condition (a) for each $i = 0, \ldots, n$ there is a W_i with $f_0(x_i) \cup f_1(x_i) \subset W_i$ so that $W \cap W_i \neq 0$, which means $\bigcup_0^r W_i \subset U(n)$ for some $U(n)$ and therefore

$$\bigcup_0^r f_0(x_i) \cup f_1(x_i) \subset U(n).$$

Proceed by induction. Assume an extension F_{k-1} on $X \times 0 \cup X \times 1 \cup N(V)_{k-1}$ which is a partial realization of $N(V)$ relative to $\{U(n-k+1)\}$; F_{k-1} extends over each closed k-cell $\bar{\sigma}$ with image in a set $U(n-k)$. Let $\eta(\bar{\sigma})$ be the infimum of the diameters of all the possible extensions; select an extension with diam $F_k(\bar{\bar{\sigma}}) < 2\eta(\bar{\sigma})$ and $F_k(\bar{\sigma}) \subset$ some $U(n-k)$. This process yields a partial realization of $N(V)$ relative to $\{U(n-k)\}$ and the map is continuous on $X \times 0 \cup X \times 1 \cup N(V)_k$ as in 3.22.

The required homotopy is $\Phi = F_{n+1}\, \mu$; the required condition is satisfied because, by the above, all vertices corresponding to sets lying in a strip $G \times I$ have images in one set $\{U(n)\}$ so that in extending the partial realization the images of all cells lie in one $\{U(0)\}$. $2 \Rightarrow 1$. As in 4.2

6. Characterization of LC^n by "factorization".

THEOREM 6.1. Let Y be metric. The following two properties are equivalent:

6.11 Y is LC^n, n finite.

6.12 For each open covering $\{U\}$ there exists a polytope P, dim $P \leqq n$ and a $g : P \to Y$ with the property: For any metric X, dim $X \leqq n$ and any $f : X \to Y$ there is a $\varphi : X \to P$ with $f \simeq g\varphi$, and the homotopy Φ can be chosen so that $\Phi(x, I)$ lies in a single U for each x.

PROOF. $1 \Rightarrow 2$. Let $\{W\}$ be a refinement satisfying 5.12 relative to $\{U\}$ let $\{\hat{V}\}$ be a refinement of $\{W\}$ with property 4.22, and finally let $\{V\}$ be a star refinement of $\{\hat{V}\}$; $\{V\}$ can be assumed *nbd*-finite. Let $P = N(V)_n$ and define $\varphi_0 : P_0 \to Y$ by sending each p_V to a $y_V \in V$; this is obviously a partial realization relative to $\{\hat{V}\}$; φ_0 extends to $\varphi : P \to Y$ with the image of each n-cell in some W.

Let X be metric, dim $X \leqq n$ and $f : X \to Y$. Form $\{f^{-1}(V)\}$ and let $\{G\}$ be a star *nbd*-finite refinement of order $\leqq n+1$. Let $K : X \to N(G)$ be the Alexandroff map and define $\gamma : N(G) \to P$ as follows: for each p_G select a V with $G \subset f^{-1}(V)$ and set $\gamma(p_G) = p_V$. This is simplicial, and extending linearly gives a continuous map (2.4). f will be shown homotopic to $\varphi\gamma K$ in the required way.

Let $x \in G_1 \cap \ldots \cap G_s$ and only these sets; then $K(x) \in \overline{(p_{G_1}, \ldots p_{G_s})}$ so $\gamma K(x) \in \overline{(p_{V_1} \ldots p_{V_s})}$; since $g(p_{V_i}) \in V_i$ and $\bigcap_1^s V_i \neq 0$, one finds $g\gamma K(x)$ lying in some W satisfying $\bigcup_1^n V_i \subset W$. Again, since $f(G_i) \subset V_i$, $i = 1, \ldots, s$ one has $f(x) \in \bigcap_1^s V_i \subset W$ also, so for each x $g\gamma K(x)$ and $f(x)$ are in a common W hence are homotopic in the required fashion.

$2 \Rightarrow 1$. Assume Y not k-LC at y_0, $k \leqq n$. There is an $S(y_0, \alpha)$ and a sequence of $f_i : S_i^k \to S(y_0, \alpha/4_i)$ with no f_i nullhomotopic in $S(y_0, \alpha)$. As in 3.2, form $\hat{S} = \hat{x} \cup \bigcup_1^\infty S_i^k$ and define $\hat{f} : \hat{S} \to Y$ by $\hat{f}(\hat{x}) = y_0$, $\hat{f} \mid \hat{S}_i^k = f_i$.

Cover Y by $S(y_0, \alpha/2)$ and $Y - \overline{S(y_0, \alpha/3)}$, and let P be the polytope corresponding to this covering. Since dim $\hat{S} \leqq n$ [13] there is a $\varphi : \hat{S} \to P$ with $f \simeq g\varphi$, and because $f(\hat{S}) \subset S(y_0, \alpha/4)$ the homotopy ϕ satisfies $\phi(\hat{S} \times I) \subset S(y_0, \alpha/2)$ hence $g\varphi(\hat{S}) \subset S(y_0, \alpha/2)$. In particular $\varphi(\hat{x}) \subset g^{-1}(S(y_0, \alpha/2))$. Cover P by $g^{-1}(S(y_0, \alpha))$ and $P - \overline{g^{-1}(S(y_0, \alpha/2))}$; subdivide P so each closed vertex star lies in one of these open sets (2.4). Let St p be the star containing $\varphi(\hat{x})$. Then $g^{-1}(S(y_0, \alpha))$ is open, contains St p and $\varphi(S_i^k) \subset$ St p for large i. Since $\varphi \mid S_i^k$ is nullhomotopic in St p (by radial contraction) and $f_i \simeq g\varphi \mid S_i^k$ in $S(y_0, \alpha/2)$ one finds $f_i \simeq 0$ in $S(y_0, \alpha)$, a contradiction.

7. The property LC.

DEFINITION 7.1. A space Y is locally contractible at $y \in Y$ if for each *nbd* $U \supset y$ there is a *nbd* V, $y \in V \subset U$, contractible to a point over U, i.e. the identity map of V is nullhomotopic in U. Y is LC if it is locally contractible at every point.

Every LC space is clearly LC^∞; the converse is not true, even in separable metric spaces [10; 273]. However,

THEOREM 7.2. Let Y be a finite-dimensional metric space. The following three properties are equivalent:

7.21 Y is LC^n, n some integer $\geqq \dim Y$.

7.22 Y is LC.

7.23 Y is LC^∞.

PROOF. Only $1 \Rightarrow 2$ requires proof. Given $U \supset y$ let V be a *nbd* satisfying 3.24. Since $\dim V \leqq n$ (2.5) the identity map of V is nullhomotopic in U.

8. The properties C^n and C.

DEFINITION 8.1. Y is connected in dimension n (written: $n-C$) if each $f : S^n \to Y$ is nullhomotopic. Y is C^n if it is $i-C$ for all $0 \leqq i \leqq n$ Y is C^∞ if it is C^n for all n.

Y is C^0 is equivalent with Y arcwise connected; Y is $n-C$ is equivalent with the $n-th$ homotopy group $\pi_n(Y) = 0$.

DEFINITION 8.2. Y is contractible (symbolism: Y is C) if the identity map of Y is nullhomotopic.

Clearly Y is C implies Y is C^∞; the converse is not true [10; 273]. The following theorem is well-known (see for example [5; 241]).

THEOREM 8.3. Let Y be any (not necessarily metric) space, and n finite.

The following five properties are equivalent:

8.31 Y is C^n.

8.32 If P is any polytope, and $Q \subset P$ a subpolytope, any $f : Q \to Y$ extends over $Q \cup P_{n+1}$, where P_s denotes the s-skeleton of P.

8.33 If P is any polytope, $Q \subset P$ a subpolytope, and $f_0, f_1 : P \to Y$ satisfy $f_0 \mid Q \simeq f_1 \mid Q$, then $f_0 \mid Q \cup P_n \simeq f_1 \mid Q \cup P_n$.

8.34 If P is a polytope with $\dim P \leqq n$, any $f : P \to Y$ is nullhomotopic.

8.35 If P is any polytope, any $f : P \to Y$ is homotopic to an $f_1 : P \to Y$ sending P_n to a single point.

This will be used in the next section.

REMARK 8.4. If $n = \infty$, no restriction need be placed on $\dim P$ in 8.3.

9. The properties C^n and LC^n together.

THEOREM 9.1. Let the metric space Y be LC^n. The following three properties are equivalent:
9.11 Y is C^n.
9.12 If X is metric, $A \subset X$ closed, and $\dim (X-A) \leqq n+1$, then any $f : A \to Y$ extends over the entire X.
9.13 If X is metric, $\dim X \leqq n$, then any $f : X \to Y$ is null-homotopic.

PROOF. Only $1 \Rightarrow 2$ requires proof. Using $A \cup N(U)$ and regarding f as defined on $A \subset A \cup N(U)$, the problem is to extend f over $A \cup N(U)$. Since Y is LC^n, the proof of 3.22 gives an extension f' over a *nbd* $W' \supset A$ in $A \cup N(U)$. Let Q be the union of all closed cells of $N(U)$ on which f' is defined; Q is a subpolytope of $N(U)$ and no point of A is a limit point of $N(U)-Q$. The extension of $f' \mid Q$ over $N(U)$ guaranteed by 8.32, together with $f' \mid A \cup Q$ is the desired map.

THEOREM 9.2. Let Y be a finite-dimensional metric LC^r space, where $r \geqq \dim Y$. The following three properties are equivalent:
9.21 Y is C^n for some $n \geqq \dim Y$.
9.22 Y is contractible.
9.23 Y is C^∞.

PROOF. As in 7.2.

REMARK 9.3. Theorem 9.2 is not true if $\dim Y = \infty$, even if Y is separable metric. [2]

10. Absolute neighborhood retracts and absolute retracts.

DEFINITION 10.1. An arbitrary space Y is an absolute *nbd* retract for a class $\mathfrak{A}$ of spaces (written: Y is an $ANR\,\mathfrak{A}$) if for any closed subset A of any $X \in \mathfrak{A}$, and any $f : A \to Y$, there is an extension $F : U \to Y$ of f over a *nbd* $U \supset A$.

Y is an absolute retract for the class $\mathfrak{A}$ (symbol: $AR\,\mathfrak{A}$) if for any closed subset A of any $X \in \mathfrak{A}$ and any $f : A \to Y$, there is an extension $F : X \to Y$.

In the following, $\mathfrak{M}$ denotes the class of metric spaces, $\mathfrak{P}$ the class of polytopes.

An immediate consequence of 2.63, as was pointed out in [3; 9], [4; 357] is

THEOREM 10.2. Any convex set C of either (a): A locally convex linear space, or (b): A real vector space with finite topology, is an $AR\,\mathfrak{M}$.

11. The properties ANR and AR in polytopes.

In [3; 10] it was shown that

11.1 Any polytope P can be embedded as a *nbd* retract in a polytope K spanning a convex subset of a real vector space with finite topology.

This leads at once to

THEOREM 11.2. Any polytope is an $ANR\,\mathfrak{M}$ and also an $ANR\,\mathfrak{P}$. It is an $AR\,\mathfrak{M}$ if and only if it is an $AR\,\mathfrak{P}$.

PROOF. The $ANR\,\mathfrak{M}$ follows from 11.1 and 10.2; this was proved in [3; 10a]. The $ANR\,\mathfrak{P}$ has also been proved in [3; 10a]. $AR\,\mathfrak{P}$ implies $AR\,\mathfrak{M}$ since P would be a retract of K in 11.1, and one then applies 10.2. $AR\,\mathfrak{M}$ implies $AR\,\mathfrak{P}$ is proved inductively exactly as [3; 10a]

THEOREM 11.3. Let P be a polytope. Taking $\mathfrak{A} = \mathfrak{M}$ or $\mathfrak{P}$, the following three properties are equivalent:

11.31 P is an $AR\,\mathfrak{A}$.

11.32 P is contractible.

11.33 P is C^∞.

PROOF. Only $3 \Rightarrow 1$ requires proof; P will be shown $AR\,\mathfrak{M}$. From 8.4, 8.32, P is a retract of the K in 11.1, so the result follows from 10.2.

12. The properties ANR and AR in metric spaces.

THEOREM 12.1. For metric Y the following are equivalent:

12.11 Y is an $ANR\,\mathfrak{M}$ (an $AR\,\mathfrak{M}$).

12.12 If Z is metric and $Y \subset Z$ closed, then Y is a *nbd* retract (retract) in Z.

12.13 Y can be embedded in the Banach space $B(Y)$ as a *nbd* retract (retract) of its convex hull $H(Y)$.

PROOF. $1 \Rightarrow 2 \Rightarrow 3$ is trivial; $3 \Rightarrow 1$ is analogous to the proof given in 11.3.

THEOREM 12.2. If Y is a metric space, and Y is an $ANR\,\mathfrak{M}$ ($AR\,\mathfrak{M}$), then Y is also an $ANR\,\mathfrak{P}$ ($AR\,\mathfrak{P}$).

This is proved in [3; 10b].

THEOREM 12.3. Let Y be metric. If Y is an $ANR\,\mathfrak{M}$ ($AR\,\mathfrak{M}$), then Y is LC (LC and C), hence also LC^∞ (LC^∞ and C^∞).

PROOF. Note first that a convex subset of a Banach space is contractible and locally contractible. The theorem follows from 12.13 by the trivial remark that contractibility is preserved under retraction and local contractibility is preserved under *nbd* retraction.

12.4. THEOREM For metric Y, the following properties are equivalent:

12.41 Y is an $AR\mathfrak{M}$.

12.42 Y is a contractible $ANR\mathfrak{M}$.

12.43 Y is a C^∞ $ANR\mathfrak{M}$.

PROOF. $1 \Rightarrow 2 \Rightarrow 3$ is trivial. $3 \Rightarrow 4$ follows as in 9.12, using 8.4, 2.63.

13. Characterization of ANR $\mathfrak{M}$ by partial realization.

The following Lemma is due to Kuratowski [8; 122].

LEMMA 13.1. Let D be an arbitrary non-empty subset of a metric Z. Let $\{U\}$ be a covering of D by sets open in D. Then there exists a collection $\{\text{Ext } U\}$ of sets open in Z with

13.11 $U = D \cap \text{Ext } U$ for each U.

13.12 The nerve of $\{U\}$ is homeomorphic to the nerve of $\{\text{Ext } U\}$

In fact, one defines

$$\text{Ext } U = \{z \in Z \mid d(z, U) < d(z, D-U)\}.$$

REMARK 13.2. If $U \subset U'$ then $\text{Ext } U \subset \text{Ext } U'$.

REMARK 13.3. Given $z \in \text{Ext } U$; if $\zeta \in D$ is to be chosen to satisfy $d(z, \zeta) < 2d(z, D)$ one can always find such an ζ in U.

THEOREM 13.4. For metric Y, the following properties are equivalent:

13.41 Y is an $ANR\mathfrak{M}$.

13.42 For each open covering $\{U\}$ of Y there is a refinement $\{V\}$ with the property: Every partial realization of any polytope relative to $\{V\}$ extends to a full realization in $\{U\}$.

PROOF. $2 \Rightarrow 1$. For each $n = 1, 2, \ldots$ define inductively an open cover $V(n)$ as follows:

(a) $\{\tilde{V}(1)\}$ is an open cover of mesh < 1, i.e. sup diam $\{\tilde{V}(1)\} < 1$

(b) $\{\tilde{V}'(1)\}$ satisfies 13.42 relative to $\{\tilde{V}(1)\}$

(c) $\{\tilde{V}''(1)\}$ is a star refinement of $\{\tilde{V}'(1)\}$

(d) $\{V''(1)\}$ satisfies 13.42 relative to $\{\tilde{V}''(1)\}$

(e) $\{V(1)\}$ is a star refinement of $\{V''(1)\}$

If $\{V(n-1)\}$ is defined, let $\tilde{V}(n)$ be a refinement of mesh $< \dfrac{1}{n}$ and go through (a) — (e) again to obtain $\{V(n)\}$. Note that $\{V(n)\}$ is a refinement of $\{V(n-1)\}$.

Embed Y in $H(Y)$; to obtain a retraction of a *nbd* of Y onto Y some further constructions are needed.

(α) Form the open sets $\text{Ext } V(n)$ in $H(Y)$; from 13.2,

13.5: Each $\text{Ext } V(n)$ is contained in some $\text{Ext } V(n-1)$.

(β) Let $\{U\}$ be a canonical cover of $H(Y)-Y$.

(γ) Define a sequence of *nbds* $W_i \supset Y$ in $H(Y)$ by induction: Set $W_1 = \cup \text{Ext}\, V(1)$. If W_{n-1} is defined, normality gives an open G_n with $Y \subset G_n \subset \overline{G}_n \subset W_{n-1}$. For each $y \in Y$ choose a *nbd* of form $G_n \cap \text{Ext}\, V(n)$; by 2.6, y has a *nbd* $W_n(y)$ such that $U \cap W_n(y) \neq 0$ implies $U \subset G_n \cap \text{Ext}\, V(n)$. Define $W_n = \bigcup_y W_n(y)$ to complete the inductive step.

Clearly $\overline{W}_n \subset W_{n-1}$ and one can assume $d(Y, H(Y) - W_n) < \frac{1}{n}$.

Furthermore,

13.6 $W_1 \supset W_2 \supset \ldots$ and $\bigcap_1^\infty W_i = Y$.

13.7 Let $n \geqq 3$. If $U \cap \overline{W}_n \neq 0$ then $U \subset W_{n-1} \cap \text{Ext}\, V(n)$ for some $V(n)$ and therefore $U \cap (\overline{W}_{n-2} - W_{n-1}) = 0$.

Indeed, $U \cap \overline{W}_n \neq 0$ implies $U \cap W_n \neq 0$ because U is open; thus $U \cap W_n(y) \neq 0$ for some y and the result follows.

(δ) To each U assign an integer n_U as follows: if $U \cap \overline{W}_3 = 0$ set $n_U = 0$; if $U \cap \overline{W}_3 \neq 0$ set $n_U = \sup \{i \mid U \cap \overline{W}_i \neq 0\}$; the finiteness of n_U follows from 13.6 and 2.6.

From 13.7 and 13.5 follows

13.8 If $n_U \geqq 3$, then $U \subset$ some $\text{Ext}\, V(n_U)$ and for each $3 \leqq k \leqq n_U$ there is a $V(k) \supset V(n_U)$ with $U \subset \text{Ext}\, V(k)$.

The constructions are now complete.

Form $Y \cup N(U)$ and map the vertices $\{p_U\}$ of $N(U)$ into Y as follows: In each U choose a point z_U; for each z_U select a $y_U \in Y$ satisfying $d(y_U, z_U) < 2\, d(z_U, Y)$. By 13.3 and 13.8, if $n_U \geqq 3$ the y_U can be assumed to lie in a set $V(n_U)$ such that $U \subset \text{Ext}\, V(n_U)$. Define $r : Y \cup N(U)_0 \to Y$ by

$$\begin{aligned} r(p_U) &= y_U \\ r(a) &= a \qquad a \in A. \end{aligned}$$

Continuity follows from 2.62.

Form the "rings" $R_m = \overline{W}_m - W_{m+1}$ and let P_m be the subpolytope of $N(U)$ formed by all the sets U intersecting R_m.

13.9 For each $m \geqq 3$, r is a partial realization of P_m in $\{V''(m)\}$. In fact, if $(p_{U1}, \ldots, p_{Us})$ is a cell of P_m, then according to 13.8 there are $V_i(m) \supset r(p_{Ui})$ with $\text{Ext}\, V_i(m) \supset U_i$, $i = 1, \ldots, s$; since $U_1 \cap \ldots \cap U_s \neq 0$ the $\text{Ext}\, V_i(m)$, hence also the $V_i(m)$, have a non-vacuous intersection. The union of the $V_i(m)$, hence $\bigcup_1^s r(p_{Ui})$, is contained in some $V''(m)$.

By 13.7, $P_m \cap P_n = 0$ for $|m - n| \geqq 2$. For each $n = 1, 2, \ldots$

extend the partial realization r on P_{2n+1} to a full realization in $\{\tilde{V}''(2n+1)\}$ and denote this extension by $\tilde{r}$.

13.10 For each $n = 1, 2, \ldots, \tilde{r}$ is a partial realization of P_{2n+2} relative to $\{\tilde{V}'(2n+1)\}$.

Indeed, let $\bar{\sigma} = (p_{U1}, \ldots, p_{Us})$ be a cell of P_{2n+2}; by **13.9**, $\tilde{r}$ sends all the vertices of $\bar{\sigma}$ into a single $V''(2n+2)$ hence also into a single set $\tilde{V}''_0(2n+1)$. Since the faces of $\bar{\sigma}$ can belong only to P_{2n+1}, P_{2n+2}, P_{2n+3}, $\tilde{r}$ sends any realized face to a $\tilde{V}''(2n+1)$ intersecting $\tilde{V}''_0(2n+1)$, so that $\tilde{r}(\bar{\sigma})$ is contained in some $\tilde{V}'(2n+1)$.

$\tilde{r}$ therefore extends to $r' : \bigcup_3^\infty P_i \to Y$ with $r' \mid P_{2n+1} \cup P_{2n+2}$ being a realization relative to $\{\tilde{V}(2n+1)\}$.

Setting

$$\eta(y) = \begin{cases} y & y \in Y \\ r'(y) & y \in \bigcup_3^\infty P_i \end{cases}$$

one has $\eta : Y \cup \bigcup_3^\infty P_i \to Y$, the continuity at points of Y following from mesh $\tilde{V}(k) < 1/k$ and the continuity of r.

Let $W = \bigcup_5^\infty W_i$ and $\mu : H(Y) \to Y \cup N(U)$ the canonical map. Then $\eta\mu \mid W$ retracts W onto Y and by **12.13** Y is an $ANR\mathfrak{M}$.

$1 \Rightarrow 2$. Embed Y in $H(Y)$; since Y is an ANR, there is a retraction r of a *nbd* $V \supset Y$ in $H(Y)$ onto Y. To simplify the terminology, a spherical *nbd* in $H(Y)$ means the intersection of a spherical *nbd* in the Banach space $B(Y)$ with $H(Y)$. For each $y \in Y$ choose a spherical *nbd* $S(y)$ of y in $H(Y)$ satisfying $S(y) \subset V$ and $S(y) \cap Y \subset$ some set $\{U\}$ of the given open covering. Finally choose a spherical *nbd* $T(y)$ in $H(Y)$ with $T(y) \subset V$ and $r(T(y)) \subset S(y)$. The desired refinement is $\{T(y) \cap Y\}$. Let f be a partial realization of P relative to $\{T(y) \cap Y\}$ defined on $Q \supset P_0$. For each closed r-cell $\bar{\sigma}$ let $Z(\bar{\sigma}) = f(Q \cap \bar{\sigma})$ and $\widehat{Z}(\bar{\sigma})$ be the convex closure of $Z(\sigma)$. The missing faces of P are now inserted so that the image of each $\bar{\sigma}$ lies in $\widehat{Z}(\bar{\sigma})$. Since $Q \supset P_0$, proceed by induction.: if all faces of dimension $< r$ have been inserted as required, for any r-cell $\bar{\sigma}$, *bdry* $\bar{\sigma}$ is a subset of $\widehat{Z}(\bar{\sigma})$; taking the join of $q \in \widehat{Z}(\bar{\sigma})$ with $f($*bdry* $\bar{\sigma})$ gives an extension over $\bar{\sigma}$ with the required property. Repeating for each r-cell completes the inductive step. If F is the full realization obtained, $F(\bar{\sigma}) \subset Z(\bar{\sigma}) \subset T(y) \subset V$ for each $\bar{\sigma}$, so rF is a full realization relative to $\{U\}$.

Remark 13.11. The implication $1 \Rightarrow 2$ remains true in case Y

is a polytope. Indeed, given an open covering $\{U\}$, subdivide to get Y' with each of its closed vertex stars lying in some set of $\{U\}$. Embed Y' in the K' of 11.1. A *nbd* W of Y' in K' of which Y' is a retract is obtained [12; 292] by taking the barycentric subdivision K'' of K' and letting W be the union of all the vertex stars in K'' which have center a vertex of Y''. Each such star is convex, and these play the role of $\widehat{Z}$ in $1 \Rightarrow 2$ of 13.4.

14. Characterization of ANR$\mathfrak{M}$ by "factorization".

LEMMA 14.1. Let Y be either an *ANR*$\mathfrak{M}$ metric space or a polytope. For each open covering $\{U\}$ of Y there exists a refinement $\{W\}$ with the property: If X is any metric space, and $f_0, f_1 : X \to Y$ are such that $f_0(x)$, $f_1(x)$ lie in a common W for each $x \in X$ then $f_0 \simeq f_1$ and the homotopy ϕ can be so chosen that $\phi(x \times I)$ lies in a set U for each $x \in X$ [4; 363].

PROOF. With the notations in $1 \Rightarrow 2$ of 13.4, $\{T(y) \cap Y\}$ is shown to be the required open cover. If f_0, f_1 are as in the statement of the Lemma relative to $\{T(y) \cap Y\}$, then for each x, $f_0(x)$ and $f_1(x)$ can be joined by a line segment lying in $T(y)$, hence in V; letting $\phi(x, t) = tf_0(x) + (1-t) f_1(x)$ gives $f_0 \simeq f_1$ in the required fashion. The proof for Y a polytope is similar.

It will be necessary to use the trivial

14.2 Let Q be a subpolytope of K. There is a retraction $r : K \times I \to K \times 0 \cup Q \times I$. Furthermore, for each cell $\bar{\sigma}$ of K, $r(\bar{\sigma} \times 1) \subset \bar{\sigma} \times I$.

This result is well known; see, for example [14; 84]. It follows by a simple induction based on the observation that $[bdry\, \bar{\sigma}] \times I \cup \bar{\sigma} \times 0$ is a retract of $\bar{\sigma} \times I$.

The following theorem is also given by Hanner [6; 358]; his proof is different from the one that appears here.

THEOREM 14.3. Let Y be a metric space. The following properties are equivalent:

14.31 Y is an *ANR*$\mathfrak{M}$.

14.32 For each open covering $\{U\}$ of Y, there exists a polytope P and $\lambda : Y \to P$, $g : P \to Y$ such that $g\lambda \simeq$ identity map of Y, and the homotopy ϕ can be so chosen that, for each y the set $\phi(y \times I)$ lies in a set U.

PROOF. $1 \Rightarrow 2$. This is similar to $1 \Rightarrow 2$ of 6.1. See also [4; 365]. $2 \Rightarrow 1$. 13.42 will be shown to hold. Given $\{U\}$ select a star refinement $\{V\}$. Let P be the polytope satisfying 14.32 relative to $\{V\}$. Since $\{g^{-1}(V)\}$ is an open cover of P, apply 13.11 to get a refinement $\{W\}$ having the partial realization property 13.11 relative to $\{g^{-1}(V)\}$. Let $\{\tilde{W}\}$ be a common refinement of $\{\lambda^{-1}(W)\}$ and $\{V\}$.

Let f be a partial realization of a polytope K relative to $\{\tilde{W}\}$. f is defined on $Q \supset K_0$. λf is a partial realization relative to $\{W\}$ hence extends to a full realization F' in P relative to $\{g^{-1}(V)\}$; $gF^1 : K \to Y$ sends each cell of K into a set V, and $gF' \mid Q = g \lambda f$. Deform $g \lambda$ to the identity according to 14.22; in obvious fashion this yields a map $\Delta : K \times 0 \cup Q \times I \to Y$ with $\Delta \mid Q \times 1 = f$, and by the 14.22, $\Delta(\bar{\sigma} \times I) \subset$ some U for each $\bar{\sigma}$. With the retraction r of 14.2, $\Delta r \mid K \times 1$ is an extension of f over K with the image of each cell in a set U. By 13.42, Y is an $ANR\mathfrak{M}$.

15. ANR$\mathfrak{M}$ and AR$\mathfrak{M}$ in finite-dimensional metric spaces.

THEOREM 15.1. Among the finite-dimensional metric spaces (whether separable or not), the locally contractible ones are the $ANR\mathfrak{M}$. Precisely, if Y is metric and dim Y finite, the following three properties are equivalent:

15.11 Y is LC^n for some integer $n \geqq \dim Y$.

15.12 Y is locally contractible.

15.13 Y is an $ANR\mathfrak{M}$.

PROOF. 1 ⇒ 2 by 7:2. 2 ⇒ 3: Since Y is LC, it is also LC^∞. If dim $Y \leqq r$, applying 6.12 with the identity map of Y, 14.3 shows Y an $ANR\mathfrak{M}$. 3 ⇒ 1 by 12.3.

For infinite dimensional spaces, this theorem is not true, even with the added hypothesis of separability.

THEOREM 15.2. Among the finite-dimensional arbitrary metric spaces, the contractible and locally contractible ones are the $AR\mathfrak{M}$. Precisely, if Y is metric and dim Y is finite, the following properties are equivalent:

15.21 Y is C^n and LC^n for some $n \geqq \dim Y$.

15.22 Y is contractible and locally contractible.

15.23 Y is an $AR\mathfrak{M}$.

PROOF. 1 ⇒ 2 from 7.2 and 9.2. 2 ⇒ 3: By 15.1, Y is an $ANR\mathfrak{M}$; use 12.4 to find Y is an $AR\mathfrak{M}$. 3 ⇒ 1 by 12.3.

The following theorem characterizes the finite-dimensional $AR\mathfrak{M}$ solely by a special type of contractibility.

THEOREM 15.3. Let Y be metric and dim Y finite. The following two properties are equivalent:

15.31 Y is an $AR\mathfrak{M}$.

15.32 Given any $y_0 \in Y$, Y is contractible to this point in such a way that during the entire process of deformation, y_0 remains fixed.

PROOF. $1 \Rightarrow 2$ is trivial.

$2 \Rightarrow 1$. One need only show that Y is locally contractible. This follows trivially by selecting a contraction 15.32 to any point y_0 and using the continuity of this contraction at y_0.

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