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Absolute neighborhood retracts and local connectedness in arbitrary metric spaces

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1. Introduction.

For separable metric spaces, the following implications and equivalences are well-known:

I. $\text{ANR} \Rightarrow$ local contractibility $\Rightarrow LC^\infty$. These concepts are equivalent only in the finite-dimensional spaces.

II. $\text{AR} \Rightarrow$ contractibility and local contractibility $\Rightarrow C^\infty$ and $LC^\infty$. These notions coincide only in the finite-dimensional spaces.

III. $n$-dimensional and $LC^n$ ($n$ finite) $\equiv \text{ANR}$.

IV. $\text{AR} \equiv$ contractible $\text{ANR} \equiv C^\infty \text{ANR}$.

A unified account is given in [9; Chap. VII] 1). The proofs of the equivalences in I—III are based on embedding into Euclidean spaces; those in IV on embedding into the Hilbert cube.

Upon attempting to determine the interrelations of the above concepts in non-separable metric spaces, one meets (1): the question of what definition of dimension to adopt, and (2): that the embeddings considered above are never possible, since the image spaces mentioned are separable. The theory, therefore, does not give any information about I—IV in non-separable metric spaces.

The main object of this paper is to show that the complete statements I—IV are valid in all metric spaces, if the covering definition of dimension be used. In the course of this, various alternative characterizations of $LC^n$ and $\text{ANR}$ are derived, so well as other subsidiary results. Only the point-set aspects are given here; homology in such spaces will be considered elsewhere.

2. Preliminaries.

$E^n$ denotes Euclidean $n$-space; $H^n \subset E^n$ is

$\{(x_1, \ldots, x_n) \in E^n \mid \sum_{i=1}^{n} x_i^2 \leq 1\}$ and the $(n-1)$-sphere $S^{n-1} =$ boundary $H^n$. $I$ represents $\{x \in E^1 \mid 0 \leq x \leq 1\}$.

1) Numbers in square brackets are references to the bibliography.
"An open covering of a space X" will always mean "a covering of X by open sets". \( \{U\} \) is a \textit{nbhd}-finite open covering of X if each \( x \in X \) has a \textit{nbhd} intersecting at most finitely many sets \( U \); X is paracompact if each open covering has a \textit{nbhd}-finite refinement.

2.1 Every metric space is paracompact. [15; 972]

2.2 In paracompact spaces, each open covering \( \{U\} \) has a star refinement \( \{V\} \), i.e. for each \( V \), \( \bigcup \{V_\alpha \mid V_\alpha \cap V \neq 0\} \subseteq \) some set \( U \). [16; 45]

"A continuous map \( f \) of a space X into a space Y" is written \( f : X \to Y \). The homotopy of \( f_0, f_1 : X \to Y \) is symbolized \( \sim \); if the homotopy \( \Phi \) satisfies \( \Phi(X \times I) \subseteq B \) where \( B \subseteq Y \), \( f_0 \) and \( f_1 \) are called \"homotopic in B\". \( f \) is nullhomotopic (notation: \( \sim 0 \)) if it is homotopic to a constant map.

2.3 \( f : S^n \to Y \) is nullhomotopic in \( B \) if and only if \( f \) has an extension \( F : H^{n+1} \to Y \) with \( F(H^{n+1}) \subseteq B \). [1; 501]

A polytope is a point set composed of an arbitrary collection of closed Euclidean cells (higher-dimensional analogs of a tetrahedron) satisfying: 1. Every face of a cell of the collection is itself a cell of the collection and 2. The intersection of any two cells is a face of both of them. The dimensions of the cells need not have a finite upper bound, and the set of cells incident with any given one may have any finite or transfinite cardinal. The CW topology [17; 316] is always used: \( U \subseteq P \) is open if and only if for each closed cell \( \bar{\sigma} \), \( U \cap \bar{\sigma} \) is open in the Euclidean topology of \( \bar{\sigma} \).

2.4 The open sets of \( P \) are invariant under subdivision. Stars of vertices are open sets. \( f : P \to Y, Y \) any space, if and only if \( f \mid \bar{\sigma} \) is continuous for each cell \( \bar{\sigma} \). If \( \{U\} \) is an open covering of \( P \), there is a subdivision \( P' \) of \( P \) having each closed vertex-star contained in some set of \( \{U\} \). [17].

For a metric space \( X \), define \( \dim X \leq n \) if each open covering \( \{U\} \) has a refinement \( \{V\} \) in which no more than \( n+1 \) sets \( V \) have a non-vacuous intersection. (i.e. \( \{V\} \) is of order \( \leq n+1 \).

2.5 \( \dim X \times I \leq \dim X + 1 \). For any set \( E \subseteq X \), \( \dim E \leq \dim X \). [18]

In a space \( X \) with metric \( d \), \( S(x_0, \varepsilon) \) denotes \( \{x \in X \mid d(x, x_0) < \varepsilon\} \). Let \( A \subseteq X \) be closed, cover \( X - A \) by \( \{S(x, \frac{1}{2}d(x, A)) \mid x \in X - A\} \) and take a \textit{nbhd}-finite refinement \( \{U\} \). Then [4; 354]

2.6 \( \{U\} \) has the properties: 1. Each \textit{nbhd} of \( a \in [A-\text{interior } A] \) contains infinitely many sets \( U \); 2. For each \textit{nbhd} \( W \supset a \) there is a \textit{nbhd} \( W' \), \( a \in W' \subseteq W \), such that whenever \( U \cap W' \neq 0 \) then \( U \subseteq W \); 3. For each \( U, d(U, A) > 0 \).
Open coverings of $X-A$ with the properties 1–3 above are called canonical. One can always assume order${}\{U\} \leq \dim(X-A) + 1$.

If $N(U)$ is the polytope nerve of a canonical cover of $X-A$, the set $\tilde{X} = A \cup N(U)$ is given a Hausdorff topology by taking:

(a). $N(U)$ with the $CW$ topology and (b). A subbasis for the nbd of $a \in A \subset \tilde{X}$ to consist of all sets with form

$$\{W \cap A\} \cup \{\text{all vertex stars in } N(U) \text{ corresponding to sets } U \subset W\}$$

where $W$ is a nbd of $a$ in $X$.

2.61 Let $K : X-A \to N(U)$ be the Alexandroff map.

The (canonical) map

$$\mu(x) = \begin{cases} x & \text{if } x \in A \\ K(x) & \text{if } x \in X-A \end{cases}$$

of $X$ into $\tilde{X}$ is continuous, and $\mu | A$ a homeomorphism. [4; 356]

2.62 If $N(U)_0$ is the zero-skeleton of $N(U)$, then $A \subset \tilde{X}$ is a retract of $A \cup N(U)_0$.

A retraction $r$ is obtained [4; 357] by first choosing a point $x_U \in U$; for each $x_U$ select $a_U \in A$ with $d(a_U, x_U) < 2d(x_U, A)$ and set

$$\begin{cases} r(p_U) = a_U, & p_U \text{ the vertex of } N(U) \text{ corresponding to} \\ r(a) = a, & a \in A. \end{cases}$$

2.63 An $f : A \to E$ has an extension $F : A \cup N(U) \to E$ if either

(a). $E$ is a convex subset of a locally convex linear space [4; 357] or

(b). $E$ is a convex subset of a real vector space $L$ having the finite topology: $G \subset L$ is open if and only if for each finite-dimensional linear subspace $K$, $G \cap K$ is open in the Euclidean topology of $K$. [8; 9]

$B(Z)$ denotes the Banach space of all bounded continuous real-valued functions on $Z$.

2.7 Each metric $X$ can be embedded in $B(X)$ [11; 543] as a closed subset of its convex hull $H(X)$ [18; 186].

**Lemma 2.8.** Let $X$, $Y$ be metric, $A \subset X$ closed and $f : A \to Y$. Then there exists a metric $Y_1 \supset Y$ and an extension $F : X \to Y_1$ such that $Y$ is closed in $Y_1$ and $F | X-A$ is a homeomorphism of $X-A$ with $Y_1-Y$.

**Proof.** Embed $Y$ in $H(Y)$; 2.61 and 2.63 yield and extension $F^+ : X \to H(Y)$. Regard $X \subset B(X)$ and form the cartesian product $H(Y) \times E^1 \times B(X)$; with Kuratowski [8; 189], $F(x) = [F^+(x), d(x, A), x \cdot d(x, A)]$ is the desired map and $Y_1 = F(X)$.

3. The property $LC^a$.

**Definition 3.1.** A space is $n$-locally connected (symbol: $n-LC$)
at $y \in Y$ if for each nbd $U \supset y$ there is a nbd $V$, $y \in V \subset U$, such that every $f : S^n \to V$ has an extension $F : H^{n+1} \to U$, or, equivalently (2.3) every $f : S^n \to V$ is nullhomotopic in $U$.

$Y$ is $LC^n$ if it is $i$–$LC$ at each point for all $0 \leq i \leq n$

$Y$ is $LC^\infty$ if it is $LC^n$ for every $n$.

The following is the generalization of Kuratowski's theorem [10; 273] to the non-separable metric case; the proof is similar.

**Theorem 3.2.** Let $Y$ be metric and $n$ finite. The following four properties are equivalent:

1. $Y$ is $LC^n$, $n$ finite.
2. If $X$ is metric, $A \subset X$ closed, and $\dim (X-A) \leq n+1$, then every $f : A \to Y$ can be extended over a nbd $W \supset A$ in $X$.
3. For each $y \in Y$ and nbd $U \supset y$ there is a nbd $V$, $y \in V \subset U$ such that: if $X$ is metric, $A \subset X$ closed, and $\dim (X-A) \leq n+1$, then every $f : A \to V$ has an extension $F : X \to U$.
4. For each $y \in Y$ and nbd $U \supset y$ there is a nbd $V$, $y \in V \subset U$ such that: if $X$ is metric, $\dim X \leq n$, every $f : X \to V$ is nullhomotopic in $U$.

**Proof.**

1 $\Rightarrow$ 2: Form $\tilde{X} = A \cup N(U)$ of 2.61 with $\dim N(U) \leq n+1$. Regarding $f$ defined on $A \subset \tilde{X}$ one need only construct an open $W$, $A \subset W \subset \tilde{X}$ and an extension $F : W \to Y$; $F_{\mu} | \mu^{-1}(W)$ will then be the required extension.

Let $N(U)_k$ be the $k$-skeleton of $N(U)$; for $k = 0, 1, \ldots, n+1$, open $W_k$, $A \subset W_k \subset \tilde{X}$, and extensions $f_k : [A \cup N(U)_k] \cap W_k \to Y$ will be defined inductively; the result follows by taking $W = W_{n+1}$ and $f = f_{n+1}$.

Take $W_0 = \tilde{X}$ by 2.62, $f_0 = fr$ is the extension.

Suppose $k-1 \leq n$ and $W_{k-1}$, $f_{k-1}$ constructed. For each $a \in A$ let $W_a$ be a nbd of $f_{k-1}(a) = f(a)$ such that any $f : S^{k-1} \to W_a$ extends to a continuous map of $H^k$ into $Y$. From 2.6 and continuity of $f_{k-1}$ there is a nbd $\tilde{V}_a \supset a$ such that each closed $k$-cell $\tilde{\sigma}$ in the vertex-stars that form $\tilde{V}_a$ satisfies (1) $\tilde{\sigma} \subset W_{k-1}$ and

(2) $f_{k-1}(\bdry \tilde{\sigma}) \subset W_a$.

Set $W_k = \bigcup a \tilde{V}_a$.

To define $f_k$, let $\tilde{\sigma}$ be any closed $k$-cell in a vertex-star forming $W_k$; by (2) $f_{k-1} | \bdry \tilde{\sigma}$ has an extension over $\tilde{\sigma}$. Let $\eta(\tilde{\sigma})$ be the infimum of the diameters of the images of $\tilde{\sigma}$ taken over all possible extensions, and define $f_k | \tilde{\sigma}$ to be an extension having image diameter $< 2 \eta(\tilde{\sigma})$. Proceeding in this way yields and extension $f_k$ of $f_{k-1}$ over $(A \cup N(U)_k) \cap W_k$. Continuity need be proved
only at \( A \), and follows at once by observing that \( Y \) is \((k-1)-LC\) and \( f_{k-1} \) is continuous.

2 \( \Rightarrow \) 3. Assume 3 not true at \( y_0 \). Then there is an \( S(y_0, \alpha) = U \), a sequence \( X_i \) of metric spaces, a sequence \( A_i \subset X_i \) of closed subsets with \( \dim (X_i - A_i) \leq n+1 \), and a sequence of \( f_i : A_i \to S(y_0, \alpha/i) \) none of which is extendable over \( X_i \) with values in \( U \).

The points and metric of \( X_i \) will carry the subscript \( i \). One can assume all the metrics bounded: \( d_i(x_i, x'_i) < 1 \). Construct a metric space \( \hat{X} \) consisting of a point \( \hat{x} \) and the pairwise disjoint union of the \( X_i \) by setting

\[
\hat{d}(x_i, x'_i) = 2^{-i} d_i(x_i, x'_i) \\
\hat{d}(x_i, x_j) = 2^{-\min(i,j)} \quad i \neq j \\
\hat{d} (\hat{x}, x_i) = 2^{-i}
\]

\( \hat{A} = \hat{x} \cup \bigcup A_i \) is closed \( \hat{X} \) and \( \dim (\hat{X} - \hat{A}) \leq n+1 \) [13]. Defining \( \hat{f} : \hat{A} \to Y \) by \( \hat{f}(a_i) = f(a_i), \hat{f}(\hat{x}) = y_0, \hat{f} \) is continuous, so by 2 there is an open \( \hat{W} \supset \hat{A} \) and an extension \( \hat{F} : \hat{W} \to Y \). Because \( \hat{x} \in \hat{W} \) and \( \text{diam} (\hat{x} \cup X_i) < 2^{-i} \), \( \hat{W} \) contains all \( X_i \) for \( i \) large. Using the continuity of \( \hat{F} \) at \( \hat{x} \) one easily finds \( \text{diam} (y_0 \cup \hat{F}(X_i) < \frac{\alpha}{2} \) for large \( i \), so that then \( \hat{F}(X_i) \subset U \). Since \( \hat{F} | X_i \) is an extension of \( f_i \) this contradiction proves the assertion.

3 \( \Rightarrow \) 4. This is the homotopy theorem corresponding to 3: set \( \hat{X} = X \times I \) and \( A = X_0 \cup X \times 1 \) then 3 applies since \( \dim (X \times I) \leq n+1 \).

4 \( \Rightarrow \) 1. Let \( X = S^r, 0 \leq r \leq n \).

Remark 3.3. In case \( n = \infty \), then 3.2 holds is one restricts \( X-A \) to having finite dimension.

Theorem 3.4. Let \( Y \) be metric. The following two properties are equivalent.,

3.41 \( Y \) is \( LC^n \) \( n \) finite.

3.42 If \( Z \) is metric, \( Y \subset Z \) is closed, and \( \dim (Z-Y) \leq n+1 \), then \( Y \) is a nbd retract in \( Z \).

Proof. 1 \( \Rightarrow \) 2 is obvious from 3.22.

2 \( \Rightarrow \) 1. One shows 3.22 valid. Given \( f : A \to Y \) as in 3.22, use 2.8 to obtain a metric \( Y_1 \supset Y \), \( Y \) closed in \( Y_1 \), and an extension \( F : X \to Y_1 \).

Since \( X-A \) is homeomorphic to \( Y_1 - Y \), \( \dim (Y_1 - Y) \leq n+1 \) so that there is an open \( U \supset Y \) in \( Y_1 \) and a retraction \( r : U \to Y \). \( rF | F^{-1}(U) \) is the desired extension of \( f \).
4. Characterization of LC\(^n\) by partial realization.

**Definition 4.1.** Let \(\{U\}\) be a covering of a space \(Y\). Let \(P\) be a polytope and \(Q\) a subpolytope of \(P\) containing the zero-skeleton of \(P\). An \(f : Q \to Y\) is called a partial realization of \(P\) relative to \(\{U\}\) if for each closed cell \(\sigma\) of \(P\), \(f(\sigma \cap \bar{Q}) \subseteq \text{some set } U\).

**Theorem 4.2.** Let \(Y\) be metric. The following two properties are equivalent:

4.21 \(Y\) is \(LC^n\), \(n\) finite.

4.22 Each open covering \(\{U\}\) of \(Y\) has a nbd-finite refinement \(\{V\}\) with the property: Every partial realization of any polytope \(P\) \(\dim P \leq n + 1\), relative to \(\{V\}\) extends to a full realization of \(P\) relative to \(\{U\}\).

**Proof.** 1 \(\Rightarrow\) 2. For each \(y \in Y\) select a \(U \ni y\); choose an open \(V, y \in V \subseteq U\) satisfying the definition of \(n\)-LC and let \(\{V(n)\}\) be a star-refinement. Repeat, using \(\{V(n)\}\) and \((n-1)\)-LC to obtain \(\{V(n-1)\}\). A nbd-finite refinement of \(\{V(0)\}\) satisfies the requirements.

2 \(\Rightarrow\) 1. Let \(y \in Y\). For any \(\epsilon > 0\) take the open covering: \(S(y, \epsilon), Y - S(y, \epsilon/2)\). Let \(\{V\}\) be the refinement satisfying (2), and choose \(V \ni y\). Then \(V \subseteq S(y, \epsilon)\) and, \(S^k\) being a subpolytope of \(H^{k+1}\), any \(f : S^k \to V\) extends to \(F : H^{k+1} \to S(y, \epsilon), k \leq n\). This suffices to prove the assertion.

**Remark 4.3.** 4.2 is valid even if the polytopes of 4.22 are restricted to be finite.

5. Characterization of LC\(^n\) by homotopy.

**Theorem 5.1.** Let \(Y\) be metric. The following two properties are equivalent:

5.11 \(Y\) is \(LC^n\), \(n\) finite.

5.12 Each open covering \(\{U\}\) has a nbd-finite refinement \(\{W\}\) with the property: For any metric \(X\), \(\dim X \leq n\), and any \(f_0, f_1 : X \to Y\) satisfying:

(a). \(f_0(x)\) and \(f_1(x)\) belong to a common \(W\) for each \(x \in X\).

Then \(f_0 \simeq f_1\) and the homotopy \(\Phi\) can be chosen so that \(\Phi(x, I)\) lies in a set \(U\) for each \(x\).

**Proof.** 1 \(\Rightarrow\) 2. Let \(\{U(0)\}\) be given. Construct successive star refinements \(\{U(i)\}, i = 0, 1, \ldots, n\) with \(\{U(i+1)\}\) having property 4.22 relative to \(\{U(i)\}\). Let \(\{W\}\) be a star refinement of \(\{U(n)\}\); by 2.1 \(\{W\}\) may be assumed nbd-finite. This is the desired open covering.

Let \(f_0, f_1\) satisfy (a) and let \(\{G\}\) be a common refinement of \(\{f_0^{-1}(W)\}\) and \(\{f_1^{-1}(W)\}\). Take a common refinement of a canonical
cover of $X \times I - [X \times 0 \cup X \times 1]$ and $\{G \times I\}$, and let $\{V\}$ be a star refinement of this resulting cover. $\{V\}$ can be assumed nbd-finite and of order $\leq n+1$; it is clearly also a canonical cover.

Form $X \times I = X \times 0 \cup X \times 1 \cup N(V)$ and regard $f_0, f_1$ defined on the subsets $X \times 0, X \times 1$, respectively, of $X \times I$. Construct an extension $F_0 : X \times 0 \cup X \times 1 \cup N(V)_0 \to Y$ as follows: given the vertex $p_V$ select $x_V \times i_V \in V$; if $i_V \leq \frac{1}{2}$ set $F_0(p_V) = f_0(x_V)$, otherwise $F_0(p_V) = f_1(x_V)$. $F_0$ is continuous.

Let $(p_0, \ldots, p_r)$ be any $r$-cell of $N(V)$; all these vertices will be shown to map into a single $\{U(n)\}$. Since $V_0 \cap \ldots \cap V_r \neq 0$, $\bigcup_0^r V_i \subset G \times I$ for some $G$ so that $\bigcup_0 x_i \subset G$. Since $G \subset f_0^{-1}(W)$ for some $W$, $\bigcup_0 f_0(x_i) \subset W$. By condition (a) for each $i = 0, \ldots, n$ there is a $W_i$ with $f_0(x_i) \cup f_1(x_i) \subset W_i$ so that $W \cap W_i \neq 0$, which means $\bigcup_0 W_i \subset U(n)$ for some $U(n)$ and therefore

$$\bigcup_0 f_0(x_i) \cup f_1(x_i) \subset U(n).$$

Proceed by induction. Assume an extension $F_{k-1}$ on $X \times 0 \cup X \times 1 \cup N(V)_{k-1}$ which is a partial realization of $N(V)$ relative to $\{U(n-k+1)\}$; $F_{k-1}$ extends over each closed $k$-cell $\bar{\sigma}$ with image in a set $U(n-k)$. Let $\eta(\bar{\sigma})$ be the infimum of the diameters of all the possible extensions; select an extension with $\text{diam } F_k(\bar{\sigma}) < 2\eta(\bar{\sigma})$ and $F_k(\bar{\sigma}) \subset$ some $U(n-k)$. This process yields a partial realization of $N(V)$ relative to $\{U(n-k)\}$ and the map is continuous on $X \times 0 \cup X \times 1 \cup N(V)_k$ as in 3.22.

The required homotopy is $\Phi = F_{n+1} \mu$; the required condition is satisfied because, by the above, all vertices corresponding to sets lying in a strip $G \times I$ have images in one set $\{U(n)\}$ so that in extending the partial realization the images of all cells lie in one set $\{U(0)\}$. $2 \Rightarrow 1$. As in 4.2

6. Characterization of $\text{LC}^n$ by “factorization”.

**Theorem 6.1.** Let $Y$ be metric. The following two properties are equivalent:

6.11 $Y$ is $\text{LC}^n$, $n$ finite.

6.12 For each open covering $\{U\}$ there exists a polytope $P$, $\dim P \leq n$ and a $g : P \to Y$ with the property: For any metric $X$, $\dim X \leq n$ and any $f : X \to Y$ there is a $\varphi : X \to P$ with $f \simeq g \varphi$, and the homotopy $\Phi$ can be chosen so that $\Phi(x, I)$ lies in a single $U$ for each $x$.

**Proof.** $1 \Rightarrow 2$. Let $\{W\}$ be a refinement satisfying 5.12 relative
to \( \{U\} \) let \( \{\bar{V}\} \) be a refinement of \( \{W\} \) with property 4.22, and finally let \( \{V\} \) be a star refinement of \( \{\bar{V}\} \); \( \{V\} \) can be assumed \( nbd\)-finite. Let \( P = \text{N}(V) \) and define \( \varphi_0 : P_0 \to Y \) by sending each \( p_U \) to a \( y_V \in V \); this is obviously a partial realization relative to \( \{\bar{V}\} \); \( \varphi_0 \) extends to \( \varphi : P \to Y \) with the image of each \( n \)-cell in some \( W \).

Let \( X \) be metric, \( \text{dim } X \leq n \) and \( f : X \to Y \). Form \( \{f^{-1}(V)\} \) and let \( \{G\} \) be a star \( nbd\)-finite refinement of order \( \leq n+1 \). Let \( K : X \to \text{N}(G) \) be the Alexandroff map and define \( \gamma : \text{N}(G) \to P \) as follows: for each \( p_G \) select a \( V \) with \( G \subseteq f^{-1}(V) \) and set \( \gamma(p_G) = p_V \). This is simplicial, and extending linearly gives a continuous map (2.4). \( f \) will be shown homotopic to \( \varphi \gamma K \) in the required way.

Let \( x \in G_1 \cap \ldots \cap G_s \) and only these sets; then \( K(x) \in (p_{G_1}, \ldots, p_{G_s}) \) so \( \gamma K(x) \in (p_{V_1}, \ldots, p_{V_s}) \); since \( g(p_{V_i}) \in V_i \), \( \gamma \) is well defined. Let \( \gamma K(x) \) lying in some \( W \) satisfying \( \bigcap_i V_i \subseteq W \). Again, since \( f(G_i) \subseteq V_i \), \( i = 1, \ldots, s \) one has \( f(x) \in (V_1 \cap \ldots \cap V_s) \subseteq W \). Hence \( \gamma K(x) \) and \( f(x) \) are in a common \( W \) hence are homotopic in the required fashion.

2 \( \Rightarrow \) 1. Assume \( Y \) not \( k\)-LC at \( y_0, k \leq n \). There is an \( S(y_0, r) \) and a sequence of \( f_i : S_i^k \to S(y_0, r/4) \) with no \( f_i \) nullhomotopic in \( S(y_0, r) \). As in 3.2, form \( S = \hat{x} \cup \bigcap_{i=1}^{\infty} S_i^k \) and define \( \hat{f} : S \to Y \) by \( \hat{f}(\hat{x}) = y_0 \). Cover \( Y \) by \( S(y_0, r/2) \) and \( Y - S(y_0, r/3) \), and let \( P \) be the polytope corresponding to this covering. Since \( \text{dim } S \leq n \) there is a \( \varphi : S \to P \) with \( f \simeq g \varphi \), and because \( f(S) \subseteq S(y_0, r/4) \) the homotopy \( \varphi \) satisfies \( \varphi(S \times I) \subseteq S(y_0, r/2) \) hence \( g \varphi(S) \subseteq S(y_0, r/2) \). In particular \( \varphi(\hat{x}) \subseteq g^{-1}(S(y_0, r/2)) \). Cover \( P \) by \( g^{-1}(S(y_0, r)) \) and \( P - g^{-1}(S(y_0, r/2)) \); subdivide \( P \) so each closed vertex star lies in one of these open sets (2.4). Let \( S_t \) be the star containing \( \varphi(\hat{x}) \). Then \( g^{-1}(S(y_0, r)) \) is open, contains \( S_t \) and \( \varphi(S_t^k) \subseteq S_t \) for large \( i \). Since \( \varphi | S_t^k \) is nullhomotopic in \( S_t \) (by radial contraction) and \( f_i \simeq g \varphi | S_t^k \) in \( S(y_0, r/2) \) one finds \( f_i \simeq 0 \) in \( S(y_0, r) \), a contradiction.

7. The property LC.

**Definition 7.1.** A space \( Y \) is locally contractible at \( y \in Y \) if for each \( nbd \ U \ni y \) there is a \( nbd \ V, y \in V \subseteq U \), contractible to a
point over $U$, i.e. the identity map of $V$ is nullhomotopic in $U$. $Y$ is $LC$ if it is locally contractible at every point.

Every $LC$ space is clearly $LC^\infty$; the converse is not true, even in separable metric spaces [10; 273]. However,

**Theorem 7.2.** Let $Y$ be a finite-dimensional metric space. The following three properties are equivalent:

7.21 $Y$ is $LC^n$, $n$ some integer $\geq \dim Y$.
7.22 $Y$ is $LC$.
7.23 $Y$ is $LC^\infty$.

**Proof.** Only $1 \Rightarrow 2$ requires proof. Given $U \supset B$ let $V$ be a nbd satisfying 3.24. Since $\dim V \leq n$ (2.5) the identity map of $V$ is nullhomotopic in $U$.

8. The properties $C^n$ and $C$.

**Definition 8.1.** $Y$ is connected in dimension $n$ (written: $n-C$) if each $f : S^n \to Y$ is nullhomotopic. $Y$ is $C^n$ if it is $i-C$ for all $0 \leq i \leq n$ $Y$ is $C^\infty$ if it is $C^n$ for all $n$.

$Y$ is $C^0$ is equivalent with $Y$ arewise connected; $Y$ is $n-C$ is equivalent with the $n-th$ homotopy group $\pi_n(Y) = 0$.

**Definition 8.2.** $Y$ is contractible (symbolism: $Y$ is $C$) if the identity map of $Y$ is nullhomotopic.

Clearly $Y$ is $C$ implies $Y$ is $C^\infty$; the converse is not true [10; 273]. The following theorem is well-known (see for example [5; 241]).

**Theorem 8.3.** Let $Y$ be any (not necessarily metric) space, and $n$ finite.

The following five properties are equivalent:

8.31 $Y$ is $C^n$.
8.32 If $P$ is any polytope, and $Q \subset P$ a subpolytope, any $f : Q \to Y$ extends over $Q \cup P_{n+1}$, where $P_s$ denotes the $s$-skeleton of $P$.
8.33 If $P$ is any polytope, $Q \subset P$ a subpolytope, and $f_0, f_1 : P \to Y$ satisfy $f_0 | Q \simeq f_1 | Q$, then $f_0 | Q \cup P_n \simeq f_1 | Q \cup P_n$.
8.34 If $P$ is a polytope with $\dim P \leq n$, any $f : P \to Y$ is nullhomotopic.
8.35 If $P$ is any polytope, any $f : P \to Y$ is homotopic to an $f_1 : P \to Y$ sending $P_n$ to a single point.

This will be used in the next section.

**Remark 8.4.** If $n = \infty$, no restriction need be placed on $\dim P$ in 8.3.
9. The properties $C^n$ and $LC^n$ together.

**Theorem 9.1.** Let the metric space $Y$ be $LC^n$. The following three properties are equivalent:

9.11 $Y$ is $C^n$.
9.12 If $X$ is metric, $A \subset X$ closed, and $\dim (X-A) \leq n+1$, then any $f : A \to Y$ extends over the entire $X$.
9.13 If $X$ is metric, $\dim X \leq n$, then any $f : X \to Y$ is null-homotopic.

**Proof.** Only $1 \Rightarrow 2$ requires proof. Using $A \cup N(U)$ and regarding $f$ as defined on $A \subset A \cup N(U)$, the problem is to extend $f$ over $A \cup N(U)$. Since $Y$ is $LC^n$, the proof of 3.22 gives an extension $f'$ over a nbd $W' \supset A$ in $A \cup N(U)$. Let $Q$ be the union of all closed cells of $N(U)$ on which $f'$ is defined; $Q$ is a subpolytope of $N(U)$ and no point of $A$ is a limit point of $N(U) - Q$. The extension of $f' \mid Q$ over $N(U)$ guaranteed by 8.32, together with $f' \mid A \cup Q$ is the desired map.

**Theorem 9.2.** Let $Y$ be a finite-dimensional metric $LC^r$ space, where $r \geq \dim Y$. The following three properties are equivalent:

9.21 $Y$ is $C^n$ for some $n \geq \dim Y$.
9.22 $Y$ is contractible.
9.23 $Y$ is $C^\infty$.

**Proof.** As in 7.2.

**Remark 9.3.** Theorem 9.2 is not true if $\dim Y = \infty$, even if $Y$ is separable metric. [2]

10. Absolute neighborhood retracts and absolute retracts.

**Definition 10.1.** An arbitrary space $Y$ is an absolute nbd retract for a class $\mathcal{A}$ of spaces (written: $Y$ is an ANR $\mathcal{A}$) if for any closed subset $A$ of any $X \in \mathcal{A}$, and any $f : A \to Y$, there is an extension $F : U \to Y$ of $f$ over a nbd $U \supset A$.

$Y$ is an absolute retract for the class $\mathcal{A}$ (symbol: $AR \mathcal{A}$) if for any closed subset $A$ of any $X \in \mathcal{A}$ and any $f : A \to Y$, there is an extension $F : X \to Y$.

In the following, $\mathfrak{M}$ denotes the class of metric spaces, $\mathfrak{B}$ the class of polytopes.

An immediate consequence of 2.63, as was pointed out in [3; 9], [4; 357] is

**Theorem 10.2.** Any convex set $C$ of either (a): A locally convex linear space, or (b): A real vector space with finite topology, is an $AR \mathfrak{M}$. 


11. The properties ANR and AR in polytopes.

In [3; 10] it was shown that

11.1 Any polytope $P$ can be embedded as a \textit{nbd} retract in a polytope $K$ spanning a convex subset of a real vector space with finite topology.

This leads at once to

Theorem 11.2. Any polytope is an \textit{ANR} $\mathfrak{M}$ and also an \textit{ANR} $\mathfrak{B}$.

It is an \textit{AR} $\mathfrak{M}$ if and only if it is an \textit{AR} $\mathfrak{B}$.

Proof. The \textit{ANR} $\mathfrak{M}$ follows from 11.1 and 10.2; this was proved in [3; 10a]. The \textit{ANR} $\mathfrak{B}$ has also been proved in [3; 10a]. \textit{AR} $\mathfrak{B}$ implies \textit{AR} $\mathfrak{M}$ since $P$ would be a retract of $K$ in 11.1, and one then applies 10.2. \textit{AR} $\mathfrak{M}$ implies \textit{AR} $\mathfrak{B}$ is proved inductively exactly as [3; 10a].

Theorem 11.3. Let $P$ be a polytope. Taking $\mathfrak{A} = \mathfrak{M}$ or $\mathfrak{B}$, the following three properties are equivalent:

11.31 $P$ is an \textit{AR} $\mathfrak{A}$.
11.32 $P$ is contractible.
11.33 $P$ is $C^\infty$.

Proof. Only 3 $\Rightarrow$ 1 requires proof; $P$ will be shown \textit{AR} $\mathfrak{M}$. From 8.4, 8.32, $P$ is a retract of the $K$ in 11.1, so the result follows from 10.2.

12. The properties ANR and AR in metric spaces.

Theorem 12.1. For metric $Y$ the following are equivalent:

12.11 $Y$ is an \textit{ANR} $\mathfrak{M}$ (an \textit{AR} $\mathfrak{M}$).
12.12 If $Z$ is metric and $Y \subset Z$ closed, then $Y$ is a \textit{nbd} retract (retract) in $Z$.
12.13 $Y$ can be embedded in the Banach space $B(Y)$ as a \textit{nbd} retract (retract) of its convex hull $H(Y)$.

Proof. 1 $\Rightarrow$ 2 $\Rightarrow$ 3 is trivial; 3 $\Rightarrow$ 1 is analogous to the proof given in 11.3.

Theorem 12.2. If $Y$ is a metric space, and $Y$ is an \textit{ANR} $\mathfrak{M}$ (\textit{AR} $\mathfrak{M}$), then $Y$ is also an \textit{ANR} $\mathfrak{B}$ (\textit{AR} $\mathfrak{B}$).

This is proved in [3; 10b].

Theorem 12.3. Let $Y$ be metric. If $Y$ is an \textit{ANR} $\mathfrak{M}$ (\textit{AR} $\mathfrak{M}$), then $Y$ is \textit{LC} ($\textit{LC}$ and $C$), hence also $\textit{LC}^\infty$ ($\textit{LC}^\infty$ and $C^\infty$).

Proof. Note first that a convex subset of a Banach space is contractible and locally contractible. The theorem follows from 12.13 by the trivial remark that contractibility is preserved under retraction and local contractibility is preserved under \textit{nbd} retraction.
12.4. Theorem For metric $Y$, the following properties are equivalent:
12.41 $Y$ is an $AR\mathcal{M}$.
12.42 $Y$ is a contractible $ANR\mathcal{M}$.
12.43 $Y$ is a $C^\infty ANR\mathcal{M}$.

Proof. $1 \Rightarrow 2 \Rightarrow 3$ is trivial. $3 \Rightarrow 4$ follows as in 9.12, using 8.4, 2.63.

13. Characterization of $ANR\mathcal{M}$ by partial realization.

The following Lemma is due to Kuratowski [8; 122].

Lemma 13.1. Let $D$ be an arbitrary non-empty subset of a metric $Z$. Let \{U\} be a covering of $D$ by sets open in $D$. Then there exists a collection \{Ext \ U\} of sets open in $Z$ with
13.11 $U = D \cap \text{Ext } \mathcal{U}$ for each $U$.
13.12 The nerve of \{U\} is homeomorphic to the nerve of \{Ext \ U\}

In fact, one defines
\[
\text{Ext } \mathcal{U} = \{z \in Z | d(z, \mathcal{U}) < d(z, D-\mathcal{U})\}.
\]

Remark 13.2. If $U \subseteq U'$ then Ext $\mathcal{U} \subseteq \text{Ext U'}$.

Remark 13.3. Given $z \in \text{Ext U}$; if $\zeta \in D$ is to be chosen to satisfy $d(z, \zeta) < 2d(z, D)$ one can always find such an $\zeta$ in $U$.

Theorem 13.4. For metric $Y$, the following properties are equivalent:
13.41 $Y$ is an $ANR\mathcal{M}$.
13.42 For each open covering \{U\} of $Y$ there is a refinement \{V\} with the property: Every partial realization of any polytope relative to \{V\} extends to a full realization in \{U\}.

Proof. $2 \Rightarrow 1$. For each $n = 1, 2, \ldots$ define inductively an open cover $V(n)$ as follows:
(a) $\{\tilde{V}(1)\}$ is an open cover of mesh $< 1$, i.e. sup diam $\{\tilde{V}(1)\} < 1$
(b) $\{\tilde{V}'(1)\}$ satisfies 13.42 relative to $\{\tilde{V}(1)\}$
(c) $\{\tilde{V}''(1)\}$ is a star refinement of $\{\tilde{V}'(1)\}$
(d) $\{V''(1)\}$ satisfies 13.42 relative to $\{\tilde{V}''(1)\}$
(e) $\{V(1)\}$ is a star refinement of $\{V''(1)\}$

If $\{V(n-1)\}$ is defined, let $V(n)$ be a refinement of mesh $\frac{1}{n}$ and go through (a) — (e) again to obtain $\{V(n)\}$. Note that $\{V(n)\}$ is a refinement of $\{V(n-1)\}$.

Embed $Y$ in $H(Y)$; to obtain a retraction of a nbd of $Y$ onto $Y$ some further constructions are needed.

(a) Form the open sets Ext $V(n)$ in $H(Y)$; from 13.2,
13.5: Each Ext $V(n)$ is contained in some Ext $V(n-1)$.

(β) Let \{U\} be a canonical cover of $H(Y)-Y$. 
(γ) Define a sequence of nbds \( W_i \supset Y \) in \( H(Y) \) by induction:

Set \( W_1 = \cup \text{Ext } V(1) \). If \( W_{n-1} \) is defined, normality gives an open \( G_n \) with \( Y \subset G_n \subset G_n \subset W_{n-1} \). For each \( y \in Y \) choose a bdd of form \( G_n \cap \text{Ext } V(n) \); by 2.6, \( y \) has a bdd \( W_n(y) \) such that \( U \cap W_n(y) \neq 0 \) implies \( U \subset G_n \cap \text{Ext } V(n) \). Define \( W_n = \cup_y W_n(y) \) to complete the inductive step.

Clearly \( \overline{W}_n \subset W_{n-1} \) and one can assume \( d(Y, H(Y) - W_n) < \frac{1}{n} \).

Furthermore,
13.6 \( W_1 \supset W_2 \supset \ldots \) and \( \bigcap_1^\infty W_i = Y \).

13.7 Let \( n \geq 3 \). If \( U \cap \overline{W}_n \neq 0 \) then \( U \subset W_{n-1} \cap \text{Ext } V(n) \) for some \( V(n) \) and therefore \( U \cap (\overline{W}_{n-2} - W_{n-1}) = 0 \).

Indeed, \( U \cap \overline{W}_n \neq 0 \) implies \( U \cap W_n = 0 \) because \( U \) is open; thus \( U \cap W_n(y) \neq 0 \) for some \( y \) and the result follows.

(δ) To each \( U \) assign an integer \( n_U \) as follows: if \( U \cap \overline{W}_3 = 0 \) set \( n_U = 0 \); if \( U \cap \overline{W}_3 \neq 0 \) set \( n_U = \sup \{ i \mid U \cap \overline{W}_i \neq 0 \} \); the finiteness of \( n_U \) follows from 13.6 and 2.6.

From 13.7 and 13.5 follows
13.8 If \( n_U \geq 3 \), then \( U \subset \text{some Ext } V(n_U) \) and for each \( 3 \leq k \leq n_U \) there is a \( V(k) \supset V(n_U) \) with \( U \subset \text{Ext } V(k) \).

The constructions are now complete.

Form \( Y \cup N(U) \) and map the vertices \( \{p_U \} \) of \( N(U) \) into \( Y \) as follows: In each \( U \) choose a point \( z_U \); for each \( z_U \) select a \( y_U \in Y \) satisfying \( d(y_U, z_U) < 2 \, d(z_U, Y) \). By 13.3 and 13.8, if \( n_U \geq 3 \) the \( y_U \) can be assumed to lie in a set \( V(n_U) \) such that \( U \subset \text{Ext } V(n_U) \). Define \( r : Y \cup N(U)_0 \to Y \) by

\[
\begin{align*}
  r(p_U) &= y_U \\
  r(a) &= a \quad a \in A.
\end{align*}
\]

Continuity follows from 2.62.

Form the "rings" \( R_m = \overline{W}_m - W_{m+1} \) and let \( P_m \) be the subpolytope of \( N(U) \) formed by all the sets \( U \) intersecting \( R_m \).
18.9 For each \( m \geq 3 \), \( r \) is a partial realization of \( P_m \) in \( \{V''(m)\} \).

In fact, if \( \{p_{U_1}, \ldots, p_{U_s}\} \) is a cell of \( P_m \), then according to 13.8 there are \( V_i(m) \supset r(p_{U_i}) \) with \( \text{Ext } V_i(m) \supset U_i, i = 1, \ldots, s \); since \( U_1 \cap \ldots \cap U_s \neq 0 \) the \( \text{Ext } V_i(m) \), hence also the \( V_i(m) \), have a non-vacuous intersection. The union of the \( V_i(m) \), hence \( \bigcup_1^s r(p_{U_i}) \), is contained in some \( V''(m) \).

By 13.7, \( P_m \cap P_n = 0 \) for \( |m-n| \geq 2 \). For each \( n = 1, 2, \ldots \),
extend the partial realization \( r \) on \( P_{2n+1} \) to a full realization in \( \{ \hat{V}'(2n+1) \} \) and denote this extension by \( \hat{r} \).

**13.10** For each \( n = 1, 2, \ldots, \) \( \hat{r} \) is a partial realization of \( P_{2n+2} \) relative to \( \{ \hat{V}'(2n+1) \} \).

Indeed, let \( \bar{\sigma} = (p_{U_1}, \ldots, p_{U_s}) \) be a cell of \( P_{2n+2} \); by 13.9, \( \hat{v} \) sends all the vertices of \( \bar{\sigma} \) into a single \( V''(2n+2) \) hence also into a single set \( \hat{V}_0''(2n+1) \). Since the faces of \( \bar{\sigma} \) can belong only to \( P_{2n+1}, P_{2n+2}, P_{2n+3} \), \( \hat{r} \) sends any realized face to a \( \hat{V}''(2n+1) \) intersecting \( \hat{V}_0''(2n+1) \), so that \( \hat{r}(\bar{\sigma}) \) is contained in some \( \hat{V}'(2n+1) \).

\( \hat{r} \) therefore extends to \( r' : \bigcup_{i} P_i \to Y \) with \( r' \mid P_{2n+1} \cup P_{2n+2} \) being a realization relative to \( \{ \hat{V}'(2n+1) \} \).

Setting

\[
\eta(y) = \begin{cases} 
  y & y \in Y \\
  r'(y) & y \in \bigcup_{i} P_i 
\end{cases}
\]

one has \( \eta : Y \cup \bigcup_{i} P_i \to Y \), the continuity at points of \( Y \) following from mesh \( \hat{V}(k) < 1/k \) and the continuity of \( r \).

Let \( W = \bigcup_{i} W_i \) and \( \mu : H(Y) \to Y \cup N(U) \) the canonical map. Then \( \eta \mu \mid W \) retracts \( W \) onto \( Y \) and by 12.13 \( Y \) is an ANR

1 \( \Rightarrow \) 2. Embed \( Y \) in \( H(Y) \); since \( Y \) is an ANR, there is a retraction \( r \) of a nbhd \( V \supset Y \) in \( H(Y) \) onto \( Y \). To simplify the terminology, a spherical nbhd in \( H(Y) \) means the intersection of a spherical nbhd in the Banach space \( B(Y) \) with \( H(Y) \). For each \( y \in Y \) choose a spherical nbhd \( S(y) \) of \( y \) in \( H(Y) \) satisfying \( S(y) \subset V \) and \( S(y) \cap Y \subset Y \) some set \( \{ U \} \) of the given open covering. Finally choose a spherical nbhd \( T(y) \) in \( H(Y) \) with \( T(y) \subset V \) and \( r(T(y)) \subset S(y) \). The desired refinement is \( \{ T(y) \cap Y \} \). Let \( f \) be a partial realization of \( P \) relative to \( \{ T(y) \cap Y \} \) defined on \( Q \supset P_0 \). For each closed \( r \)-cell \( \bar{\sigma} \) let \( Z(\bar{\sigma}) = f(Q \cap \bar{\sigma}) \) and \( \hat{Z}(\bar{\sigma}) \) be the convex closure of \( Z(\sigma) \). The missing faces of \( P \) are now inserted so that the image of each \( \bar{\sigma} \) lies in \( \hat{Z}(\bar{\sigma}) \). Since \( Q \supset P_0 \), proceed by induction.; if all faces of dimension \( < r \) have been inserted as required, for any \( r \)-cell \( \bar{\sigma} \), \( bdry \bar{\sigma} \) is a subset of \( \hat{Z}(\bar{\sigma}) \); taking the join of \( q \in \hat{Z}(\sigma) \) with \( f(bdry \bar{\sigma}) \) gives an extension over \( \bar{\sigma} \) with the required property. Repeating for each \( r \)-cell completes the inductive step. If \( F \) is the full realization obtained, \( F(\bar{\sigma}) \subset Z(\bar{\sigma}) \subset T(y) \subset V \) for each \( \bar{\sigma} \), so \( rF \) is a full realization relative to \( \{ U \} \).

**Remark 13.11.** The implication \( 1 \Rightarrow 2 \) remains true in case \( Y \)
is a polytope. Indeed, given an open covering \{U\}, subdivide to get \(Y'\) with each of its closed vertex stars lying in some set of \{U\}. Embed \(Y'\) in the \(K'\) of 11.1. A nbd \(W\) of \(Y'\) in \(K'\) of which \(Y'\) is a retract is obtained [12; 292] by taking 'the barycentric sub-
division \(K''\) of \(K'\) and letting \(W\) be the union of all the vertex stars in \(K''\) which have center a vertex of \(Y''\). Each such star is convex, and these play the role of \(\widetilde{Z}\) in 1 \(\Rightarrow\) 2 of 13.4.

14. Characterization of \(\text{ANR} \mathbb{R}\) by "factorization".

**LEMMA 14.1.** Let \(Y\) be either an \(\text{ANR} \mathbb{R}\) metric space or a polytope. For each open covering \{\(U\)\} of \(Y\) there exists a refinement \{\(W\)\} with the property: If \(X\) is any metric space, and \(f_0, f_1 : X \to Y\) are such that \(f_0(x), f_1(x)\) lie in a common \(W\) for each \(x \in X\) then \(f_0 \simeq f_1\) and the homotopy \(\phi\) can be so chosen that \(\phi(x \times I)\) lies in a set \(U\) for each \(x \in X\) [4; 363].

**PROOF.** With the notations in 1 \(\Rightarrow\) 2 of 13.4, \(\{T(y) \cap Y\}\) is shown to be the required open cover. If \(f_0, f_1\) are as in the statement of the Lemma relative to \(\{T(y) \cap Y\}\), then for each \(x, f_0(x)\) and \(f_1(x)\) can be joined by a line segment lying in \(T(y)\), hence in \(V\); letting \(\phi(x, t) = tf_0(x) + (1-t)f_1(x)\) gives \(f_0 \simeq f_1\) in the required fashion. The proof for \(Y\) a polytope is similar.

It will be necessary to use the trivial

14.2 Let \(Q\) be a subpolytope of \(K\). There is a retraction \(r : K \times I \to K \times 0 \cup Q \times I\). Furthermore, for each cell \(\bar{\sigma}\) of \(K\), \(r(\bar{\sigma} \times 1) \subset \bar{\sigma} \times I\).

This result is well known; see, for example [14; 84]. It follows by a simple induction based on the observation that 

\([\text{bdry } \bar{\sigma}] \times I \cup \bar{\sigma} \times 0\) is a retract of \(\bar{\sigma} \times I\).

The following theorem is also given by Hanner [6; 358]; his proof is different from the one that appears here.

**THEOREM 14.3.** Let \(Y\) be a metric space. The following properties are equivalent:

14.31 \(Y\) is an \(\text{ANR} \mathbb{R}\).

14.32 For each open covering \{\(U\)\} of \(Y\), there exists a polytope \(P\) and \(\lambda : Y \to P, g : P \to Y\) such that \(g\lambda \simeq\) identity map of \(Y\), and the homotopy \(\phi\) can be so chosen that, for each \(y\) the set \(\phi(y \times I)\) lies in a set \(U\).

**PROOF.** 1 \(\Rightarrow\) 2. This is similar to 1 \(\Rightarrow\) 2 of 6.1. See also [4; 365].

2 \(\Rightarrow\) 1. 13.42 will be shown to hold. Given \{\(U\)\} select a star refinement \{\(V\)\}. Let \(P\) be the polytope satisfying 14.32 relative to \{\(V\)\}. Since \(\{g^{-1}(V)\}\) is an open cover of \(P\), apply 13.11 to get a refinement \{\(W\)\} having the partial realization property 13.11 relative
to \(\{g^{-1}(V)\}\). Let \(\{\hat{W}\}\) be a common refinement of \(\{\lambda^{-1}(W)\}\) and \(\{V\}\).

Let \(f\) be a partial realization of a polytope \(K\) relative to \(\{\hat{W}\}\). \(f\) is defined on \(Q \supset K_0\). \(\lambda f\) is a partial realization relative to \(\{W\}\) hence extends to a full realization \(F'\) in \(P\) relative to \(\{g^{-1}(V)\}\); \(gF' : K \to Y\) sends each cell of \(K\) into a set \(V\), and \(gF' | Q = g \lambda f\).

Deform \(g \lambda\) to the identity according to 14.22; in obvious fashion this yields a map \(\Delta : K \times 0 \cup Q \times I \to Y\) with \(\Delta | Q \times 1 = f\), and by the 14.22, \(\Delta(\tau \times I) \subseteq U\) for each \(\tau\). With the retraction \(r\) of 14.2, \(\Delta r | K \times 1\) is an extension of \(f\) over \(K\) with the image of each cell in a set \(U\). By 13.42, \(Y\) is an \(ANR\).

15. \(ANR\) and \(AR\) in finite-dimensional metric spaces.

**Theorem 15.1.** Among the finite-dimensional metric spaces (whether separable or not), the locally contractible ones are the \(ANR\). Precisely, if \(Y\) is metric and \(\dim Y\) finite, the following three properties are equivalent:

15.11 \(Y\) is \(LC^n\) for some integer \(n \geq \dim Y\).
15.12 \(Y\) is locally contractible.
15.13 \(Y\) is an \(ANR\).

**Proof.** 1 \(\Rightarrow\) 2 by 7.2. 2 \(\Rightarrow\) 3: Since \(Y\) is \(LC\), it is also \(LC^\infty\). If \(\dim Y \leq r\), applying 6.12 with the identity map of \(Y\), 14.3 shows \(Y\) an \(ANR\). 3 \(\Rightarrow\) 1 by 12.3.

For infinite dimensional spaces, this theorem is not true, even with the added hypothesis of separability.

**Theorem 15.2.** Among the finite-dimensional arbitrary metric spaces, the contractible and locally contractible ones are the \(AR\). Precisely, if \(Y\) is metric and \(\dim Y\) is finite, the following properties are equivalent:

15.21 \(Y\) is \(C^n\) and \(LC^n\) for some \(n \geq \dim Y\).
15.22 \(Y\) is contractible and locally contractible.
15.23 \(Y\) is an \(AR\).

**Proof.** 1 \(\Rightarrow\) 2 from 7.2 and 9.2. 2 \(\Rightarrow\) 3: By 15.1, \(Y\) is an \(ANR\); use 12.4 to find \(Y\) is an \(AR\). 3 \(\Rightarrow\) 1 by 12.3.

The following theorem characterizes the finite-dimensional \(AR\) solely by a special type of contractibility.

**Theorem 15.3.** Let \(Y\) be metric and \(\dim Y\) finite. The following two properties are equivalent:

15.31 \(Y\) is an \(AR\).
15.32 Given any \(y_0 \in Y\), \(Y\) is contractible to this point in such a
way that during the entire process of deformation, $y_0$ remains fixed.

**Proof.** $1 \Rightarrow 2$ is trivial.

$2 \Rightarrow 1$. One need only show that $Y$ is locally contractible. This follows trivially by selecting a contraction 15.32 to any point $y_0$ and using the continuity of this contraction at $y_0$.

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