Composition Mathematica

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Compositio Mathematica, tome 13 (1956-1958), p. 247-256
<http://www.numdam.org/item?id=CM_1956-1958__13__247_0>

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The congruence of tangents to a system of curves on a surface

by

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1. Let the surface of reference be taken as the given surface S, the curvilinear co-ordinates of any point on which are u and v, We consider in this paper the congruence of tangents to a given system of curves on the surface. In particular, some interesting properties are set forth regarding the congruence of tangents to the lines of curvature and the asymptotic lines of the surface.

Notations

$\mathbf{r}$ = position vector of any point of the surface.
$\mathbf{n}$ = the unit normal vector at the point.
$\mathbf{d}$ = the unit tangent to the curve of the congruence.

a, b, b^1, c and e, f, g are the co-efficients of the first and the second quadratic forms associated with the congruence.

The suffixes 1 and 2 denote differentiation with respect to u and v respectively. E, F, G, and L, M, N are the magnitudes of the first and second orders for the surface.

We have

$$\mathbf{n} \cdot \mathbf{d} = 0$$

$$\therefore\ (\mathbf{r}_1 \times \mathbf{r}_2) \cdot (\mathbf{d}_1 \times \mathbf{d}_2) = 0$$

$$\text{i.e. } (\mathbf{r}_1 \cdot \mathbf{d}_1)(\mathbf{r}_2 \cdot \mathbf{d}_2) - (\mathbf{r}_2 \cdot \mathbf{d}_1)(\mathbf{r}_1 \cdot \mathbf{d}_2) = 0 \tag{1}$$

$$\text{i.e. } ac - bb^1 = 0$$

Through each line of a rectilinear congruence, there pass two ruled surfaces whose lines of striction lie on the surface of reference. These are given by the equations [1].

$$a + (b + b^1)m + cm^2 = 0, m = \frac{dv}{du} \tag{2}$$

If θ is the angle between the two lines of striction, we easily obtain

$$\tan \theta = \frac{H(b - b^1)}{cE - F(b + b^1) + aG}, \text{ where } H = \sqrt{EG - F^2} \tag{3}$$

$b = b^1$ is the necessary and sufficient condition that the congruence be normal.

Hence we have

Theorem 1. *The necessary and sufficient condition that the congruence of tangents to a system of curves on a surface form a normal congruence is that the two ruled surfaces whose lines of striction lie on the surface coincide.*

2. The distances r_1, r_2 of the limit points of any ray of the congruence are given by [2].

$$(4) \qquad h^2 r^2 + [ag + ce - f(b + b^1)]r + ac - \frac{(b + b^1)^2}{4} = 0,$$

where $h^2 = eg - f^2$

$$(5) \qquad \therefore r_1 r_2 = -\frac{(b - b^1)^2}{4h^2}, \text{ in virtue of (1)}$$

The two limit points are therefore on opposite sides of S. Similarly the distances of the focal points are given by

$$(6) \qquad h^2 \rho^2 + [ag + ce - f(b + b^1)]\rho + ac - bb^1 = 0$$

$$(7) \qquad \text{This gives } \rho = 0 \text{ and } \rho = \frac{f(b + b^1) - ag - ce}{h^2}$$

One of the focal points thus lies on the surface, which fact is obvious from geometrical considerations.

The roots m_1 and m_2 of the quadratic (2) are

$$(8) \qquad m_1 = -\frac{b}{c}, \; m_2 = -\frac{b^1}{c}.$$

The expression for the parameter of distribution is given by [3]

$$(9) \quad \beta = \frac{(af - b^1 e) + (ag + bf - b^1 f - ce)\,m + (bg - cf)m^2}{h(e + 2fm + gm^2)}.$$

The value m_1 of m gives $\beta = 0$. One of the surfaces whose lines of striction lie on the given surface is therefore a developable. This fact also follows easily from geometrical considerations.

3. Choose the given system of curves as $v =$ constant and their orthogonal trajectories as $u =$ constant. We have [4]

$$(10) \qquad d_1 = \frac{L}{\sqrt{E}}\mathbf{n} - \frac{E_2}{2H}\frac{\mathbf{r}_2}{\sqrt{G}}, \quad d_2 = \frac{M}{\sqrt{E}}\mathbf{n} + \frac{G_1}{2H}\frac{\mathbf{r}_2}{\sqrt{G}}$$

$$\therefore a = 0, \; b = -\frac{E_2\sqrt{G}}{2H}, \quad b^1 = 0, \quad c = \frac{G_1\sqrt{G}}{2H}.$$

Substituting in (3) we get

$$\tan\theta = -\frac{\sqrt{G}}{\sqrt{E}}\frac{E_2}{G_1}. \tag{11}$$

The congruence is normal if and only if $E_2 = 0$ i.e. if E is a function of u only. This is the condition that the given curves $v =$ constant are geodesics. This is a well known result.

θ is a right angle if G is a function of v only, which is the condition that the curves $v =$ constant form a system of geodesic parallels.

Hence we have

THEOREM 2. *For the congruence of tangents to a family of geodesic parallels on a surface, the two ruled surfaces whose lines of striction lie on the surface intersect at right angles at the point of the ray on the surface.*

If β_1 and β_2 are the principal parameters then

$$\beta_1+\beta_2=\frac{e(bg-cf)+g(af-b^1e)-f(ag+bf-b^1f-ce)}{h(eg-f^2)} \tag{12}$$

which reduces to

$$\beta_1+\beta_2=\frac{b}{h}$$

If r_1 and r_2 are the distances of the limit points

$$r_1r_2=-\frac{b^2}{4h^2}$$

Hence we have

$$r_1r_2=-\left(\frac{\beta_1+\beta_2}{2}\right)^2=-\bar{\beta}^2 \tag{13}$$

The product of the distances of the limit points from the surface is numerically equal to the square of the mean parameter.

It is known [5] — that for any rectilinear congruence

$$\beta_1\beta_2=-\frac{\rho^2}{4}$$

$$\beta_1\beta_2=-\left(\frac{r_1+r_2}{2}\right)^2 \text{ by (4) and (6)} \tag{14}$$

using (13) and (14), we get

$$\beta_1-\beta_2=r_1-r_2.$$

Hence we have

THEOREM 3. *In the congruence of tangents to a system of curves on a surface, the difference between the principal parameters of any ray is equal to the difference between the distances of the limit points measured from the surface.*

4. We shall now consider the two congruences formed by the tangents to the system of curves $v =$ constant and the tangents to their orthogonal trajectories $u =$ constant. For the congruence of tangents to $v =$ constant the surfaces whose lines of striction lie on the given surface are using (2) and (5) given by:

$$- E_2 \sqrt{G}\, du\, dv + G_1 \sqrt{G}\, dv^2 = 0$$

i.e. $v =$ constant and $E_2\, du - G_1\, dv = 0$.

Similarly for the congruence of tangents to $u =$ constant if λ is the unit vector along the tangent, then [6]

$$\lambda_1 = \frac{M}{\sqrt{G}}\mathbf{n} + \frac{E_2}{2H}\frac{\mathbf{r}_1}{\sqrt{E}}, \quad \lambda_2 = \frac{N}{\sqrt{G}}\mathbf{n} - \frac{G_1}{2H}\frac{\mathbf{r}_1}{\sqrt{E}}$$

$$\therefore a = \frac{E_2\sqrt{E}}{2H}, \quad b = 0, \quad b^1 = -\frac{G_1\sqrt{E}}{2H}, \quad c = 0$$

The surfaces of the congruence whose lines of striction lie on the given surface are given by

$$u = \text{constant and } E_2\, du - G_1\, dv = 0$$

We note that the equation $E_2 du - G_1 dv = 0$ is common to the two congruences.

Hence, we have

THEOREM 4. *For the congruence of tangents to a system of curves on a surface, one of the two systems of surfaces whose lines of striction lie on the surface coincides with one of the two similar systems for the congruence of tangents to the orthogonal trajectories of the given curves.*

In §§ 5—9 we set forth some properties of the congruence of tangents to the system of lines of curvature of a surface.

5. Choosing $u =$ constant and $v =$ constant as the lines of curvature, we have for the congruence of tangents to the system $v =$ constant [7]

$$a = 0, \; b = \frac{-E_2\sqrt{G}}{2H}, \; b^1 = 0, \; c = \frac{G_1\sqrt{G}}{2H}$$

$$e = \frac{L^2}{E} + \frac{E_2^2}{4H^2}, \; f = \frac{-E_2 G_1}{4H^2}, \; g = \frac{G_1^2}{4H^2}, \; h^2 = \frac{L^2 G_1^2}{4H^2 E}.$$

From (4) and (6), we easily obtain

$$r_1 - r_2 = \frac{H\sqrt{E_2^2 + 4GL^2}}{G_1 L}, \quad \rho_1 - \rho_2 = \frac{-2H\sqrt{G}}{G_1}$$

If ϕ is the angle between the focal planes, it is well known that

$$\sin\varphi = \frac{\rho_1 - \rho_2}{r_1 - r_2}.$$

Hence we get

$$\tan\varphi = \frac{2L\sqrt{G}}{E_2}.$$

If β_1 and β_2 are the parameters of distribution corresponding to the mean rules surfaces, we have using (7), (12) and (14)

$$\beta_1 + \beta_2 = \frac{-E_2 H}{G_1 L}$$

$$\beta_1\beta_2 = \frac{-H^2 G}{G_1^2}$$

It is known [8] if

$$\mathcal{J} = edu^2 + 2f\,du\,dv + gdv^2$$

$$\text{and } \phi \equiv \begin{vmatrix} Edu + Fdv & Fdu + Gdv \\ edu + fdu & fdu + gdv \end{vmatrix}$$

the differential equation

$$J[\phi, J(\mathcal{J}, \phi)] = 0$$

Where J stands for the Jacobian, represents the characteristic ruled surfaces of the congruence. The parameter of distribution of the characteristic ruled surfaces of any ray given by [9]

$$\frac{2\beta_1\beta_2}{\beta_1 + \beta_2} \text{ which in the present case reduces to } \frac{2HGL}{G_1 E_2}$$

If the congruence is normal, $E_2 = 0$, the curves $v =$ constant will be geodesics as well as lines of curvature and hence plane curves. The characteristic ruled surfaces coincide with the surfaces whose spherical representations are minimal lines. The same result is true when $G_1 = 0$ i.e. when $v =$ constant are geodesic parallels i.e. when $u =$ constant are geodesics.

6. The equation giving the developable surfaces of the congruence,

viz. $(af - b^1 e)du^2 + (ag + bf - b^1 f - ce)\,dudv + (bg - cf)dv^2 = 0$

reduces in virtue of (16) to $dudv = 0$. The developables therefore meet the given surface along its lines of curvature, and are therefore the same for the two congruences defined by the tangents to the two systems of lines of curvature. The given surface itself is one focal surface, while the other focal surface is defined by

$$\mathbf{R} = \mathbf{r} - \frac{2H\sqrt{G}}{G_1}\mathbf{d} = \mathbf{r} - \frac{2H\sqrt{G}}{G_1}\frac{\mathbf{r}_1}{\sqrt{E}}$$

It is known [10] that the lines of the congruence are tangents to the curves u = constant on this focal surface, so that $\mathbf{R}_2 \cdot \mathbf{r}_2 = 0$

This can be verified directly, as follows:

$$\mathbf{R}_2 = \mathbf{r}_2 - \frac{\partial}{\partial v}\left(\frac{2H\sqrt{G}}{G_1}\right)\mathbf{d} - \frac{2H\sqrt{G}}{G_1}\mathbf{d}_2$$

$$\therefore \mathbf{R}_2 \cdot \mathbf{r}_2 = G - \frac{2H\sqrt{G}}{G_1}c = 0$$

Suppose now that the focal surfaces correspond with orthogonality of corresponding linear elements, so that $\mathbf{R}_1 \cdot \mathbf{r}_1 = 0$

Now $\mathbf{R}_1 = \mathbf{r}_1 - \dfrac{\partial}{\partial u}\left(\dfrac{2H\sqrt{G}}{G_1}\right)\mathbf{d} - \dfrac{2H\sqrt{G}}{G_1}\mathbf{d}_1$

$\therefore \mathbf{R}_1 \cdot \mathbf{r}_1 = E - \dfrac{\partial}{\partial u}\left(\dfrac{2H\sqrt{G}}{G_1}\right)\sqrt{E}$

$\therefore \mathbf{R}_1 \cdot \mathbf{r}_1 = 0$ if and only if $\dfrac{H}{G_1}$ is a function of v only.

$\therefore \mathbf{R}_1, \mathbf{R}_2 = 0$

i.e. $F = 0$ on the second focal surface. Hence the congruence of tangents to a system of lines of curvature is a Guichard congruence. Conversely if the congruence of tangents to a system of lines of curvature is a Guichard congruence, then

$$\mathbf{R}_1 \cdot \mathbf{R}_2 = 0$$

It follows H/G_1 is a function of v alone

$$\therefore \mathbf{R}_1 \cdot \mathbf{r}_1 = 0$$

Hence we have

THEOREM 5. *A necessary and sufficient condition that the congruence of tangents to a system of lines of curvature on a surface S form a Guichard congruence is that the surface S and the other focal surface should correspond with orthogonality of corresponding linear elements.*

7. Let us consider now the congruences of tangents to both systems of lines of curvature on a surface S. We take the lines of curvature as u = constant and v = constant and we shall denote

by the subscripts u and v the functions corresponding to the two congruences.

From formulae (5), (10) and (16), we obtain

$$(r_1 r_2)_v = -\frac{H^2 E_2^2}{4L^2 G_1^2}$$

Similarly for the congruence of tangents to u — constant

$$(r_1 r_2)_u = -\frac{H^2 G_1^2}{4N^2 E_2^2}$$

$$\therefore (r_1 r_2)_v \cdot (r_1 r_2)_u = \frac{H^4}{16L^2 N^2} = \frac{1}{16K^2},$$

where K is the Gaussian curvature of S.

Hence we have

THEOREM 6. *Considering the congruences of tangents to the two systems of lines of curvature on a surface, the product of the distances of the two pairs of limit points of the two rays which meet at a point on S, is equal to* $\frac{1}{16K^2}$.

Using (13) we also get that *the product of the sums of the parameters of distribution of the mean ruled surfaces is equal to* $\frac{1}{K}$.

8. If the parameter of distribution corresponding to the characteristic ruled surfaces of the two congruences are $\bar{\bar{\beta}}_v$ and $\bar{\bar{\beta}}_u$

$$\bar{\bar{\beta}}_v + \bar{\bar{\beta}}_u = \frac{2HGL}{G_1 E_2} + \frac{2HEN}{G_1 E_2}$$

$$= \frac{2H^3}{G_1 E_2} J = -\frac{2H}{f} J$$

where J is the first curvature of S.

If ϕ is the angle between the focal planes of the first congruence

$$\tan \varphi = \frac{2L\sqrt{G_1}}{E_2}$$

similarly for the other congruence

$$\tan \varphi^1 = \frac{2N\sqrt{E}}{G_1}$$

Hence

$$\tan \varphi \tan \varphi^1 = \frac{4LNH}{E_2 G_1} = -\frac{H}{f} K$$

$$\therefore \bar{\bar{\beta}}_v + \bar{\bar{\beta}}_u = 2\frac{J}{K} \tan \varphi \tan \varphi^1$$

Hence, we have

THEOREM 7. *The sum of the parameters of distribution of the characteristic ruled surfaces of the two congruences, at a point on S is equal to* $2\frac{J}{K}\tan\varphi\tan\varphi^1$. *This vanishes when the surface S is a minimal surface.*

The two focal planes at a point are the osculating plane to the curve and the tangent plane to the surface. Hence ϕ will be equal to the angle between the binormal to the curve and the normal to the surface at the point. Therefore $\varphi = \frac{\Pi}{2} - \omega$ where ω is the angle between the normal to the surface and the principal normal

$$\bar{\bar{\beta}}_v + \bar{\bar{\beta}}_u = 2\frac{J}{K}\cot\omega\cot\omega^1.$$

We have also

$$\frac{\bar{\bar{\beta}}_v}{\bar{\bar{\beta}}_u} = \frac{GL}{EN} = \frac{\frac{L}{E}}{\frac{N}{C}} = \frac{K_1}{K_2}$$

where K_1 and K_2 are the principal curvatures at the point. Hence, we have

THEOREM 8. *The parameters of distribution of the characteristic ruled surfaces of the two congruences at a point are proportional to the principal curvatures at the point.*

9. The ratio of the focal distances of the two congruences is equal to

$$\frac{\rho_v}{\rho_u} = \sqrt{\frac{G}{E}\frac{E_2}{G_1}} = -\tan\theta_v$$

where θ_v is the angle between the two lines of striction of the congruence of tangents to $v =$ constant.

Similarly

$$\frac{\rho_u}{\rho_v} = -\tan\theta_u$$

$$\therefore \tan\theta_v\tan\theta_u = 1$$

$$\theta_v = \frac{\Pi}{2} - \theta_u$$

This is implicitly contained in Theorem 4.

10. *The congruence of tangents to a system of asymptotic lines on a surface S.*

This congruence has been briefly studied in [11]. We add one or two properties here.

Let v — constant be one system of asymptotic lines and let their orthogonal trajectories be u — constant. We have [12]

$$a = 0,\ b = -\frac{E_2}{2\sqrt{E}},\quad b^1 = 0,\ c = \frac{G_1}{2\sqrt{E}}$$

$$e = \frac{E_2^2}{4EG},\ f = \frac{-E_2 G_1}{4EG},\ g = \frac{4M^2 G + G_1^2}{4EG},\ h^2 = \frac{M^2 E_2^2}{4E^2 G}.$$

The parameters of distribution for the mean ruled surfaces are obtained from the expression

$$\beta = \frac{(bf - ce)m + (bg - cf)m^2}{h(e + 2fm + gm^2)}$$

which gives the two values

$$\beta_1 = 0,\quad \beta_2 = \frac{b}{h}$$

Hence, one of the mean ruled surfaces is a developable.

The parameter of distribution corresponding to the characteristics ruled surfaces vanishes and therefore the characteristic ruled surfaces are identical with the developables of the congruence.

$$\text{Also } \beta_2^2 = \frac{b^2}{h^2} = \frac{H^2}{M^2} = -\frac{1}{K}$$

$$\beta_2 = \pm\frac{1}{\sqrt{-K}}.$$

Hence, the parameters of distribution corresponding to the mean ruled surfaces are respectively equal to zero and the radius of torsion of the asymptotic line.

We have also

$$\beta_2 = \frac{1}{\sqrt{-K}} = r.$$

If r is the distance between the limit points, since the point on S is the middle point, we have

$$r = \beta_1 + \beta_2 = \beta_2 = \frac{1}{\sqrt{-K}}.$$

Hence, we have

THEOREM 9. *In the congruence of tangents to a system of asymptotic lines on a surface, the distance between the limit points of any ray is equal to the radius of torsion of the corresponding asymptotic line.* [13]

I express my thanks to Dr. C. N. Srinivasiengar for having discussed this paper with me, and for helpful suggestions.

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