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On Certain Periodic Characteristic Functions

by

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1. Introduction.

The purpose of this note is to derive certain properties of periodic characteristic functions and to determine those distributions whose characteristic functions are entire periodic functions.

Let $F(x)$ be a probability distribution, that is, a never-decreasing, right continuous function such that $\lim_{x \rightarrow -\infty} F(x) = 0$ and $\lim_{x \rightarrow \infty} F(x) = 1$. The Fourier transform of $F(x)$, that is the function

$$(1) \quad f(z) = \int_{-\infty}^{\infty} e^{izx} dF(x)$$

is called the characteristic function of $F(x)$.

A distribution is called a *lattice distribution* if it is purely discontinuous and if its discontinuity points form a (proper or improper) subset of a sequence of equidistant points.

A characteristic function is said to be an *analytic characteristic function* if it coincides with an analytic function in some neighborhood of the origin.

We prove the following theorems:

THEOREM 1. *An analytic characteristic function which is single valued and periodic has either a real or a purely imaginary period. The period is real if, and only if, the characteristic function belongs to a lattice distribution.*

THEOREM 2. *A characteristic function is an entire periodic function (not $\equiv 1$) if, and only if, it is the characteristic function of a lattice distribution.*

2. The lemmas.

For the proof of these theorems we need two lemmas which we derive in this section.

LEMMA 1. *A characteristic function $f(t)$ assumes the value 1 for some real $t_0 \neq 0$ if, and only if, it is the characteristic function of a lattice distribution.*

PROOF: Let us first assume that for some real $t_0 \neq 0$ the value $f(t_0) = f(0) = 1$. We see then from (1) that

$$\int_{-\infty}^{\infty} (1 - e^{it_0x}) dF(x) = 0$$

and therefore also

$$\int_{-\infty}^{\infty} (1 - \cos t_0x) dF(x) = 0$$

Since the function $1 - \cos t_0x$ is continuous and nonnegative this relation can hold only if $F(x)$ is a purely discontinuous distribution such that its discontinuity points are contained in the point set $2\pi s/t_0$ ($s = 0, \pm 1, \pm 2, \dots$) where $1 - \cos t_0x$ vanishes. The distribution function $F(x)$ has therefore necessarily the form

$$(2) \quad F(x) = \sum_{s=-\infty}^{\infty} p_s \varepsilon(x - 2\pi s/t_0)$$

where

$$p_s \geq 0, \quad \sum_{s=-\infty}^{\infty} p_s = 1$$

and where

$$(3) \quad \varepsilon(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$$

is the degenerate distribution. On the other hand, if $F(x)$ has the form (2) then

$$(4) \quad f(t) = \sum_{s=-\infty}^{\infty} p_s \exp(2\pi its/t_0)$$

so that $f(t_0) = 1$.

Lemma 1 is a particular case of a somewhat more general result of A. Wintner ([2], p. 48). Wintner proves that $|f(\xi)| = 1$ for some real $\xi \neq 0$ if, and only if, $F(x)$ is a lattice distribution.

LEMMA 2. *Let $f(z)$ be an analytic characteristic function with the strip of convergence¹⁾ $-\alpha < \text{Im}(z) < \beta$ and assume that there are three real numbers n_1, n_2, n_3 such that*

- (i) $-\alpha < n_1 < n_2 < n_3 < \beta$,
- (ii) $n_1 + n_3 = 2n_2$,
- (iii) $f(in_1) = f(in_2) = f(in_3)$,

Then $f(z) \equiv 1$.

¹⁾ For the properties of analytic characteristic functions see [1].

PROOF: We rewrite (iii) as

$$(5) \quad \begin{cases} f(in_2) - f(in_1) = 0 \\ f(in_3) - f(in_2) = 0 \end{cases}$$

We conclude from (i) that the three points in_j ($j = 1, 2, 3$) are in the interior of the strip of convergence of $f(z)$ so that the integral representation (1) is valid at these points. Then

$$f(in_j) = \int_{-\infty}^{\infty} e^{-xn_j} dF(x) \text{ for } j = 1, 2, 3$$

and

$$f(in_{j+1}) - f(in_j) = \int_{-\infty}^{\infty} [e^{-xn_{j+1}} - e^{-xn_j}] dF(x) \text{ for } j = 1, 2.$$

If we put

$$(6) \quad \begin{cases} g_j(x) = e^{-xn_{j+1}} - e^{-xn_j} & (j = 1, 2) \\ I_j = \int_{-\infty}^{\infty} g_j(x) dF(x) & (j = 1, 2) \end{cases}$$

We can write (5) as

$$(7) \quad I_1 = I_2 = 0$$

We carry the proof of lemma 2 indirectly and assume therefore tentatively that $f(z) \not\equiv 1$ and show that this leads to a contradiction with (7). If $f(z) \not\equiv 1$, then also $f(t) \not\equiv 1$ for real t and the corresponding distribution function is nondegenerate and therefore there exists a real $h > 0$ such that at least one of the two relations

$$(8a) \quad \int_{+h}^{+\infty} dF(x) = 1 - F(h) > 0$$

$$(8b) \quad \int_{-\infty}^{-h} dF(x) = F(-h) > 0$$

is satisfied.

From (ii) we see that $n_3 - n_2 = n_2 - n_1$ and obtain therefore from (6)

$$g_1(x) - g_2(x) = -e^{-xn_1} [e^{-x(n_3 - n_1)} - 1]^2.$$

Therefore

$$(9) \quad \begin{cases} g_1(x) < g_2(x) \text{ if } x \neq 0 \\ g_1(0) = g_2(0) = 0 \end{cases}$$

Let us first assume that (8a) is satisfied and choose h accordingly. We see then from (9) that

$$(10a) \quad \int_0^h g_1(x) dF(x) \leq \int_0^h g_2(x) dF(x) \leq 0$$

and

$$(11) \quad 0 \leq \int_{-\infty}^0 g_1(x) dF(x) \leq \int_{-\infty}^0 g_2(x) dF(x)$$

From (8a) we conclude that there exists a finite $A > h$ such that $\int_h^A dF(x) > 0$. We see from (9) that $g_2(x) - g_1(x) > 0$ for $0 < h \leq x \leq A$ and there exists a real number $\mu > 0$ such that $g_2(x) - g_1(x) \geq \mu > 0$ for $h \leq x \leq A$. Since $g_2(x) - g_1(x)$ is positive for $x \geq h$, we see that

$$\int_h^{\infty} [g_2(x) - g_1(x)] dF(x) \geq \int_h^A [g_2(x) - g_1(x)] dF(x) \geq \mu \int_h^A dF(x) > 0$$

and therefore

$$(10b) \quad \int_h^{\infty} g_1(x) dF(x) < \int_h^{\infty} g_2(x) dF(x)$$

Adding (10a), (10b) and (11) we obtain

$$\int_{-\infty}^{+\infty} g_1(x) dF(x) < \int_{-\infty}^{+\infty} g_2(x) dF(x)$$

or

$$(12) \quad I_1 < I_2.$$

In case (8b) is valid we obtain by a similar reasoning again (12). Thus if at least one of the conditions (8a) or (8b) is satisfied, relation (12) must be valid in contradiction to (7). Therefore no h exists such that at least one of the relations (8a) or (8b) is satisfied. But this means that $1 - F(h) = F(-h) = 0$ for any $h > 0$, in other words, $\int_{-h}^{+h} dF(x) = 1$ for any $h > 0$, i.e. the distribution $F(x)$ is necessarily equal to the degenerate distribution $\varepsilon(x)$.

In this proof condition (iii) is apparently not used fully. In the argument only the relation $f(in_2) - f(in_1) = f(in_3) - f(in_2) = M$ was used. Nevertheless (iii) can not be weakened since for $M \neq 0$ the function $f(t)$ could not be a characteristic function.

3. Proof of the theorems.

We next prove theorem 1. Let us therefore assume that $f(z)$ is a single valued analytic characteristic function which has the period

$$(13) \quad \omega = \xi + i\eta \quad (\xi, \eta \text{ real}).$$

It is known (see for instance [1]) that the relation

$$(14) \quad f(-\bar{z}) = \overline{f(z)}$$

is satisfied for every analytic characteristic function in its do-

main of regularity. On account of the periodicity of $f(z)$ we see that

$$f(-\xi - i\eta) = f(0) = 1,$$

we deduce then from (14) that

$$f(\xi - i\eta) = 1.$$

Adding to the argument the period $\omega = \xi + i\eta$, we obtain

$$(15) \quad f(2\xi) = 1.$$

We consider first the case where $\xi \neq 0$. In this case we see from (15) and from lemma 1 that $f(z)$ is the characteristic function of a lattice distribution. We see from (2) and (4) that its characteristic function is given by

$$(16) \quad f(t) = \sum_{s=-\infty}^{\infty} p_s \exp(it\pi s/\xi) \quad (p_s \geq 0, \sum_{-\infty}^{+\infty} p_s = 1)$$

Therefore $f(t)$ is a simply periodic function with the real period 2ξ while $\eta = 0$. If on the other hand $\xi = 0$, then (15) is satisfied and we see from (18) that $\omega = i\eta$, i.e. that the period is purely imaginary. This establishes theorem 1.

The case of a characteristic function which has a purely imaginary period can actually occur. As an example we mention the well known characteristic function $f(t) = \frac{1}{\cosh t}$.

From the proof of theorem 1 we obtain the following corollary:

COROLLARY TO THEOREM 1. *A characteristic function which does not reduce to a constant can not be doubly periodic.*

We proceed to prove theorem 2. Let therefore $f(t)$ be an entire characteristic function which does not reduce to a constant and assume that it is periodic. From theorem 1 we see that $f(t)$ must be simply periodic with either a real or a purely imaginary period. From lemma 2 we see that $f(t)$ can not have a purely imaginary period. The theorem follows then from lemma 1.

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