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A New Generalization of a Problem of F. Lukács

by

Louis Brickman

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Introduction

Let $\chi(x)$ be a real valued non-decreasing function with infinitely many points of increase in the finite or infinite interval (a, b), and let the moments

$$\int_a^b x^\nu d\chi(x), \qquad (\nu = 0, 1, 2, \ldots),$$

exist. Then there exists a set of polynomials $\{\phi_\nu(x)\}_0^\infty$ uniquely determined by the following conditions:

(a) $\phi_\nu(x)$ is a polynomial of precise degree ν in which the coefficient of x^ν is positive.

(b) The system $\{\phi_\nu(x)\}$ is orthonormal, i.e.,

$$\int_a^b \phi_\mu(x)\phi_\nu(x)d\chi(x) = \delta_{\mu\nu}, \qquad (\mu, \nu = 0, 1, \ldots)$$ [1])

The natural number n being given, we denote by Π_n the class of real polynomials $f(x)$ of degree at most n satisfying the following two conditions:

$$(1) \qquad f(x) \geqq 0, \qquad a < x < b,$$

$$(2) \qquad \int_a^b f(x)d\chi(x) = 1.$$

This class has received much attention in connection with the problem of determining

$$(3) \qquad M_n(z) = \max_{f \in \Pi_n} f(z),$$

where z is a real number usually, but not always, assumed to be in (a, b).

F. Lukács determined in 1918 the value of $M_n(+1)$ for all n for the special case $(a, b) = (-1, +1)$ and $\chi(x) = x$.[2]) For this pur-

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[1]) G. Szegö, *Orthogonal Polynomials*, pp. 24—25.

[2]) F. Lukács, „Verschärfung des ersten Mittelwertsatzes der Integralrechnung für rationale Polynome", *Mathematische Zeitschrift*, 2: 295—305, 1918. See also G. Polya and G. Szegö, *Aufgaben und Lehrsätze aus der Analysis*, Problem 108, p. 96.

pose Lukács rediscovers the quadrature formulae of Radau [5], apparently unaware of Radau's memoir of 1880. Slight variations of the Radau formulae appear in this paper in (1:1.10), (1:1.12), and (1:2.7) and play an important part in the discussion.

M. Riesz, in 1922, without the use of mechanical quadratures, determined $M_n(z)$ for all n and all z for the case $(a, b) = (-\infty, +\infty)$, $d\chi(x)$ arbitrary.[3]) See also O. Bottema [1] for an approach with quadrature formulae.

In 1925 G. Polya and G. Szegö extended Lukács result to a variety of intervals and distributions in a series of problems.[4])

Using a method based on a parametric representation of polynomials belonging to Π_n, G. Szegö in 1939, computed max $|f(z)|$, $f(x) \in \Pi_n$, where (a, b) is finite and z is any real number.[5])

Finally, in a paper [7] of 1959, I. Schoenberg and G. Szegö completely determine the set

$$R_z = \{f(z) \mid f \in \Pi_n\} \tag{8}$$

for any real z and for arbitrary (a, b) and $d\chi(x)$. Their method involves the parametric representation mentioned above.

The situation for real z thus being completely described, it is the purpose of this paper to determine the set (8) when z is imaginary. Two properties of R_z can be established immediately. Firstly, since Π_n is evidently a convex class, R_z is a convex set of complex numbers. Secondly, R_z is compact. To see this we first show that Π_n, regarded as a subset of $n+1$ dimensional Euclidean space, is compact. We need the Gauss-Jacobi quadrature formula:

$$\sum_{\nu=0}^{n} \lambda_\nu f(x_\nu) = \int_a^b f(x)\, d\chi(x), \tag{9}$$

where the x_ν are the zeros of $\phi_{n+1}(x)$, $\lambda_\nu > 0$ $(\nu = 0, \ldots, n)$, and $f(x) \in \pi_{2n+1}$.[6]) (Following Szegö, [8], we write $f(x) \in \pi_m$ to indicate that $f(x)$ is a polynomial of degree not exceeding m.) To indicate that (9) is valid if $f(x) \in \pi_{2n+1}$ we shall say that (9) is *of degree* $2n+1$. Applying (9) to an arbitrary member of Π_n, we obtain

$$\sum_{\nu=0}^{n} \lambda_\nu f(x_\nu) = 1, \tag{10}$$

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[3]) M. Riesz, „Sur le problème des moments," 3me Note, *Arkiv for Matematik, Astronomi och Fysik,* 17: 19—20, 1922.

[4]) G. Polya and G. Szegö, op. cit., Problems 103—13, pp. 95—97.

[5]) Szegö, op. cit., pp. 173—78.

[6]) Ibid., p. 46.

a condition equivalent to (2). Since all x_ν are in (a, b), it follows that

$$0 \leqq f(x_\nu) \leqq \frac{1}{\min \lambda_\nu} \qquad (\nu = 0, 1, \ldots, n).$$

Now, a variable polynomial $f(x) \in \pi_n$ which is bounded at $n+1$ points must have bounded coefficients.[7]) Hence Π_n is bounded in E_{n+1}, Properties (1) and (10) are evidently preserved in passing to the limit of a sequence in E_{n+1}, and so Π_n is compact. Finally, since the mapping

$$(a_0, a_1, \ldots, a_n) \to \sum_{\nu=0}^{n} a_\nu z^\nu$$

from Π_n to R_z is continuous, R_z is also compact.

As a consequence of the possession of these two properties, R_z can be completely described by its function of support. This will be obtained with the aid of quadrature formulae especially constructed in terms of z. We shall prove that for $n > 2$, R_z is the set bounded by a certain ellipse with a focus at zero if $(a, b) = (-\infty, \infty)$, and is the convex hull of the union of two such sets otherwise.

We shall make use of four orthonormal systems, depending on z, with properties analogous to (a) and (b). Viz.

$$\{p_\nu(x)\}, \; \{q_\nu(x)\}, \; \{r_\nu(x)\}, \; \{s_\nu(x)\}$$

defined by

$$\int_a^b p_\mu(x) p_\nu(x)(x-z)(x-\bar{z})\,d\chi(x) = \delta_{\mu\nu}, \tag{11}$$

$$\int_a^b q_\mu(x) q_\nu(x)(x-a)(b-x)(x-z)(x-\bar{z})\,d\chi(x) = \delta_{\mu\nu}, \tag{12}$$

$$\int_a^b r_\mu(x) r_\nu(x)(x-a)(x-z)(x-\bar{z})\,d\chi(x) = \delta_{\mu\nu}, \tag{13}$$

$$\int_a^b s_\mu(x) s_\nu(x)(b-x)(x-z)(x-\bar{z})\,d\chi(x) = \delta_{\mu\nu}, \tag{14}$$

respectively. These systems exist (provided that the quantities a and b which appear in the integrands are finite) because the weight function in each case is a polynomial which is positive in (a, b).

The paper contains four sections. In § 1 we assemble all the necessary quadrature formulae; several classes of formulae are needed depending upon whether (a, b) is a finite interval, a half-line, or the whole real axis, and upon whether n is odd or even.

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[7]) C. de la Vallee Poussin, *Leçons sur l'approximation des fonctions d'une variable réelle*, p. 74.

In § 2 these formulae are applied to obtain a description of R_z. In particular, the value of

$$\max |f(z)|, \qquad f(x) \in \Pi_n$$

(which appears in [6] for the special case $(a, b) = (-\infty, +\infty)$ [8]) is obtained in terms of the system $\{\phi_\nu(x)\}$. The third section contains a discussion of the exceptional cases $n = 1, 2$, and the concluding section is devoted to a proof that R_z varies continuously with z, for all complex z.

§ 1. Some Special Classes of Quadrature Formulae

1:1 *A class of formulae of open type of degree 2k.*

THEOREM I. Let $-\infty \leqq a < b \leqq +\infty$, let $\{p_\nu(x)\}$ be the orthonormal system defined by (11). Then for any real number c, the zeros ξ_ν of

$$\omega(x) = p_k(x) - c p_{k-1}(x), \qquad k \geqq 1, \tag{1:1.1}$$

are real and distinct. To these knots there corresponds a formula

$$\sum_{\nu=1}^{k} P_\nu f(\xi_\nu) + P f(z) + \bar{P} f(\bar{z}) = \int_a^b f(x) d\chi(x) \text{ of deg } 2k. \tag{1:1.2}$$

The coefficients P_ν satisfy

$$P_\nu > 0 \qquad (\nu = 1, 2, \ldots, k). \tag{1:1.3}$$

P is a linear fractional function of c which maps the real axis onto a circle of the complex plane containing the origin as an interior point. For increasing c, P moves clockwise if $\mathscr{I}m\, z > 0$, counterclockwise if $\mathscr{I}m\, z < 0$.

PROOF. The zeros of (1:1.1) are readily studied:[9]) $p_k(x)$ and $p_{k-1}(x)$ have only real and simple zeros, all lying in (a, b), the zeros of $p_{k-1}(x)$ separating those of $p_k(x)$. Hence if

$$p_{k-1}(x) = C_{k-1}(x-\gamma'_1) \ldots (x-\gamma'_{k-1}), \quad (C_{k-1} > 0),$$

we have the partial fraction decomposition

$$p_k(x)/p_{k-1}(x) = Kx + L - \sum_{\nu-1}^{k=1} a_\nu/(x-\gamma'_\nu),$$

where K and all a_ν are positive. Moreover

$$\frac{d}{dx}\{p_k(x)/p_{k-1}(x)\} = K + \sum_{\nu-1}^{k=1} a_\nu/(x-\gamma'_\nu)^2 > 0$$

[8]) Riesz, op. cit., pp. 20—21.
[9]) Ibid., pp. 14—18. See also Szegö, op. cit., p. 45, Theorem 3.3.4.

for all real x where the derivative exists. A graph now reveals the following facts which we state as a lemma for repeated future use.

LEMMA 1. *The zeros ξ_ν of (1:1.1) are real and simple for all real c. Letting $\xi_1 < \xi_2 < \ldots < \xi_k$, we have the further inequalities*

$$-\infty < \xi_1 < \gamma_1' < \xi_2 < \gamma_2' < \ldots < \xi_{k-1} < \gamma_{k-1}' < \xi_k < +\infty.$$

As c increases from $-\infty$ to $+\infty$, each ξ_ν increases continuously taking on all values in the open interval to which it is restricted.

For any real c and $f(x) \in \pi_{k+1}$, let us write the Lagrange interpolation formula based on the $k+2$ points

$$\xi_1 < \xi_2 < \ldots < \xi_k, z, \bar{z}$$

as

$$f(x) = \sum_{\nu=1}^{k} L_\nu(x) f(\xi_\nu) + \frac{\omega(x)(x-\bar{z})}{\omega(z)(z-\bar{z})} f(z) + \frac{\omega(x)(x-z)}{\omega(\bar{z})(\bar{z}-z)} f(\bar{z}),$$

where $L_\nu(x)$ is a polynomial independent of $f(x)$ of degree $k+1$. Integrating this identity we obtain formula (1:1.2) of degree $k+1$, where

$$P_\nu = \int_a^b L_\nu(x) d\chi(x), \qquad P = \int_a^b \frac{\omega(x)(x-\bar{z})}{\omega(z)(z-\bar{z})} d\chi(x).$$

To show that (1:1.2) is actually of degree $2k$, we use Jacobi's classical argument. Let $f(x) \in \pi_{2k}$ and let

$$f(x) = \omega(x)(x-z)(x-\bar{z}) g_{k-2}(x) + h_{k+1}(x) \tag{1:1.4}$$

be the result of dividing $f(x)$ by $\omega(x)(x-z)(x-\bar{z})$, subscripts indicating maximum degree. By (11), (1:1.2), and (1:1.4),

$$\int_a^b f(x) d\chi(x) = \int_a^b h(x) d\chi(x) = \sum_{\nu=1}^{k} P_\nu h(\xi_\nu) + P h(z) + \bar{P} h(\bar{z})$$
$$= \sum_{\nu=1}^{k} P_\nu f(\xi_\nu) + P f(z) + \bar{P} f(\bar{z}).$$

The decisive point in Jacobi's proof is that $\omega(x)$ is orthogonal to an arbitrary $g_{k-2}(x)$ with respect to the "distribution" $(x-z)(x-\bar{z})d\chi(x)$. Hence we may note that the zeros ξ_ν of (1:1.1) are the only knots for which (1:1.2) is of degree $2k$.

To prove (1:1.3), apply (1:1.2) to

$$f(x) = \left(\frac{\omega(x)}{x-\xi_\nu}\right)^2 (x-z)(x-\bar{z}) \in \pi_{2k}, \quad (\nu = 1, 2, \ldots, k).$$

There results

$$P_\nu f(\xi_\nu) = \int_a^b f(x)d\chi(x) > 0,$$

and (1:1.3) follows.

Next we study the mapping

$$(1{:}1.5)\quad P(c) = \frac{\int_a^b p_k(x)(x-\bar{z})\,d\chi(x) - c\int_a^b p_{k-1}(x)(x-\bar{z})\,d\chi(x)}{\left(p_k(z)-cp_{k-1}(z)\right)(z-\bar{z})}.$$

To this end, let c' and c'' be unequal real numbers, and let

$$\sum_{\nu=1}^{k} P'_\nu f(\xi'_\nu) + P'f(z) + \overline{P'}f(\bar{z}) = \int_a^b f(x)d\chi(x) \text{ of degree } 2k$$

and

$$\sum_{\nu=1}^{k} P''_\nu f(\xi''_\nu) + P''f(z) + \overline{P''}f(\bar{z}) = \int_a^b f(x)d\chi(x) \text{ of degree } 2k$$

be the corresponding quadrature formulas. Applying both of these to

$$f(x) = \prod_{\nu=1}^{k} (x-\xi'_\nu)^2 \in \pi_{2k}$$

we obtain

$$(1{:}1.6)\quad \begin{aligned} 0 < \int_a^b f(x)d\chi(x) &= P'f(z) + \overline{P'}f(\bar{z}) \\ &= \sum_{\nu=1}^{k} P''_\nu f(\xi''_\nu) + P''f(z) + \overline{P''}f(\bar{z}). \end{aligned}$$

From (1:1.3) and Lemma 1 it follows that

$$(1{:}1.7)\qquad P''f(z) + \overline{P''}f(\bar{z}) < P'f(z) + \overline{P'}f(\bar{z}).$$

Hence the transformation (1:1.5) is non-constant. Another application of the lemma shows that the pole of (1:1.5) is imaginary. Therefore, (1:1.5) maps the real axis onto a bounded circle. The origin is not outside this circle, for then we could choose c' and c'' so that $DP' = P''$ with $D > 1$, and this would contradict (1:1.7). To prove that the origin is not on the circle, we first observe by means of (1:1.6) that $P' \neq 0$ i.e. P does not vanish for finite c. Hence we need only show that

$$P(\infty) = \frac{\int_a^b p_{k-1}(x)(x-\bar{z})\,d\chi(x)}{p_{k-1}(z)(z-\bar{z})} \neq 0.$$

However, this is precisely the expression for $P(0)$ if k is replaced by $k-1$. (If $k = 1$, there is no "lower" formula but then the inequality is obvious.)

The orientation of the circle depends only on the location of the pole $p_k(z)/p_{k-1}(z)$ of (1:1.5). Namely, if $\mathcal{I}m\, p_k(z)/p_{k-1}(z) > 0$, P moves clockwise with increasing c, and if $\mathcal{I}m\, p_k(z)/p_{k-1}(z) < 0$, P moves counterclockwise. The partial fraction decomposition of $p_k(x)/p_{k-1}(x)$ shows that

$$\operatorname{sgn} \mathcal{I}m\, p_k(z)/p_{k-1}(z) = \operatorname{sgn} \mathcal{I}m\, z$$

and this completes the proof.

REMARK 1. For $c = 0$ and $f(x) \in \pi_{2k+1}$, (1:1.4) becomes

$$f(x) = p_k(x)(x-z)(x-\bar{z})g_{k-1}(x)+h_{k+1}(x).$$

It follows that (1:1.2) is of degree $2k+1$ for this value of c. This formula is especially important and is here recorded as

$$(1:1.8) \quad \sum_{\nu=1}^{k} G_\nu f(\gamma_\nu)+Gf(z)+\bar{G}f(\bar{z}) = \int_a^b f(x)d\chi(x) \text{ of degree } 2k+1,$$

γ_ν being the zeros of $p_k(x)$. We shall refer to (1:1.8) as *Gauss's formula* because of the similarity with (9).

REMARK 2. There are two other formulae in the class (1:1.2) which are especially noteworthy. If $-\infty < a$, Lemma 1 shows that there exists $c_1 < 0$ for which $\xi_1 = a$. (In fact $c_1 = p_k(a)/p_{k-1}(a)$. Denoting the remaining zeros of (1:1.1) by

$$(1:1.9) \qquad \xi_2 = \alpha_1,\ \xi_3 = \alpha_2, \ldots, \xi_k = \alpha_{k-1},$$

we write the special formula so obtained as

$$(1:1.10) \quad A_0 f(a) + \sum_{\nu=1}^{k-1} A_\nu f(\alpha_\nu)+Af(z)+\bar{A}f(\bar{z}) = \int_a^b f(x)d\chi(x) \quad \text{of degree } 2k.$$

The α_ν are evidently the zeros of

$$f_{k-1}(x) = \frac{p_{k-1}(a)p_k(x)-p_k(a)p_{k-1}(x)}{(x-a)} \in \pi_{k-1}.$$

But $f_{k-1}(x)(x-a)$ is clearly orthogonal to any polynomial of degree $k-2$ with respect to $(x-z)(x-\bar{z})d\chi(x)$. Therefore $f_{k-1}(x) = r_{k-1}(x)$ up to a numerical factor, and the α_ν are the zeros of $r_{k-1}(x)$. Similarly, if $b < +\infty$, there exists $c_2 > 0$ for which $\xi_k = b$. Writing

$$(1:1.11) \qquad \xi_1 = \beta_1,\ \xi_2 = \beta_2, \ \ldots, \xi_{k-1} = \beta_{k-1},$$

the corresponding formula will be written

$$(1{:}1.12)\quad \sum_{\nu=1}^{k-1} B_\nu f(\beta_\nu)+B_k f(b)+Bf(z)+\bar{B}f(\bar{z}) = \int_a^b f(x)\,d\chi(x)$$

of degree $2k$.

By an argument similar to the one just used we see that the knots β_ν are the roots of $s_{k-1}(x)$. We shall call (1:1.10) and (1:1.12) the *left-sided and right-sided formulae of Radau* because quite similar formulas were first discovered by Radau.[10]) Finally, let us record here the following inequalities which are evident in view of Lemma 1.

$$(1{:}1.13)\quad \alpha<\beta_1<\gamma_1'<\alpha_1<\beta_2<\gamma_2'<\alpha_2<\ldots<\beta_{k-1}<\gamma_{k-1}'<\alpha_{k-1}<b.$$

1:2 *A class of formulae of closed type of degree 2k.*

THEOREM II. Let $-\infty < a < b < +\infty$, let $\{q_\nu(x)\}$ be the orthonormal system defined by (12). There are two real numbers d_1 and d_2, $d_1 < 0 < d_2$, such that the zeros δ_ν of

$$(1{:}2.1)\qquad \psi(x) = q_{k-1}(x)-dq_{k-2}(x), \qquad k \geqq 2,$$

agree with the knots β_ν and α_ν of the Radau formulae (1:1.12) and (1:1.10) for $d = d_1$ and $d = d_2$ respectively. For every d in the range

$$(1{:}2.2)\qquad d_1 < d < d_2$$

the δ_ν satisfy

$$(1{:}2.3)\qquad \beta_\nu < \delta_\nu < \alpha_\nu \qquad (\nu = 1, 2, \ldots, k-1).$$

To these knots there corresponds a formula

$$(1{:}2.4)\quad Q_0 f(a)+\sum_{\nu=1}^{k-1} Q_\nu f(\delta_\nu)+Q_k f(b)+Qf(z)+\bar{Q}f(\bar{z}) = \int_a^b f(x)d\chi(x)$$

of degree $2k$.

The coefficients Q_ν satisfy

$$(1{:}2.5)\qquad Q_\nu > 0 \qquad (\nu = 0, 1, \ldots, k).$$

Q is a linear fractional function of d which maps the real axis onto a circle of the complex plane containing the origin as an interior point. For increasing d, Q moves clockwise if $\mathscr{Im}\, z > 0$, counterclockwise if $\mathscr{Im}\, z < 0$.

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[10]) R. Radau, „Étude sur les formules d'approximation qui servent à calculer la valeur numérique d'une intégrale définie," *Journal des Mathématiques pures et appliquées,* 3me série, 6: 296, 1880.

The circles described by P and Q intersect at $A = P(c_1) = Q(d_2)$ and $B = P(c_2) = Q(d_1)$ forming a lens containing the origin. The sides of the lens correspond to the ranges $c_1 < c < c_2$ and (1:2.2) respectively.

Proof. To demonstrate the existence of numbers d_1 and d_2 having the stated properties, we use an argument found in [7]. Expanding $r_{k-1}(x)$ in terms of the system $\{q_\nu(x)\}$ yields

$$r_{k-1}(x) = \sum_{\nu=0}^{k-1} \left\{ \int_a^b r_{k-1}(x) q_\nu(x)(x-a)(b-x)(x-z)(x-\bar{z}) dx(x) \right\} q_\nu(x).$$

By the orthogonality properties of $r_{k-1}(x)$, only the last two terms survive. Thus

$$r_{k-1}(x) = Eq_{k-1}(x) + Fq_{k-2}(x),$$

where $E > 0$ and

$$F = \int_a^b r_{k-1}(x) q_{k-2}(x)(x-a)(b-x)(x-z)(x-\bar{z}) d\chi(x).$$

But on expanding $q_{k-2}(x)(b-x)$ in terms of the system $\{r_\nu(x)\}$ we find

$$q_{k-2}(x)(b-x) = Fr_{k-1}(x) + \dots$$

Thus $F < 0$ and $d_2 = -F/E > 0$. A similar argument deals with d_1. Now the polynomial (1:2.1) has properties analogous to (1:1.1). Hence, by Lemma 1, (1:2.3) holds for d in the range (1:2.2). For any such d, integration of Lagrange's formula based on the $k+3$ points

$$a < \delta_1 < \delta_2 < \dots < \delta_{k-1} < b, z, \bar{z}$$

leads to (1:2.4) of degree $k+2$. For $f(x) \in \pi_{2k}$ we have

$$\text{(1:2.6)} \quad f(x) = \psi(x)(x-a)(b-x)(x-z)(x-\bar{z}) g_{k-3}(x) + h_{k+2}(x).$$

By (12), (1:2.4), and (1:2.6) we obtain

$$\int_a^b f(x) d\chi(x) = \int_a^b h(x) d\chi(x)$$
$$= Q_0 h(a) + \sum_{\nu=1}^{k-1} Q_\nu h(\delta_\nu) + Q_k h(b) + Qh(z) + \bar{Q}h(\bar{z})$$
$$= Q_0 f(a) + \sum_{\nu=1}^{k-1} Q_\nu f(\delta_\nu) + Q_k f(b) + Qf(z) + \bar{Q}f(\bar{z}).$$

Thus (1:2.4) is of degree $2k$.

Applying (1:2.4) to

$$f(x)=\left(\frac{\psi(x)}{x-\delta_\nu}\right)^2(x-a)(b-x)(x-z)(x-\bar{z})\in\pi_{2k},\quad(\nu=1,2,\ldots,k-1),$$

yields

$$Q_\nu f(\delta_\nu)=\int_a^b f(x)d\chi(x)>0.$$

Since $a<\delta_\nu<b$, $f(\delta_\nu)>0$. Therefore

$$Q_\nu>0\qquad(\nu=1,2,\ldots,k-1).$$

Now

$$Q_0(d)=\int_a^b\frac{\psi(x)(b-x)(x-z)(x-\bar{z})}{\psi(a)(b-a)(a-z)(a-\bar{z})}d\chi(x)$$

$$=\frac{\int_a^b q_{k-1}(x)(b-x)(x-z)(x-\bar{z})d\chi(x)-d\int_a^b q_{k-2}(x)(b-x)(x-z)(x-\bar{z})d\chi(x)}{(q_{k-1}(a)-dq_{k-2}(a))(b-a)(a-z)(a-\bar{z})}$$

Notice that the denominator vanishes only for a value of d less than d_1. Recalling that the zeros of (1:2.1) are the β_ν for $d=d_1$, we obtain

$$Q_0(d_1)=\int_a^b\frac{s_{k-1}(x)(b-x)(x-z)(x-\bar{z})}{s_{k-1}(a)(b-a)(a-z)(a-\bar{z})}d\chi(x)=0,$$

and by applying (1:1.10) we obtain

$$Q_0(d_2)=\int_a^b\frac{r_{k-1}(x)(b-x)(x-z)(x-\bar{z})}{r_{k-1}(a)(b-a)(a-z)(a-\bar{z})}d\chi(x)=A_0>0.$$

(Observe that the degree of the integrand does not exceed $2k$.) But a real linear fractional function is monotone in any range not including the pole. Hence (1:2.2) implies

$$Q_0>0.$$

A similar argument shows that Q_k decreases from B_k to 0 in the range (1:2.2).

$$Q(d)=\int_a^b\frac{\psi(x)(x-\bar{z})(x-a)(b-x)}{\psi(z)(z-\bar{z})(z-a)(b-z)}d\chi(x)$$

$$=\frac{\int_a^b q_{k-1}(x)(x-\bar{z})(x-a)(b-x)d\chi(x)-d\int_a^b q_{k-2}(x)(x-\bar{z})(x-a)(b-x)d\chi(x)}{(q_{k-1}(z)-dq_{k-2}(z))(z-\bar{z})(z-a)(b-z).}$$

This is the same type of transformation as (1:1.5) with $d\chi(x)$

replaced by $(x-a)(b-x)d\chi(x)$ and k replaced by $k-1$. Hence the stated properties of Q are consequences of Theorem I. Now

$$Q(d_1) = B = P(c_2),$$

the integral being evaluated by (1:1.12), and

$$Q(d_2) = A = P(c_1)$$

by (1:1.10).

To prove the existence of a lens with the described properties, choose d in the range (1:2.2) so that $\arg Q(d) \neq \arg P(\infty)$. Next choose a real c so that

$$\arg P(c) = \arg Q(d).$$

Applying the corresponding formulae (1:2.4) and (1:1.2) to

$$f(x) = \prod_{\nu=1}^{k} (x-\xi_\nu)^2 \in \pi_{2k}$$

we obtain

$$0 < \int_a^b f(x)d\chi(x) = Q_0 f(a) + \sum_{\nu=1}^{k-1} Q_\nu f(\delta_\nu) + Q_k f(b) + Qf(z) + \bar{Q}f(\bar{z})$$
$$= Pf(z) + \bar{P}f(\bar{z}).$$

By (1:2.3) and (1:2.5) we conclude

$$Qf(z) + \bar{Q}f(\bar{z}) < Pf(z) + \bar{P}f(\bar{z})$$

or

$$Q/P \cdot \{Pf(z) + \bar{P}f(\bar{z})\} < Pf(z) + \bar{P}f(\bar{z}).$$

Therefore

$$|Q| < |P|.$$

Stated geometrically, Q describes one side of the lens formed by the intersecting circles, going from B to A as d increases from d_1 to d_2.

It is possible to prove in a similar way that

$$|P| < |Q|$$

if P corresponds to c satisfying $c_1 < c < c_2$ and $\arg Q = \arg P$. However, the rest of the theorem now follows easily from the fact that both circles have the same orientation. Namely, as c increases from c_1 to c_2, P describes a circular arc from A to B. Since the orientation is the same as that of the arc described by Q, this arc must be the other side of the lens.

REMARK 3. For $d = 0$ and $f(x) \in \pi_{2k+1}$ (1:2.6) becomes

$$f(x) = q_{k-1}(x)(x-a)(b-x)(x-z)(x-\bar{z})\,q_{k-2}(x)+h_{k+2}(x).$$

It follows that (1:2.4) is of degree $2k+1$ for $d = 0$. The knots are then the zeros of $q_{k-1}(x)$ which we denote by τ_ν. Also writing $Q_\nu = T_\nu$, $Q = T$ if $d = 0$, (1:2.4) reduces to

$$(1{:}2.7) \quad T_0 f(a)+\sum_{\nu=1}^{k-1} T_\nu f(\tau_\nu)+T_k f(b)+Tf(z)+\bar{T}f(\bar{z}) = \int_a^b f(x)d\chi(x)$$

of degree $2k+1$.

We shall refer to (1:2.7) as *Radau's two-sided formula*, for a similar one was first derived by Radau for $(a, b) = (-1, +1)$.[11])

1:3 *Two classes of formulae of half-closed type of degree* $2k+1$.

THEOREM III. Let $-\infty < a < b \leqq +\infty$, let $\{r_\nu(x)\}$ be the orthonormal system defined by (13). There exists $e_1 < 0$ such that the zeros η_ν of

$$(1{:}3.1) \qquad \phi(x) = r_k(x)-er_{k-1}(x), \qquad k \geqq 1,$$

are identical with the zeros γ_ν of $p_k(x)$ for $e = e_1$. For every e in the range

$$(1{:}3.2) \qquad e_1 < e$$

the η_ν satisfy

$$(1{:}3.3) \qquad \gamma_\nu < \eta_\nu \qquad (\nu = 1, 2, \ldots, k).$$

To these knots there corresponds a formula

$$(1{:}3.4) \quad R_0 f(a)+\sum_{\nu=1}^{k} R_\nu f(\eta_\nu)+Rf(z)+\bar{R}f(\bar{z}) = \int_a^b f(x)d\chi(x)$$

of degree $2k+1$.

The coefficients R_ν satisfy

$$(1{:}3.5) \qquad R_\nu > 0 \qquad (\nu = 0, 1, \ldots, k).$$

R is a linear fractional function of e which maps the real axis onto a circle of the complex plane containing the origin as an interior point. For increasing e, R moves clockwise if $\mathscr{I}m\, z > 0$, counterclockwise if $\mathscr{I}m\, z < 0$.

The circles described by P and R intersect at $A = P(c_1) = R(\infty)$ and $G = P(0) = R(e_1)$ forming a lens containing the origin.

---

[11]) Idem.

The sides of the lens correspond to the ranges $c_1 < c < 0$ and (1:3.2) respectively.

If $b < +\infty$, there exists $e_2 > 0$ for which

$$\eta_\nu = \tau_\nu \quad (\nu = 1, 2, \ldots, k-1), \qquad \eta_k = b,$$

and (1:3.4) is then identical with Radau's two-sided formula (1:2.7). In particular

$$R(e_2) = T = Q(0). \tag{1:3.6}$$

Proof. The existence of a negative e_1, for which the zeros of (1:3.1) are the γ_ν is proved by expanding $p_k(x)$ in terms of the system $\{r_\nu(x)\}$. (See [7]). Hence, by Lemma 1, (1:3.2) implies (1:3.3). For any e in the range (1:3.2), integration of Lagrange's formula based on the $k+3$ points

$$a < \eta_1 < \eta_2 < \ldots < \eta_k, z, \bar{z}$$

leads to (1:3.4) of degree $k+2$. For $f(x) \in \pi_{2k+1}$ we have

$$f(x) = \phi(x)(x-a)(x-z)(x-\bar{z})g_{k-2}(x)+h_{k+2}(x). \tag{1:3.7}$$

By (13), (1:3.4), and (1:3.7) we obtain

$$\int_a^b f(x)d\chi(x) = \int_a^b h(x)d\chi(x) = R_0h(a)+\sum_{\nu=1}^{k} R_\nu h(\eta_\nu)+Rh(z)+\bar{R}h(\bar{z})$$
$$= R_0f(a)+\sum_{\nu=1}^{k} R_\nu f(\eta_\nu)+Rf(z)+\bar{R}f(\bar{z}).$$

Thus (1:3.4) is of degree $2k+1$.

Applying (1:3.4) to

$$f(x) = (\phi(x)/(x-\eta_\nu))^2(x-a)(x-z)(x-\bar{z}) \in \pi_{2k+1}, \quad (\nu = 1, 2, \ldots, k),$$

yields

$$R_\nu f(\eta_\nu) = \int_a^b f(x)d\chi(x) > 0.$$

Since $\eta_\nu > a$, we obtain

$$f(\eta_\nu) > 0.$$

Therefore

$$R_\nu > 0 \qquad (\nu = 1, 2, \ldots, k).$$

Now

$$R_0(e) = \int_a^b \frac{\phi(x)(x-z)(x-\bar{z})}{\phi(a)(a-z)(a-\bar{z})}d\chi(x)$$
$$= \frac{\int_a^b r_k(x)(x-z)(x-\bar{z})d\chi(x)-e\int_a^b r_{k-1}(x)(x-z)(x-\bar{z})d\chi(x)}{(r_k(a)-er_{k-1}(a))\,(a-z)(a-\bar{z})}.$$

Observe that the denominator vanishes only for a value of e less than e_1. To prove that (1:3.2) implies $R_0(e) > 0$, we examine two special values. First

$$R_0(e_1) = \int_a^b \frac{p_k(x)(x-z)(x-\bar{z})}{p_k(a)(a-z)(a-\bar{z})} d\chi(x) = 0.$$

Next we see from (1:3.7) that (1:3.4) is of degree $2k+2$ for $e = 0$. Applying this formula to

$$f(x) = (r_k(x))^2(x-z)(x-\bar{z}) \in \pi_{2k+2},$$

we obtain

$$R_0(0)\cdot f(a) = \int_a^b f(x) d\chi(x) > 0,$$

and the rest of (1:3.5) follows

$$R(e) = \int_a^b \frac{\phi(x)(x-\bar{z})(x-a)}{\phi(z)(z-\bar{z})(z-a)} d\chi(x)$$
$$= \frac{\int_a^b r_k(x)(x-\bar{z})(x-a)d\chi(x) - e\int_a^b r_{k-1}(x)(x-\bar{z})(x-a)d\chi(x)}{(r_k(z)-er_{k-1}(z))\,(z-\bar{z})(z-a)}.$$

This is the analog of (1:1.5) with $d\chi(x)$ replaced by $(x-a)d\chi(x)$. Hence the stated properties of R are consequences of Theorem I. Now

$$R(e_1) = \int_a^b \frac{p_k(x)(x-\bar{z})(x-a)}{p_k(z)(z-\bar{z})(z-a)} d\chi(x) = G = P(0),$$

the integral being evaluated by Gauss's formula (1:1.8). (Observe that the degree of the integrand does not exceed $2k+1$.)

$$R(\infty) = \int_a^b \frac{r_{k-1}(x)(x-\bar{z})(x-a)}{r_{k-1}(z)(z-\bar{z})(z-a)} d\chi(x) = A = P(c_1),$$

this integral evaluated by Radau's left-sided formula (1:1.10). Next, choose e in the range (1:3.2) so that $\arg R(e) \neq \arg P(\infty)$. Then choose a real c so that

$$\arg P(c) = \arg R(e).$$

Applying the corresponding formulae (1:3.4) and (1:1.2) to

$$f(x) = \prod_{\nu=1}^{k} (x-\xi_\nu)^2 \in \pi_{2k},$$

we obtain

$$0 < \int_a^b f(x)d\chi(x) = R_0 f(a) + \sum_{\nu=1}^{k} R_\nu f(\eta_\nu) + Rf(z) + \bar{R}f(\bar{z}) = Pf(z) + \bar{P}f(\bar{z}).$$

By (1:3.3) and (1:3.5)

$$Rf(z) + \bar{R}f(\bar{z}) < Pf(z) + \bar{P}f(\bar{z})$$

or

$$R/P \cdot (Pf(z) + \bar{P}f(\bar{z})) < Pf(z) + \bar{P}f(\bar{z}).$$

Therefore

$$|R| < |P|.$$

Thus, as e increases from e_1 to $+\infty$, R describes one side of the lens formed by the circles of P and R. Since both circles have the same orientation, P describes the other side of this lens as c increases from c_1 to 0.

Finally, suppose $b < +\infty$. Then Lemma 1 implies the existence of $e_2 > 0$ for which $\eta_k = b$. But then (1:3.4) has the same form and degree as Radau's two-sided formula (1:2.7) and hence must be identical with it. Also directly it can be shown that

$$Eq_{k-1}(x)(x-b) = r_k(x) - Fr_{k-1}(x)$$

for suitable positive numbers E and F (See [7]).

THEOREM IV. Let $-\infty \leqq a < b < +\infty$, let $\{s_\nu(x)\}$ be the orthonormal system defined by (14). There exists $m_2 > 0$ such that the zeros μ_ν of

(1:3.8) $$s_k(x) - ms_{k-1}(x), \qquad k \geqq 1,$$

are identical with the zeros γ_ν of $p_k(x)$ for $m = m_2$. For every m in the range

(1:3.9) $$m < m_2$$

the μ_ν satisfy

(1:3.10) $$\mu_\nu < \gamma_\nu \qquad (\nu = 1, 2, \ldots, k).$$

To these knots there corresponds a formula

(1:3.11) $$\sum_{\nu=1}^{k} S_\nu f(\mu_\nu) + S_{k+1} f(b) + Sf(z) + \bar{S}f(\bar{z}) = \int_a^b f(x)d\chi(x)$$

of deg $2k+1$.

The coefficients S_ν satisfy

(1:3.12) $$S_\nu > 0, \qquad (\nu = 1, 2, \ldots, k+1).$$

S is a linear fractional function of m which maps the real axis onto a circle of the complex plane containing the origin as an interior point. For increasing m, s moves clockwise if $\mathscr{I}m\, z > 0$, counterclockwise if $\mathscr{I}m\, z < 0$.

The circles described by P and S intersect at $B = P(c_2) = S(\infty)$ and $G = P(0) = S(m_2)$ forming a lens containing the origin. The sides of the lens correspond to the ranges $0 < c < c_2$ and (1:3.9) respectively.

If $-\infty < a$, there exists $m_1 < 0$ for which

$$\mu_1 = a, \quad \mu_\nu = \tau_{\nu-1} \qquad (\nu = 2, 3, \ldots, k),$$

and (1:3.11) is then identical with Radau's two-sided formula (1:2.7). In particular

$$S(m_1) = T = Q(0). \tag{1:3.13}$$

The circles described by R and S intersect at $G = R(e_1) = S(m_2)$ and $T = R(e_2) = S(m_1)$ forming a lens containing the origin. The sides of the lens correspond to the ranges $e_1 < e < e_2$ and $m_1 < m < m_2$ respectively. The formulae (1:3.4) and (1:3.11) corresponding to these ranges have positive coefficients and knots contained in (a, b).

Proof. The only new feature is the statement concerning the sides of the lens formed by R and S. To prove this it is necessary first to observe that the class of formulae (1:3.4), which corresponds to the range (1:3.2), can be extended (at the sacrifice of (1:3.5)). Indeed, the proof of Theorem III shows that (1:3.4) holds for any real e except $e_a = r_k(a)/r_{k-1}(a)$, for which $\eta_1 = a$. Now choose m in the range $m_1 < m < m_2$ so that $\arg S(m) \neq \arg R(\infty)$, $\arg S(m) \neq \arg R(e_a)$. Next choose a real e so that

$$\arg R(e) = \arg S(m).$$

Applying the corresponding formulae from (1:3.11) and the extended class (1:3.4) to

$$f(x) = (x-a) \prod_{\nu=1}^{k} (x-\eta_\nu)^2 \in \pi_{2k+1},$$

we obtain

$$0 < \int_a^b f(x)\, d\chi(x) = \sum_{\nu=1}^{k} S_\nu f(\mu_\nu) + S_{k+1} f(b) + S f(z) + \bar{S} f(\bar{z})$$
$$= R f(z) + \bar{R} f(\bar{z}).$$

Since the μ_ν are all strictly between a and b, we conclude by means of (1:3.12) that

$$Sf(z)+\bar{S}f(\bar{z}) < Rf(z)+\bar{R}f(\bar{z})$$

or

$$S/R\cdot(Rf(z)+\bar{R}f(\bar{z})) < Rf(z)+\bar{R}f(\bar{z}).$$

Therefore

$$|S| < |R|.$$

Also

$$|R| < |S|$$

if $e_1 < e < e_2$ and $\arg R = \arg S$. This follows either from a similar argument or from our usual considerations of orientation. The proof is now complete.

1:4 *Two classes of formulae involving the leading coefficient of the integrand.*

THEOREM V. Let $-\infty \leqq a < b \leqq +\infty$, $k \geqq 1$. Then for any real c there is a formula

$$(1:4.1)\quad \sum_{\nu=1}^{k} P_\nu f(\xi_\nu)+P_{k+1}C_f+Pf(z)+\bar{P}f(\bar{z}) = \int_a^b f(x)\,d\alpha(x)$$

of deg $2k+1$,

where

$$(1:4.2)\qquad f(x) = C_f x^{2k+1}+\ldots,$$

and c, P, ξ_ν, P_ν ($\nu = 1, 2, \ldots, k$) are the quantities defined in Theorem I. P_{k+1} satisfies

$$(1:4.3)\qquad \operatorname{sgn} P_{k+1} = -\operatorname{sgn} c.$$

PROOF: Let $f(x) = C_f x^{2k+1}+\ldots \in \pi_{2k+1}$. Apply (1:1.2) to

$$f(x)-C_f x^{2k+1} \in \pi_{2k},$$

and (1:4.1) follows with

$$P_{k+1} = \int_a^b x^{2k+1}d\chi(x) - \sum_{\nu=1}^{k} P_\nu \xi_\nu^{2k+1} - Pz^{2k+1} - \bar{P}\bar{z}^{2k+1}.$$

But applying (1:4.1) to

$$f(x) = p_{k-1}(x)(p_k(x)-cp_{k-1}(x))(x-z)(x-\bar{z}) \in \pi_{2k+1}$$

yields

$$P_{k+1}\cdot C_{k-1}\cdot C_k = -c,$$

where C_{k-1}, C_k are the leading coefficients of $p_{k-1}(x)$ and $p_k(x)$

respectively. Since these coefficients are positive, (1:4.3) is proved.

THEOREM VI. Let $-\infty < a < b \leqq +\infty$, let $\{r_\nu(x)\}$ be the orthonormal system defined by (13). There exists $e'_1 < 0$ such that the zeros η'_ν of

$$(1:4.4) \qquad r_{k-1}(x) - e' r_{k-2}(x), \qquad k \geqq 2,$$

are identical with the zeros γ'_ν of $p_{k-1}(x)$ for $e' = e'_1$. For every e' in the range

$$(1:4.5) \qquad e'_1 < e' < 0$$

the η'_ν satisfy

$$(1:4.6) \qquad \gamma'_\nu < \eta'_\nu < \alpha_\nu \qquad (\nu = 1, 2, \ldots, k-1).$$

To these knots there corresponds a formula

$$(1:4.7) \quad R'_0 f(a) + \sum_{\nu=1}^{k-1} R'_\nu f(\eta'_\nu) + R'_k C_f + R' f(z) + \bar{R}' f(\bar{z}) = \int_a^b f(x) d\chi(x)$$

of deg $2k$,

where

$$(1:4.8) \qquad f(x) = C_f x^{2k} + \ldots$$

The coefficients R'_ν satisfy

$$(1:4.9) \qquad R'_\nu > 0 \qquad (\nu = 0, 1, \ldots, k).$$

R' is a linear fractional function of e' which maps the real axis onto a circle of the complex plane containing the origin as an interior point. For increasing e', R' moves clockwise if $\mathscr{I}m\, z > 0$, counterclockwise if $\mathscr{I}m\, z < 0$.

The circles described by P and R' intersect at $A = P(c_1) = R'(0)$ and $G' = P(\infty) = R'(e'_1)$ forming a lens containing the origin. The sides of the lens correspond to the ranges $c_1 < c$ and (1:4.5) respectively.

PROOF. By Theorem III, with k replaced by $k-1$, there exists $e'_1 < 0$ such that the zeros η'_ν of (1:4.4) are identical with the zeros γ'_ν of $p_{k-1}(x)$ for $e' = e'_1$. For $e' = 0$ the η'_ν are the zeros α_ν of $r_{k-1}(x)$. Thus (1:4.5) implies (1:4.6). For any e' in the range

$$e'_1 < e'$$

there is a class of formulae

$$(1:4.10) \quad R'_0 f(a) + \sum_{\nu=1}^{k-1} R'_\nu f(\eta'_\nu) + R' f(z) + \bar{R}' f(\bar{z}) = \int_a^b f(x) d\chi(x)$$

of deg $2k-1$

completely described in Theorem III.

Now let $f(x) = C_f x^{2k} + \dots \in \pi_{2k}$. Apply (1:4.10) to

$$f(x) - C_f x^{2k} \in \pi_{2k-1}$$

and (1:4.7) follows with

$$R'_k = \int_a^b x^{2k} d\chi(x) - R'_0 a^{2k} - \sum_{\nu=1}^{k-1} R'_\nu \eta'^{2k}_\nu - R' z^{2k} - \bar{R}' \bar{z}^{2k}.$$

Applying (1:4.7) to

$$f(x) = r_{k-2}(x) \cdot \left(r_{k-1}(x) - e' r_{k-2}(x)\right) (x-a)(x-z)(x-\bar{z}) \in \pi_{2k}$$

yields

$$R'_k \cdot E_{k-2} \cdot E_{k-1} = -e',$$

where E_{k-2} and E_{k-1} are the leading coefficients of $r_{k-2}(x)$ and $r_{k-1}(x)$ respectively. Hence (1:4.5) implies (1:4.9). All that remains is to discuss the relations between P and R'.

By Theorem III

$$A = P(c_1) = R(\infty) = \int_a^b \frac{r_{k-1}(x)(x-\bar{z})(x-a)}{r_{k-1}(z)(z-\bar{z})(z-a)} d\chi(x).$$

But this is precisely the expression for $R'(0)$. Also by Theorem III

$$G = P(0) = R(e_1).$$

Replacing k by $k-1$ we obtain

$$G' = P(\infty) = R'(e'_1).$$

Finally, choose e' in the range (1:4.5) and a real c so that

$$\arg\ P(c) = \arg R'(e').$$

Applying the corresponding formulae from (1:4.7) and (1:1.2) to

$$f(x) = \prod_{\nu=1}^{k} (x-\xi_\nu)^2 \in \pi_{2k},$$

we obtain

$$0 < \int_a^b f(x) d\chi(x) = R'_0 f(a) + \sum_{\nu=1}^{k-1} R'_\nu f(\eta'_\nu) + R'_k + R' f(z) + \bar{R}' f(\bar{z})$$
$$= P f(z) + \bar{P} f(\bar{z}).$$

By (1:4.9)

$$R' f(z) + \bar{R}' f(\bar{z}) < P f(z) + \bar{P} f(\bar{z})$$

or

$$R'/P \cdot \left(P f(z) + \bar{P} f(\bar{z})\right) < P f(z) + \bar{P} f(\bar{z}).$$

Therefore

$$|R'| < |P|.$$

The rest follows from the fact that P and R' describe identically oriented circles.

§ 2. Applications of § 1 to the Determination of R_z

2:1 $(a, b) = (-\infty, +\infty)$, *n even*

THEOREM VII. Let $(a, b) = (-\infty, +\infty)$, $n = 2k$, $k \geqq 1$. Then R_z is the solid ellipse with principal circle described by $1/2P$, and a focus at the origin.

PROOF. We use only Theorem I. If $f(x) \in \Pi_n$ and c is any real number, we have by (2) and (1:1.2)

$$\sum_{\nu=1}^{k} P_\nu f(\xi_\nu) + Pf(z) + \overline{Pf(z)} = 1. \tag{2:1.1}$$

By (1) and (1:1.3) we obtain

$$2\, \mathscr{Re}\, Pf(z) \leqq 1, \tag{2:1.2}$$

with equality if and only if

$$f(x) = D \cdot \prod_{\nu=1}^{k} (x - \xi_\nu)^2 \tag{2:1.3}$$

for an appropriate positive constant D.

Since P describes a circle about the origin, the same is true of $1/2P$, and if we let

$$\frac{1}{2P} = p(\theta) e^{i\theta}, \tag{2:1.4}$$

(2:1.2) becomes

$$\mathscr{Re}\, e^{-i\theta} f(z) \leqq p(\theta) \tag{2:1.5}$$

for all θ corresponding to finite values of c. But the function of support, $h(\theta)$, of R_z is given by

$$h(\theta) = \max_{f \in \Pi_n} \{\mathscr{Re}\, f(z) \cos\theta + \mathscr{Im}\, f(z) \sin\theta\} = \max_{f \in \Pi_n} \{\mathscr{Re}\, e^{-i\theta} f(z)\}.$$

Since equality is possible in (2:1.5) we obtain

$$h(\theta) = p(\theta), \tag{2:1.6}$$

and this holds for all θ by the continuity of a function of support. It follows that R_z is as described. We shall denote such an elliptical set by E_P. It is constructed from the circle described by P, which in turn depends upon (a, b), $d\chi(x)$, n, and z.

REMARK 4. R_z reduce to a circular disk if and only if the circle described by P is centered at the origin. Since ∞ corresponds to $p_k(z)/p_{k-1}(z)$ under the mapping (1:1.5), the center of this circle corresponds to $\overline{p_k(z)/p_{k-1}(z)}$. Therefore a necessary and sufficient condition for a circular disk is

$$\int_a^b (\overline{p_{k-1}(z)}p_k(x)-\overline{p_k(z)}p_{k-1}(x))\,(x-\bar{z})\,d\chi(x) = 0$$

or

$$\int_a^b (p_{k-1}(z)\,p_k(x)-p_k(z)p_{k-1}(x))\,(x-z)\,d\chi(x) = 0.$$

This is a polynomial equation in z of degree $k+1$ (with roots situated symmetrically about the real axis). Hence R_z is a circular disk if and only if z is one of the imaginary roots of this equation.

THEOREM VIII. The circle described by $1/2P$ is given by

$$\tfrac{1}{2}\{K_k(z, z)+e^{i\theta}K_k(z, \bar{z})\}, \quad \theta \text{ real},$$

where

$$K_k(x, w) = \sum_{\nu=0}^{k} \phi_\nu(x)\phi_\nu(w).$$

PROOF. The "kernel polynomial" $K_k(x, w)$ is characterized by the reproducing property

$$\int_a^b K_k(x, w)f(x)\,d\chi(x) = f(w), \qquad f(x) \in \pi_k.$$

Therefore

$$\int_a^b K_k(x, z)f(x)(x-z)(x-\bar{z})\,d\chi(x) = 0, \qquad f(x) \in \pi_{k-2}.$$

Hence $K_k(x, z)$ is a quasi-orthogonal polynomial of the form

$$Cp_k(x)+Dp_{k-1}(x) \tag{2:1.7}$$

where C and D are complex constants. Similarly $K(x, \bar{z})$ and hence all linear combinations

$$EK_k(x, z)+FK_k(x, \bar{z}) \tag{2:1.8}$$

are quasi-orthogonal. Moreover any polynomial of the form (2:1.7) can be written in the form (2:1.8), for $K_k(x, z)$ and $K_k(x, \bar{z})$ are linearly independent in the two dimensional vector space defined by (2:1.7). To see this multiply the equation

$$EK_k(x, z)+FK_k(x, \bar{z}) \equiv 0$$

by $(x-z)$ and $(x-\bar{z})$ and integrate. In (1:1.5), however, we are concerned with real polynomials. If in (2:1.8) E and F are complex conjugates, then (2:1.8) is real. Conversely if C and D are real in (2:1.7) and if

$$Cp_k(x)+Dp_{k-1}(x) = EK_k(x, z)+FK_k(x, \bar{z}),$$

then

$$\int_a^b (Cp_k(x)+Dp_{k-1}(x))\,(x-\bar{z})\,d\chi(x) = E(z-\bar{z})$$

and

$$\int_a^b (Cp_k(x)+Dp_{k-1}(x))\,(x-z)\,d\chi(x) = F(\bar{z}-z).$$

Thus $\bar{E} = F$ Therefore (1:1.5) becomes

$$P = \frac{\int_a^b (EK_k(x, z)+\bar{E}K_k(x, \bar{z}))\,(x-\bar{z})\,d\chi(x)}{(EK_k(z, z)+\bar{E}K_k(z, \bar{z}))\,(z-\bar{z})}$$

$$= \frac{E}{EK_k(z, z)+\bar{E}K_k(z, \bar{z})} = \frac{1}{K_k(z, z)+e^{i\theta}K_k(z, \bar{z})},$$

and the theorem follows.

Corollary 1.

$$\max_{f\in\Pi_n} |f(z)| = \frac{|K_k(z, z)|+K_k(z, \bar{z})}{2}.$$

Proof. By Theorem VII we have

$$\max_{f\in\Pi_n} |f(z)| = \max \left|\frac{1}{2P}\right|,$$

and by Theorem VIII we obtain

$$\max \left|\frac{1}{2P}\right| = \frac{|K_k(z, z)|+K_k(z, \bar{z})}{2}.$$

This result agrees with the remark

$$\max_{f\in\Pi_{2k}} |f(z)| = \frac{\left|\begin{vmatrix} 0 & z^j \\ z^i & c_{i+j} \end{vmatrix}_0^k\right| - \begin{vmatrix} 0 & z^j \\ \bar{z}^i & c_{i+j} \end{vmatrix}_0^k}{2|c_{i+j}|_0^k}, \quad c_\nu = \int_a^b x^\nu d\chi(x),$$

of M. Riesz,[12]) for

$$K_k(x, w) = \frac{-\begin{vmatrix} 0 & x^j \\ w^i & c_{i+j} \end{vmatrix}_0^k}{|c_{i+j}|_0^k}.$$

[12]) Riesz, op. cit., pp. 20—21.

2:2 (a, b) *finite*, n *even*, $n \geqq 4$

THEOREM IX. Let $-\infty < a < b < +\infty$, $n = 2k$, $k \geqq 2$. Then R_z is the convex hull of the union $E_P \cup E_Q$. (See Fig. 1).

PROOF. We use Theorems I and II. The analog of (2:1.1) is again valid for $f(x) \in \Pi_n$. If $c_1 < c < c_2$, then by Lemma 1 all ξ_ν are in (a, b). Hence (2:1.2) and (2:1.3) follow as before but only for these values of c. Therefore

$$h(\theta) = p(\theta), \qquad c_1 < c < c_2. \tag{2:2.1}$$

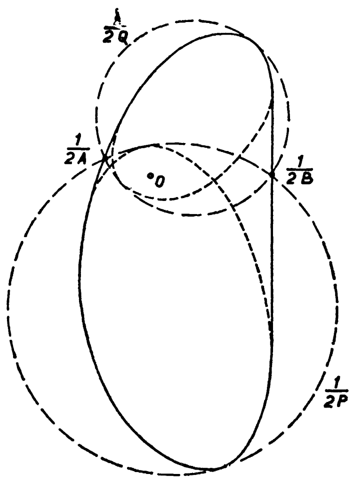

Fig. 1. R_z in the case (a, b) finite, n even, $n \geqq 4$.

If d is the range (1:2.2), we have by (1:2.4)

$$Q_0 f(a) + \sum_{\nu=1}^{k-1} Q_\nu f(\delta_\nu) + Q_k f(b) + Qf(z) + \overline{Qf(z)} = 1. \tag{2:2.2}$$

By (1:2.3) and (1:1.13) the knots δ_ν are in (a, b), and by (1:2.5) the coefficients Q_ν $(\nu = 0, 1, \ldots, k)$ are positive. Therefore

$$2\, \mathscr{Re}\, Qf(z) \leqq 1 \tag{2:2.3}$$

with equality if and only if

$$f(x) = D(x-a)(b-x) \prod_{\nu=1}^{k-1} (x-\delta_\nu)^2 \tag{2:2.4}$$

for an appropriate D. Setting

$$\frac{1}{2Q} = q(\theta) e^{i\theta} \tag{2:2.5}$$

we obtain

$$(2:2.6) \qquad h(\theta) = q(\theta), \qquad d_1 < d < d_2.$$

From the relationship between P and Q described in Theorem II it follows that $h(\theta)$ is now described for all values of θ except $\arg 1/2A$ and $\arg 1/2B$. Furthermore, since $p(\theta)$ and $q(\theta)$ are related inversely as $|P|$ and $|Q|$, (2:2.1) and (2:2.6) can be combined as

$$(2:2.7) \qquad h(\theta) = \max\,\{p(\theta), q(\theta)\},$$

and this holds for all θ by continuity of $h(\theta)$. But the function (2:2.7) is the function of support of the smallest convex set containing both E_P and E_Q, the sets associated with $p(\theta)$ and $q(\theta)$.[13])

Remark 5. It could have been proved directly that (2:2.1) holds for $c_1 \leqq c \leqq c_2$. For example, let $c = c_1$. Then (2:1.2) follows as before but (2:1.3) is now replaced by a whole set of extremal polynomials. This follows from the fact that $\xi_1 = a$ so that a double zero here is no longer necessary. (The plurality of extremal functions indicates that the boundary of R_z is straight in the direction $\theta = \arg 1/2A$.) In the future, however, it will sometimes be necessary to rely on the continuity of $h(\theta)$ in similar situations.

Corollary 2.

$$\max_{f\in\Pi_n} |f(z)| = \tfrac{1}{2}\max\Big\{|K_k(z, z)| + K_k(z, \bar{z}),\; |z-a|\cdot|b-z|\,\big(|\tilde{K}_{k-1}(z, z)| + \tilde{K}_{k-1}(z, \bar{z})\big)\Big\},$$

where $\tilde{K}_{k-1}$ is the kernel of degree $k-1$ of the distribution

$$(x-a)(b-x)d\chi(x).$$

Proof. From Corollary 1 and the formula for Q, we obtain

$$\max\left|\frac{1}{2Q}\right| = \frac{|z-a|\cdot|b-z|\,\big(|\tilde{K}_{k-1}(z, z)| + \tilde{K}_{k-1}(z, \bar{z})\big)}{2}.$$

Since

$$\max_{f\in\Pi_n} |f(z)| = \max\left\{\max\left|\frac{1}{2P}\right|,\; \max\left|\frac{1}{2Q}\right|\right\},$$

the assertion is proved. The results of corollaries 1 and 2 are gene-

[13]) G. Polya, „Untersuchungen über Lücken und Singularitäten von Potenzreihen." *Mathematische Zeitschrift*, 29: 577, 1928—9.

ralizations of formulae for real z.[14]). Similar corollaries hold for the remaining theorems of § 2.

2:3 *(a, b) half infinite, n odd, $n \geqq 3$*

THEOREM X. Let $-\infty < a < b = +\infty$, $n = 2k+1$, $k \geqq 1$. Then R_z is the convex hull of the union $E_P \cup E_R$.

PROOF. We use Theorems III and V. For $f \in \Pi_n$ and e in the range (1:3.2) we have by (2) and (1:3.4)

$$R_0 f(a) + \sum_{\nu=1}^{k} R_\nu f(\eta_\nu) + Rf(z) + \overline{Rf(z)} = 1 \tag{2:3.1}$$

By (1:3.3) and (1:3.5) we obtain

$$2\,\mathcal{R}e\, Rf(z) \leqq 1, \tag{2:3.2}$$

with equality if and only if

$$f(x) = D(x-a) \prod_{\nu=1}^{k} (x-\eta_\nu)^2. \tag{2:3.3}$$

Setting

$$\frac{1}{2R} = r(\theta) e^{i\theta} \tag{2:3.4}$$

there follows

$$h(\theta) = r(\theta), \qquad e_1 < e. \tag{2:3.5}$$

For any real c we obtain by (1:4.1)

$$\sum_{\nu=1}^{k} P_\nu f(\xi_\nu) + P_{k+1} C_f + Pf(z) + \overline{Pf(z)} = 1. \tag{2:3.6}$$

Since $f(x) \geqq 0$ for $x \geqq a$, we observe that

$$C_f \geqq 0. \tag{2:3.7}$$

For c in the range

$$c_1 < c < 0, \tag{2:3.8}$$

the knots ξ_ν are in (a, b) and the coefficients $P_\nu (\nu = 1, 2, \ldots, k+1)$ are positive by (1:1.3) and (1:4.3). Therefore

$$2\,\mathcal{R}e\, Pf(z) \leqq 1, \tag{2:3.9}$$

with equality if and only if

$$C_f = 0$$

[14]) Riesz, op. cit., p. 20. See also Szegö, op. cit., p. 178.

and hence

$$f(x) = D\prod_{\nu=1}^{k}(x-\xi_\nu)^2 \in \Pi_{n-1}. \tag{2:3.10}$$

Therefore

$$h(\theta) = p(\theta), \qquad c_1 < c < 0. \tag{2:3.11}$$

By Theorem III and the continuity of $h(\theta)$ we now can combine (2:3.5) and (2:3.11) into

$$h(\theta) = \max\{p(\theta), r(\theta)\}, \quad \text{all } \theta, \tag{2:3.12}$$

and the theorem follows.

THEOREM XI. Let $-\infty = a < b < +\infty$, $n = 2k+1$, $k \geqq 1$. Then R_z is the convex hull of the union $E_P \cup E_S$.

PROOF. We use Theorems IV and V. The analog of (2:3.6) is again valid, but (2:3.7) becomes

$$C_f \leqq 0. \tag{2:3.13}$$

For c in the range

$$0 < c < c_2 \tag{2:3.14}$$

the knots ξ_ν are in (a, b), and the coefficients are positive with the exception of P_{k+1}, which is negative. Therefore the appropriate version of (2:3.9) again follows. The remainder of the proof has no new feature.

2:4 (a, b) *finite*, n *odd*, $n \geqq 3$

THEOREM XII. $-\infty < a < b < +\infty$, $n = 2k+1$, $k \geqq 1$. Then R_z is the convex hull of the union $E_R \cup E_S$.

PROOF. We use Theorems III and IV. For $f(x) \in \Pi_n$ and e in the range (1:3.2), the analog of (2:3.1) again holds. If e also satisfies

$$e_1 < e < e_2 \tag{2:4.1}$$

then all the knots η_ν are in (a, b). Therefore we obtain

$$h(\theta) = r(\theta), \qquad e_1 < e < e_2. \tag{2:4.2}$$

Similarly, from (1:3.11) and (1:3.12) there follows

$$h(\theta) = s(\theta), \qquad m_1 < m < m_2, \tag{2:4.3}$$

where

$$\frac{1}{2S} = s(\theta)e^{i\theta}. \tag{2:4.4}$$

By the conclusing statements of Theorem IV, (2:4.2) and (2:4.3) can be combined into

$$h(\theta) = \max \{r(\theta), s(\theta)\}, \quad \text{all } \theta, \tag{2:4.5}$$

and the theorem follows.

2:5 *(a, b) half infinite, n even, $n \geqq 4$*

THEOREM XIII. Let $-\infty < a < b = +\infty$, $n = 2k$, $k \geqq 2$. Then R_z is the convex hull of the union $E_P \cup E_{R'}$.

PROOF. We use Theorems I and VI. Proceeding as in Theorem VII we obtain

$$h(\theta) = p(\theta), \qquad c_1 < c, \tag{2:5.1}$$

the restriction on c making all knots lie in (a, b). For e' in the range (1:4.5) and $f(x) \in \Pi_n$, we have by (1:4.7)

$$R_0'f(a) + \sum_{\nu=1}^{k-1} R_\nu' f(\eta_\nu') + R_k' C_f + R'f(z) + \overline{R'f(z)} = 1, \tag{2:5.2}$$

where the coefficients R_ν' $(\nu = 0, 1, \ldots, k)$ are positive according to (1:4.9). Since $C_f \geqq 0$ and all knots are in (a, b) by (1:4.6), there follows

$$2\, \mathcal{Re}\, R'f(z) \leqq 1, \tag{2:5.3}$$

with equality if and only if

$$C_f = 0$$

and

$$f(x) = D(x-a) \prod_{\nu=1}^{k-1} (x-\eta_\nu')^2 \in \Pi_{n-1}. \tag{2:5.4}$$

It follows that

$$h(\theta) = r'(\theta), \qquad e_1' < e' < 0, \tag{2:5.5}$$

where

$$\frac{1}{2R'} = r'(\theta) e^{i\theta}.$$

By the concluding statements of Theorem VI, we have

$$h(\theta) = \max \{p(\theta), r'(\theta)\}, \tag{2:5.6}$$

and the theorem follows.

REMARK 6. A similar theorem holds in the case $-\infty = a < b < +\infty$, $n = 2k$, $k \geqq 2$.

§ 3. The Special Cases n = 1, 2

3:1 $n = 1$

THEOREM XIV. For $n = 1$, R_z is the line segment joining $(z-a)/\int_a^b(x-a)d\chi(x)$ and $(b-z)/\int_a^b(b-x)d\chi(x)$, where either fraction is to be interpreted as its limit $1/\int_a^b d\chi(x)$ if the appropriate number a or b is infinite.

PROOF. We use a direct method not involving quadrature formulae: Let $f(x) = cx+d \in \Pi_n$. Using the abbreviations

$$\mu_0 = \int_a^b d\chi(x), \qquad \mu_1 = \int_a^b xd\chi(x)$$

we have

$$1 = c\mu_1+d\mu_0.$$

Elimination of d yields

$$f(x) = c(x-\mu_1/\mu_0)+1/\mu_0, \tag{3:1.1}$$

a condition equivalent to (2). Assuming first that (a, b) is finite, we observe that property (1) of the class Π_n is equivalent to the pair of inequalities

$$0 \leqq f(a), \qquad 0 \leqq f(b). \tag{3:1.2}$$

This in turn is equivalent to

$$0 \leqq c\int_a^b (a-x)d\chi(x)+1, \qquad 0 \leqq c\int_a^b (b-x)d\chi(x)+1$$

or

$$-1\Big/\int_a^b (b-x)d\chi(x) \leqq c \leqq 1\Big/\int_a^b (x-a)d\chi(x). \tag{3:1.3}$$

By (3:1.1) we conclude that R_z is the line segment

$$c(z-\mu_1/\mu_0)+1/\mu_0, \tag{3:1.4}$$

where c assumes all values in the range (3:1.3), and this is equivalent to the statement of the theorem.

If $-\infty < a < b = +\infty$, then (1) is equivalent to

$$0 \leqq f(a), \qquad 0 \leqq c \tag{3:1.5}$$

from which we obtain

$$0 \leqq c \leqq 1\Big/\int_a^b (x-a)d\chi(x). \tag{3:1.6}$$

By the above convention, (3:1.1) and (3:1.6) together are equivalent to the statement of the theorem. In a similar way the theorem is seen to hold if $-\infty = a < b < +\infty$.

If both a and b are infinite, Π_1 reduces to the trivial class Π_0, and the line segment defined above reduces to the single point $1/\int_a^b d\chi(x)$ as it should.

3:2 $n = 2$

THEOREM XV. Let $-\infty < a < b < +\infty$, $n = 2$. Then R_z is the convex hull of the union

$$E_P \cup \left\{(z-a)(b-z)\Big/\int_a^b (x-a)(b-x)d\chi(x)\right\}.$$

(See Fig. 2)

PROOF. Proceeding as in Theorem IX, we employ the formula

(3:2.1) $P_1 f(\xi_1)+Pf(z)+\overline{Pf(\bar{z})} = \int_a^b f(x)d\chi(x)$ of degree 2

to conclude

(3:2.2) $h(\theta) = p(\theta), \qquad c_1 \leqq c \leqq c_2.$

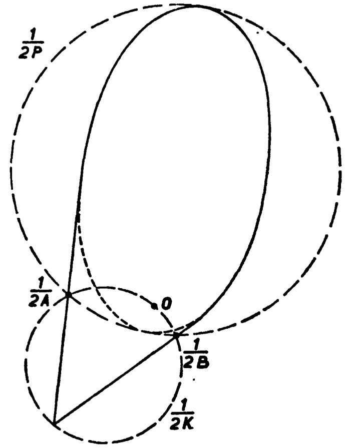

Fig. 2. R_z in the case $n = 2$, (a, b) finite.

However, since Theorem II holds only for $k \geqq 2$, a new class of formulae is needed to obtain $h(\theta)$ for other values of θ. We construct a "convex" one-parameter family of formulae which "joins continuously" with the Radau formulae

(3:2.3) $A_0 f(a)+Af(z)+\bar{A}f(\bar{z}) = \int_a^b f(x)d\chi(x)$ of degree 2

and

(3:2.4) $B_1 f(b)+Bf(z)+\bar{B}f(\bar{z}) = \int_a^b f(x)d\chi(x)$ of degree 2 associated with $c = c_1$ and $c = c_2$ respectively. For d in the range

$$0 < d < 1 \tag{3:2.5}$$

we apply (3:2.3) to $df(x)$, and (3:2.4) to $(1-d)f(x)$, and add. There results

$$K_0 f(a)+K_1 f(b)+Kf(z)+\bar{K}f(\bar{z}) = \int_a^b f(x)d\chi(x) \text{ of deg } 2, \tag{3:2.6}$$

where

$$K_0 = dA_0, \quad K_1 = (1-d)B_1, \quad K = dA+(1-d)B. \tag{3:2.7}$$

Thus the coefficients K_0 and K_1 of (3:2.6) are positive, while K describes the line segment between A and B. This line does not pass through the origin as is evident on applying (3:2.6) to

$$f(x) = (x-a)(b-x).$$

Applying (3:2.6) to the members of Π_2 and setting

$$\frac{1}{2K} = k(\theta)e^{i\theta}, \tag{3:2.8}$$

we obtain

$$h(\theta) = k(\theta), \qquad 0 < d < 1. \tag{3:2.9}$$

These values of θ must be disjoint from those in (3:2.2). Therefore $h(\theta)$ is now described for all θ. Finally, since the extremal polynomial associated with (3:2.9) is

$$f(x) = (x-a)(b-x)\Big/\int_a^b (x-a)(b-x)d\chi(x) \tag{3:2.10}$$

independent of θ, (3:2.2) and (3:2.9) combine to give the stated result.

THEOREM XVI. Let $-\infty < a < b = +\infty$, $n = 2$. Then R_z is the convex hull of the union

$$E_P \cup \left\{(z-a)\Big/\int_a^b (x-a)d\chi(x)\right\}.$$

PROOF. Beginning as before we use formula (3:2.1) to obtain

$$h(\theta) = p(\theta), \qquad c_1 \leqq c < \infty. \tag{3:2.11}$$

Next, we observe that there is a formula

$$L_1 C_f+Lf(z)+\bar{L}f(\bar{z}) = \int_a^b f(x)d\chi(x) \text{ of degree } 2, \tag{3:2.12}$$

where

$$(3{:}2.13)\qquad f(x) = C_f x^2 + \ldots, \qquad L = \int_a^b \frac{x-\bar z}{z-\bar z}\, d\chi(x),$$
$$L_1 = \int_a^b x^2 d\chi(x) - Lz^2 - \bar L \bar z^2$$

derived as in Theorem V. Applying this formula to

$$f(x) = (x-z)(x-\bar z)$$

we obtain

$$(3{:}2.14)\qquad L_1 = \int_a^b (x-z)(x-\bar z)\, d\chi(x) > 0.$$

For any e in the range

$$(3{:}2.15)\qquad 0 < e < 1$$

we apply (3:2.3) to $ef(x)$, and (3:2.12) to $(1-e)f(x)$, obtaining

$$(3{:}2.16)\quad M_0 f(a) + M_1 C_f + Mf(z) + \bar M f(\bar z) = \int_a^b f(x)\, d\chi(x) \text{ of degree 2,}$$

where

$$(3{:}2.17)\quad M_0 = eA_0, \quad M_1 = (1-e)L_1, \quad M = eA + (1-e)L.$$

Thus the coefficients M_0 and M_1 of (3:2.16) are positive, while M describes the line segment between $A = P(c_1)$ and $L = P(\infty)$. Applying (3:2.16) to

$$f(x) = (x-a),$$

we see that this line does not pass through the origin. The rest follows as in Theorem XV.

Remark 7. For the case $-\infty = a < b < +\infty$, $n = 2$ a similar argument shows that R_z is the convex hull of the union

$$E_P \cup \left\{(b-z) \Big/ \int_a^b (b-x)\, d\chi(x)\right\}.$$

The case $(a, b) = (-\infty, +\infty)$, $n = 2$ is of course included in Theorem VII.

§ 4. The Variation of Rz with z

4:1 *A lemma on continuity*

Lemma 2. Let z_0 be any complex number, let $\varepsilon > 0$. Then there exists $\delta = \delta(\varepsilon, z_0) > 0$ such that $|z - z_0| < \delta$ implies $|f(z) - f(z_0)| < \varepsilon$ for any $f \in \Pi_n$ and any complex number z.

Proof. Suppose the lemma is false. Then for some complex number z_0 there exist $\varepsilon > 0$, a sequence $\{z_\nu\}$, and a sequence

$\{f_\nu\} \subset \Pi_n$ such that $\{z_\nu\} \to z_0$, and $|f_\nu(z_\nu)-f_\nu(z_0)| \geqq \varepsilon$ for all ν. Since Π_n is compact in E_{n+1}, we can assume $\{f_\nu\} \to f \in \Pi_n$. The inequality

$$|f(z)-f_\nu(z)| = |\sum_{\mu=0}^{n} a_\mu z^\mu - \sum_{\mu=0}^{n} a_\mu^{(\nu)} z^\mu| \leqq \sum_{\mu=0}^{n} |a_\mu - a_\mu^{(\nu)}| \cdot |z|^\mu$$

now shows that $\{f_\nu(z)\} \to f(z)$ uniformly in any bounded set.

It follows that $f(z)$ is discontinuous at $z = z_0$. Indeed, let $N(z_0)$ be an arbitrary neighborhood of z_0. Choose ν so that $z_\nu \in N(z_0)$ and so that $|f(z)-f_\nu(z)| \leqq \varepsilon/3$ for all z in some compact set containing $\{z_\nu\}$. Then

$$|f(z_\nu)-f(z_0)| \geqq |f_\nu(z_\nu)-f_\nu(z_0)| - |f(z_\nu)-f_\nu(z_\nu)| - |f(z_0)-f_\nu(z_0)| \geqq \varepsilon/3.$$

This absurdity being reached, the lemma is established.

4:2 *A continuity theorem*

THEOREM XVII. If $h(\theta)$ and $h_0(\theta)$ are the functions of support of R_z and R_{z_0} respectively, then

$$\lim_{z\to z_0} h(\theta) = h_0(\theta) \qquad \text{uniformly in } \theta.$$

PROOF. Let ε be an arbitrary positive number. For any θ there exist appropriate polynomials $f(x)$ and $f_0(x)$ in the class Π_n such that

$$h(\theta) = \mathscr{Re}\, e^{-i\theta} f(z)$$

and

$$h_0(\theta) = \mathscr{Re}\, e^{-i\theta} f_0(z_0).$$

If $|z-z_0| < \delta(\varepsilon, z_0)$, then by Lemma 2 and the definition of a function of support we obtain

$$h_0(\theta) \geqq \mathscr{Re}\, e^{-i\theta} f(z_0) > \mathscr{Re}\, e^{-i\theta} f(z) - \varepsilon$$

or

$$h_0(\theta) > h(\theta) - \varepsilon.$$

Similarly

$$h(\theta) > h_0(\theta) - \varepsilon.$$

Thus

$$|z-z_0| < \delta(\varepsilon, z_0)$$

implies

$$|h(\theta)-h_0(\theta)| < \varepsilon \qquad \text{for all } \theta.$$

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