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Distribution modulo 1 of the powers of real numbers larger than 1*

by

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1. A class of algebraic integers

Since the fundamental work of H. Weyl in 1916 on the theory of uniform distribution modulo 1, exhaustive studies have been made about the distribution of \( f(n) \) when \( f(n) \) is a polynomial in \( n \), or when \( f(n) \) increases slower than a polynomial in \( n \) (e.g. \( f(n) = n^\alpha \), \( \alpha > 0 \)) or \( f(n) = \log^\alpha n \).

In comparison, very little is known on the distribution of the simpler functions increasing faster than any power of \( n \), in particular of the distribution of \( \theta^n \) when \( \theta \) is a real number larger than one. Whereas Koksma [1] proved in 1935 that \( \theta^n \) is uniformly distributed except for a set of \( \theta \) of measure zero, nothing is known for the simplest individual values of \( \theta \), such as, e.g. \( \theta = e \), or \( \theta = \frac{3}{2} \).

Pisot, in 1938 ([2]) concentrated his work on the study of those values of \( \theta \), for which, far from being uniformly distributed, \( \theta^n \) is, in a certain sense, as badly distributed as possible. It had been known, for a time, that if we denote by \( S \) the class of algebraic integers whose conjugates lie all (except \( \theta \) itself) inside the unit circle then \( \theta^n \) (mod. 1) tends to zero for \( n = \infty \). This is true also for \( \lambda \theta^n \) when \( \lambda \) is any algebraic integer of the field \( K(\theta) \) generated by \( \theta \) over the rationals.

Let us denote by \( ||a|| \) the distance between \( a \) and the nearest integer. Pisot [2] proved that if \( \theta > 1 \) is such that there exists a real \( \lambda \) with the property that

\[
\sum ||\lambda \theta^n||^2 < \infty
\]

then \( \theta \) belongs to the class \( S \) and \( \lambda \) is an algebraic number of the field \( K(\theta) \).

The question whether this important theorem is true if we replace (1) by the weaker condition

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is open, except if we know beforehand that \( \theta \) is algebraic \([8]\), in which case we can again assert that \( \theta \in S \) and \( \lambda \in K(\theta) \).

In other words, the open problem can be stated as follows: Do there exist transcendental \( \theta \) with the property that, for some \( \lambda \), (2) holds?

\[ ||\lambda \theta^n|| \to 0 \quad (n = \infty) \]

2. Another remarkable class of algebraic integers

Instead of considering the class \( S \) of algebraic integers \( \theta \) such that all the conjugates of \( \theta \) (except \( \theta \) itself) have moduli strictly inferior to 1, Salem \([4]\) has introduced the class \( T \) of algebraic integers \( \tau > 1 \) such that the conjugates \( \alpha_K \) of \( \tau \) have their moduli \( |\alpha_K| \leq 1 \), the equality being permitted.

One sees immediately, (since if \( |\alpha_K| = 1 \), \( \overline{\alpha_K} = 1/\alpha_K \)), that if \( \tau \) does not belong to \( S \), i.e. if there exist actually conjugates with moduli 1, \( \tau \) is the root of an irreducible reciprocal equation of even degree, whose roots lie all on the circumference of the unit circle, the only conjugate lying inside the unit circle being \( 1/\tau \).

The distribution modulo 1 of \( \tau^m \) has interesting properties:

1°) The numbers \( \tau^m \) reduced modulo 1 are everywhere dense.

The proof of this result is based on the following lemma. Let \( 2K \) be the degree of \( \tau \) and \( \alpha_j = e^{2\pi i \omega_j}, \overline{\alpha_j} = e^{-2\pi i \omega_j} \) (\( j = 1, \ldots, K-1 \)) be the conjugates of \( \tau \) having moduli 1. Then \( \omega_1, \ldots, \omega_{K-1} \) are linearly independent.

Let \( A_j \) be the rational integers,\n
\[ \alpha_1^{A_1} \cdots \alpha_{K-1}^{A_{K-1}} = 1 \]

is an impossible equality.

Since the equation having the root \( \tau \) is irreducible, its Galois group is transitive, i.e. there exists an automorphism \( \sigma \) of the Galois group sending, e.g. the root \( \alpha_1 \) into the root \( \tau \). This automorphism cannot send \( \alpha_j \) (\( j \neq 1 \)) into \( 1/\tau \) for since \( \sigma(\alpha_1) = \tau \), \( \sigma(1/\alpha_1) = 1/\tau \), and thus this would imply \( \alpha_j = 1/\alpha_1 \) which is not the case. Thus the automorphism applied to (8) would give

\[ \tau^{A_1} \alpha_2^{A_2} \cdots \alpha_{K-1}^{A_{K-1}} = 1 \]

if \( \sigma(\alpha_j) = \alpha'_j \) (\( j \neq 1 \)). This is clearly impossible since \( \tau > 1 \) and \( |\alpha'_j| = 1 \). Hence the proof of the linear independence of \( \alpha_1 \cdots \alpha_{K-1} \) and 1.

Now, we have modulo 1
\( \tau^m + \frac{1}{\tau^m} + \sum_{j=1}^{K-1} (e^{2\pi i m \omega_j} + e^{-2\pi i m \omega_j}) = 0 \)

or

\[
\tau^m + 2 \sum_{j=1}^{K-1} \cos 2\pi m \omega_j \to 0 \pmod{1}
\]

as \( m \to \infty \). But, by Kronecker's theorem and the linear independence just proved we can determine \( m \) arbitrarily large, such that

\[
2 \sum_{j=1}^{K-1} \cos 2\pi m \omega_j
\]

be arbitrarily close to any number given in advance (mod. 1). Take, e.g. \( m \) such that

\[
|m \omega_1 - \delta| < \varepsilon \pmod{1}
\]

\[
|m \omega_j - \frac{1}{2}| < \varepsilon \pmod{1} \quad (j = 2, 3, \ldots, K-1).
\]

Thus \( \{\tau^m\} \pmod{1} \) is everywhere dense.

(The same argument, applied to \( \lambda \tau^m \), \( \lambda \) being an integer of \( K(\tau) \) shows that \( \{\lambda \tau^m\} \) is everywhere dense in a certain interval).

2°) The powers \( \tau^m \) of a number \( \tau \) of the class \( T \), although everywhere dense modulo 1, are not uniformly distributed modulo 1.

We prove first the following lemma.

**Lemma.** If the \( p \)-dimensional vector \( \{u_j^n\}_{n=1}^{\infty} \) (\( j = 1, 2, \ldots, p \)) is uniformly distributed (mod. 1) in the \( p \)-dimensional euclidean space \( R^p \), the sequence

\[
v_n = \omega(u_1^n) + \omega(u_2^n) + \ldots + \omega(u_p^n),
\]

where \( \omega(x) \) is continuous with period 1, is uniformly distributed if and only if the condition

\[
\int_0^1 e^{2\pi i h \omega(x)} \, dx = 0
\]

is satisfied for all integers \( h \neq 0 \).

(A different form of this condition is that the distribution function of \( \omega(x) \) (mod. 1) should be linear.)

To prove the lemma, remark that \( \{v_n\} \) is uniformly distributed (mod. 1) if and only if

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i h v_n} = 0 \quad (h \text{ integer } \neq 0).
\]

But

\[
\frac{1}{N} \sum_{n=1}^{N} e^{2\pi i h \omega(u_1^n) + \ldots + \omega(u_p^n)} \to \left( \int_0^1 e^{2\pi i h \omega(x)} \, dx \right)^p.
\]
Now in our case, in view of (4) we have to prove that the sequence
\[ v_m = 2 \cos 2\pi m \omega_1 + \ldots + 2 \cos 2\pi m \omega_{K-1} \]
is not uniformly distributed (mod. 1).

In view of the lemma it is enough to prove that
\[ \int_0^1 e^{i \pi \cos 2\pi x} \, dx = J_0(4\pi h), \]
\(J_0\) being the Bessel function of order zero, does not vanish for all integers \( h \neq 0 \), which is true.

(If we use the second form of the lemma, we would have to show that the distribution function (mod. 1) of \( 2 \cos 2\pi x \) is not linear, which can be proved by direct computation.)

**Unsolved problems.**

Besides the unsolved problem quoted at the end of § 1, we quote the following one:

I) Salem has proved [5] that the set of all numbers \( \theta \) of the class \( S \) is closed.

Little is known about the set of numbers \( \tau \) of the class \( T \), except [4] that every number \( \theta \) of the class \( S \) is a limit point (on both sides) for numbers of \( T \). It is not known what are the other limit points of \( T \), if any.

II) Pisot [6] has proved that if there exists \( \lambda \geq 1 \) such that
\[ ||\lambda \theta^n|| \leq \varepsilon \quad \text{for all} \quad n = 0, 1, \ldots, \]
where
\[ \varepsilon = \frac{1}{2e \theta (\theta + 1)(1 + \log \lambda)} \]
then \( \theta \in S \cup T \).

It is still an open question whether there exists a theorem including both results (1) and (5).

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