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Metrical problems concerning continued fractions *

by

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§ 1.

A) Let θ be a real number ($0 \leqq \theta \leqq 1$).

We expand θ in the continued fraction:

$$(1) \qquad \theta = \{a_1, a_2, a_3, \ldots\} = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ldots}}}$$

with integer partial quotients $a_n \geqq 1$.

For irrational θ we define the sequence of functions θ_n as follows:

$$(2) \qquad \begin{cases} \theta_0 = \theta \\ \theta_{n+1} = \dfrac{1}{\theta_n} - \left[\dfrac{1}{\theta_n}\right] \end{cases}$$

($[\alpha]$ means the largest integer $\leqq \alpha$).

From the definition of the function θ_n it follows that:

$$(3) \qquad a_{n+1} = \left[\frac{1}{\theta_n}\right].$$

B) The first known metrical problem concerning continued fractions is due to Gauss. If $m_n(x)$ ($0 \leqq x \leqq$; $n \geqq 0$) denotes the measure of the set of those θ for which $\theta_n \leqq x$, Gauss asserts:

$$(4) \qquad \lim_{n\to\infty} m_n(x) = \frac{\log(1+x)}{\log 2}.$$

In 1928, R. Kuzmin [3] has given a proof of (4) including an estimate of the remainder which is important for further develop-

* Nijenrode lecture.

ments. It is to be remarked that in 1929 P. Lévy, independent of Kuzmin's proof, published another one.

In later years A. Khintchine [2], using Kuzmin's result, proved among other things the following theorem:

Let $f(p)$ be a nonnegative function of the positive integer p such that $f(p) \leqq K\, p^{\frac{1}{2}-\beta}$ (K and β being positive constants). Then for almost all θ we have:

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(a_n) = \sum_{p=1}^{\infty} f(p) \frac{\log\left\{1+\dfrac{1}{p(p+2)}\right\}}{\log 2}. \tag{5}$$

Putting in this theorem $f(x) = \log x$, Khintchine proved:

$$\lim_{N\to\infty} \sqrt[N]{a_1 \cdot a_2 \ldots a_N} = \prod_{p=1}^{\infty} \left\{1+\frac{1}{p(p+2)}\right\}^{\log p/\log 2} \tag{6}$$

for almost all θ.

§ 2.

A. Let $f(x)$ be a real function on $0 \leqq x \leqq 1$. Let D be a division of the interval $\langle 0, 1\rangle$ with dividing points $x_0(=0)$, x_1, x_2, . . ., x_P $(=1)$.

Further put

$$\sigma_p = \sup f(x) - \inf f(x)$$

where both, sup and inf, are to be extended over the closed interval $x_p \leqq x \leqq x_{p+1}$.

Then

$$\varphi(D) = \sum_{p=0}^{P-1} \sigma_p (x_{p+1} - x_p) \tag{7}$$

denotes the "oscillation of $f(x)$ for the division D".

In our thesis [4] we proved the following theorem:

Let D_k be the division of $\langle 0, 1\rangle$ with dividing points

$$h_0^k(=0),\ h_1^k\left(=\frac{1}{2^k}\right),\ h_2^k\left(=\frac{2}{2^k}\right),\ \ldots,\ h_{2^k}^k(=1).$$

Let $f(x)$ be a real function on $0 \leqq x \leqq 1$, such that for the sequence $\{D_k\}$ of divisions D_k of $\langle 0, 1\rangle$ the corresponding oscillations satisfy

$$\sum_{k=1}^{\infty} \varphi(D_k) < \infty. \tag{8}$$

Then for any couple of integers $M \geqq 0, N \geqq 1$ we have:

$$(9) \quad I \overset{\text{def}}{=} \int_0^1 \left\{ \sum_{n=M+1}^{M+N} \left(f(\theta_n) - \int_0^1 \frac{f(x)}{(1+x)\log 2}\, dx \right) \right\}^2 d\theta \leqq K_1 N,$$

K_1 being a constant depending only on $f(x)$.

B) We give a sketch of the proof. From the conditions imposed upon $f(x)$ it follows that $f(x)$ is bounded and that, without loss of generality, we may limit ourselves to the case $f(x) > 0$. Therefore:

$$(10) \qquad 0 < f(x) < B$$

and if we put:

$$\int_0^1 \frac{f(x)}{(1+x)\log 2}\, dx = P$$

we have:

$$(11) \qquad 0 < P < B.$$

Working out (9) we get:

$$(12) \qquad \begin{aligned} I = &\sum_{n=M+1}^{M+N} \int_0^1 (f(\theta_n) - P)^2 d\theta \\ &+ 2 \sum_{n=M+1}^{M+N-1} \sum_{\substack{m=M+2 \\ m>n}}^{M+N} \int_0^1 (f(\theta_n)-P)(f(\theta_m)-P)\, d\theta. \end{aligned}$$

Now:

$$(13) \qquad \sum_{n=M+1}^{M+N} \int_0^1 (f(\theta_n)-P)^2 d\theta \leqq B^2 N$$

and

$$(14) \qquad \sum_{\substack{n=M+1 \\ m=n+1}}^{M+N} \int_0^1 (f(\theta_n)-P)(f(\theta_m)-P)\, d\theta \leqq B^2 N.$$

Further we have:

$$(15) \qquad \begin{aligned} &\int_0^1 (f(\theta_n)-P)(f(\theta_m)-P)\, d\theta \\ &= \int_0^1 f(\theta_n) f(\theta_m)\, d\theta - P \int_0^1 f(\theta_n)\, d\theta - P \int_0^1 f(\theta_m)\, d\theta + P^2. \end{aligned}$$

If

$$m_n(x) \overset{\text{def}}{=} mE\{\theta;\, \theta_n \leqq x\},$$

R. Kuzmin [3] proved:

$$(16) \qquad \frac{dm_n(x)}{dx} = \frac{1}{(1+x)\log 2} + \delta A e^{-\alpha\sqrt{n}}$$

with $|\delta| < 1$, whereas A and α are positive constants.

Now it is easily shown that

$$(17)\qquad \int_0^1 f(\theta_n)d\theta = \int_0^1 f(x)\frac{dm_n(x)}{dx}\,dx = P+\delta_1 ABe^{-\alpha\sqrt{n}}$$

with $|\delta_1| < 1$.

As the series $\sum_{n=1}^{\infty} e^{-\alpha\sqrt{n}}$ converges we get:

$$(18)\qquad \begin{aligned}&\sum_{n=M+1}^{M+N-2}\ \sum_{\substack{m=M+3\\ m>n}}^{M+N}\left(-P\int_0^1 f(\theta_n)d\theta - P\int_0^1 f(\theta_m)d\theta + P^2\right) = \\ &\sum_{n=M+1}^{M+N-2}\ \sum_{\substack{m=M+3\\ m>n}}^{M+N}(-P^2)+\delta_2 K_2 N\end{aligned}$$

with $|\delta_2| < 1$, whereas K_2 is a positive constant.

In order to estimate

$$\sum_{n=M+1}^{M+N-2}\ \sum_{\substack{m=M+3\\ m>n}}^{M+N}\int_0^1 f(\theta_n)f(\theta_m)d\theta$$

we define:

$$(19)\qquad k = [\tfrac{1}{2}(m-n)].$$

Then we can prove:

$$(20)\qquad \begin{aligned}&\int_0^1 f(\theta_n)f(\theta_m)d\theta \\ &\quad = (P+\delta_3 ABe^{-\alpha\sqrt{m-n-k}})(P+\delta_4 ABe^{-\alpha\sqrt{n}}+2\delta_5\varphi(k))\end{aligned}$$

where $\varphi(k)$ means the oscillation of $f(x)$ for the division D_k.

Because of our choice of k(19) and of the condition (8) imposed upon $f(x)$ we get:

$$(21)\qquad \sum_{n=M+1}^{M+N-2}\ \sum_{\substack{m=M+3\\ m>n}}^{M+N}\int_0^1 f(\theta_n)f(\theta_m)d\theta = \sum_{n=M+1}^{M+N-2}\ \sum_{\substack{m=M+3\\ m>n}}^{M+N}(P^2)+\delta_6 K_3 N$$

with $|\delta_6| < 1$, whereas K_3 is a positive constant.

Combination of (12), (13), (14), (15), (18) and (21) proves our theorem.

C) On the result (9) we apply a theorem of I. S. Gàl and J. F. Koksma [1] (Theorem III with $S \equiv$ the set $0 \leqq \theta \leqq 1$, $\psi(N) = N$ and $p = 2$). Then we get:

Let $f(x)$ be a real function on $0 \leqq x \leqq 1$, satisfying the condition (8). Then for almost all θ we have:

$$(22)\qquad \sum_{n=1}^{N} f(\theta_n) - N\int_0^1 \frac{f(x)}{(1+x)\log 2}\,dx = o(N^{\frac{1}{2}}\log^{(3+\varepsilon)/2} N).$$

§ 3.

Shortly, in Indagationes Mathematicae, we will publish a note in which we shall prove the results (9) and (22) for functions satisfying somewhat weaker conditions.

As applications then we will get the following refinements of the results (5) and (6) of A. Khintchine.

1) Let $f(p)$ be a nonnegative function of the positive integer p such that $f(p) \leqq Kp^{\frac{1}{2}-\beta}$ (K and β being positive constants). Then for almost all θ we have:

$$(23) \quad \sum_{n=1}^{N} f(a_n) - N \sum_{p=1}^{\infty} f(p) \frac{\log\left\{1+\dfrac{1}{p(p+2)}\right\}}{\log 2} = o(N^{\frac{1}{2}} \log^{(3+\varepsilon)/2} N).$$

2) For almost all θ we have:

(24)

$$\sqrt[N]{a_1 \cdot a_2 \ldots a_N} - \prod_{p=1}^{\infty} \left\{1+\frac{1}{p(p+2)}\right\}^{\log p/\log 2} = o(N^{-\frac{1}{2}} \log^{(3+\varepsilon)/2} N).$$

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