COMPOSITIO MATHEMATICA

JOHANN CIGLER

Methods of summability and uniform distribution mod 1

Compositio Mathematica, tome 16 (1964), p. 44-51
<http://www.numdam.org/item?id=CM_1964__16__44_0>

Methods of summability and uniform distribution mod 1 *

by

Johann Cigler

Let $\{x_n\}$ be a sequence of real numbers. If it has a distribution function $z(x)$, i.e. if

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=N} f(x_n) = \int_0^1 f(x)dz(x)$$

exists for each continuous function $f(x)$ with period 1, then its asymptotic behavior is characterized to a certain extent by $z(x)$. If no distribution function exists, one may ask for other ways of characterizing the asymptotic behavior of the sequence $\{x_n\}$. One possibility consists in replacing the arithmetic mean in the definition of uniform distribution by other summability methods. This is what we propose to do in the sequel.

The first researches in this direction have been taken up by M. Tsuji [12]. He considered weighted means (M, λ_n). As is well-known, one says that a sequence s_n is summable (M, λ_n) to a limit s, if $\lim_{N\to\infty} (\sum_{n\leq N} \lambda_n s_n / \sum_{n\leq N} \lambda_n) = s$. With respect to questions of uniform distribution one is thus led to the following definition: A sequence $\{x_n\}$ has the (M, λ_n)-distribution function $z(x)$ mod 1, if for every continuous function $f(x)$ with period 1 the sequence $f(x_n)$ is summable (M, λ_n) to the value $\int_0^1 f(x)dz(x)$. If $z(x) = x$, then one calls the sequence $\{x_n\}$ (M, λ_n)-uniformly distributed mod 1. M. Tsuji made the assumptions $\lambda_1 \geqq \lambda_2 \geqq \ldots \geqq \lambda_n \geqq \ldots > 0$ and $\sum_{n=1}^{\infty} \lambda_n = \infty$.

In the case $\lambda_n = \lambda(n)$, where $\lambda(t) > 0$ is a continuous decreasing function with a continuous derivative $\lambda'(t)$ for $1 \leqq t < \infty$ such that

$$\sum_{k=1}^{n} \lambda_k \sim \int_1^n \lambda(t)dt \to \infty \qquad (n \to \infty),$$

he proved a generalization of a well-known theorem due to L. Fejér: [1]

* Nijenrode lecture.

[1] Fejér's theorem is stated in [10], II. Abschn., Aufg. 174.

THEOREM: *Let $f(t) > 0$ be a continuous increasing function with a continuous derivative $f'(t)$ for $1 \leqq t < \infty$ which satisfies the following conditions:*

(I) $f(t) \to \infty$, *as* $t \to \infty$

(II) $f'(t) \to 0$ *monotonically, as* $t \to \infty$

(III) $f'(t)/\lambda(t)$ *is monotone for* $t \geqq t_0 > 0$

(IV) $f'(t)/\lambda(t) \int_1^t \lambda(t)dt \to \infty$, *as* $t \to \infty$.

Then the sequence $\{f(n)\}$ *is* (M, λ_n)*-uniformly distributed mod* 1. In particular he considered the means (M, λ_n) with $\lambda_n = 1/n$, $1/n \log n, \ldots$. It follows that the sequence $\{\log n\}$ is uniformly distributed with respect to the summation method $(M, 1/n)$. On the other hand it is known that $\{\log n\}$ has no distribution function with respect to the arithmetic mean [2]. This shows that there is really a gain in considering weighted means. The proof of this theorem uses Weyl's criterion and Euler's summation formula.

Further M. Tsuji proved a generalization of van der Corput's fundamental theorem in the case of weighted means: *If for each* $h = 1, 2, 3, \ldots$ *the sequence* $\{f(n+h)-f(n)\}$ *is uniformly distributed* (M, λ_n), *then the same is true for the sequence* $\{f(n)\}$ *itself*. For the proof Tsuji used an analogue of van der Corput's fundamental inequality. Here the λ_n are supposed to be a positive decreasing sequence with $\sum \lambda_n = \infty$. [3])

E. Hlawka [7] has introduced the general concept of A-uniform distribution, where A is an arbitrary regular summability method. If $A = (a_{nk})$ then the sequence $\{x_k\}$ is said to have the A-distribution function $z(x)$, if for each continuous function $f(x)$ with period 1 the sequence $f(x_n)$ is A summable to the value $\int_0^1 f(x)dz(x)$, in other words if for each such function the relation

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk} f(x_k) = \int_0^1 f(x)dz(x)$$

holds.

It is easy to see that Weyl's criterion remains true, i.e. that the sequence $\{x_k\}$ has the distribution function $z(x)$ if and only if for each $q = 1, 2, 3, \ldots$

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk} e^{2\pi i q x_k} = \int_0^1 e^{2\pi i q x} dz(x).$$

[2] Cf. [10] II. Abschn., Aufg. 179.

[3] Tsuji has assumed a further condition, but it can be shown that this condition is unnecessary. Cf. [2].

An important task is to compare the efficiency of different methods of summability with respect to questions of the asymptotic distribution of a sequence. It is quite easy to see that the (C, k)-means and the Abel-mean are equivalent with respect to uniform distribution, that is to say that every sequence $\{x_k\}$ which has a distribution function with respect to the arithmetic mean has the same distribution function with respect to the (C, k)-means $(k > 1)$ and the Abel-mean and conversely that each sequence which has a distribution function with respect to the Abel-mean has the same distribution function with respect to the arithmetic mean. The first part is obvious, because the (C, k)-means and the Abel-mean are stronger than the arithmetic mean, in the sense that each sequence which is summable $(C, 1)$ is also summable (C, k) and A and the limits are identical. The second part follows from a well-known Tauberian theorem (cf. [4] Th. 92), because for each continuous function $f(x)$ the sequence $f(x_k)$ is bounded.

There are interesting connections between different weighted means. Hardy and Cesàro proved the following theorem: [4]

Let (M, p_n) and (M, q_n) be two weighted means, $p_n > 0$, $q_n > 0$, $\sum p_n = \infty$ and $\sum q_n = \infty$.

If (I)

$$\frac{q_{n+1}}{q_n} \leqq \frac{p_{n+1}}{p_n}$$

or (II)

$$\frac{p_{n+1}}{p_n} \leqq \frac{q_{n+1}}{q_n} \quad \text{and} \quad \frac{p_1+p_2+\cdots+p_n}{p_n} \leqq H \frac{q_1+q_2+\cdots+q_n}{q_n}$$

then (M, q_n) is stronger than (M, p_n), $(M, q_n) \supseteqq (M, p_n)$. It follows immediately that e.g. $(M, 1) \subseteqq (M, 1/n) \subseteqq (M, 1/n \log n) \subseteqq \ldots$. Another consequence is that $(M, 1)$ and (M, n^σ) are equivalent for $\sigma > -1$.

Using Euler's summation formula the following theorem can be proved (cf. [1]):

Let $g(x)$ be an increasing continuous function, whose second derivative exists and whose first derivative $g'(x)$ is decreasing, and let $f(x)$ be an increasing continuous function, whose derivative is continuous and satisfies $\lim_{x\to\infty} f'(x) = 0$. Then the sequence $\{f(g(n))\}$ has an $(M, g'(n))$-distribution function $z(x)$ if and only if the sequence

[4]) Cf. [4] Th. 14.

$\{f(n)\}$ *has the* $(M, 1)$*-distribution function* $z(x)$. It follows that if $f(x)$ fulfills the above conditions then the uniform distribution of the sequence $\{f(n)\}$ implies the uniform distribution of the sequence $\{f(n^\sigma)\}$ for all σ with $0 < \sigma \leqq 1$. For our theorem implies that the sequence $f(n^\sigma)$ is uniformly distributed with respect to the summability method $(M, n^{\sigma-1})$. On the other hand we already know that the means $(M, 1)$ and $(M, n^{\sigma-1})$ are equivalent.

Another theorem of this kind is the following one:

Let $f(x)$ *be increasing, twice continuously differentiable,* $\lim_{x\to\infty} f(x) = \infty$ *and let* $f'(x)$ *be decreasing to zero. Then the sequence* $\{f(n)\}$ *is* $(M, f'(n))$*-uniformly distributed mod* 1.

It follows from this theorem that e.g. $\{\log n\}$ is $(M, 1/n)$-uniformly distributed or that the sequence $\{n^\sigma\}$ $(0 < \sigma < 1)$ is $(M, n^{\sigma-1}) \sim (M, 1)$-uniformly distributed.

It is perhaps interesting to note that for an increasing twice continuously differentiable function $f(x)$ with $f(x) \to \infty$, $f'(x) \to 0$ monotonically, there are only two possibilities:

Either the sequence $\{f(n)\}$ *has no* $(C, 1)$*-distribution function at all or it is* $(C, 1)$*-uniformly distributed.*

In fact the following stronger theorem is true: [5]

Let $\{x_n\}$ *be a sequence of real numbers with* $\lim_{n\to\infty} x_n = \infty$ *and let* $x_{n+1}-x_n$ *tend to zero in a monotone way. If* $\underline{\lim}\, |n(x_{n+1}-x_n)| < \infty$ *then the sequence* $\{x_n\}$ *has no* $(C, 1)$*-distribution function, otherwise it is (by Fejér's theorem) uniformly distributed.*

An interesting topic is to give a characterization of all summation methods with respect to which the sequences $\{n\vartheta\}$ are uniformly distributed mod 1 for each irrational number ϑ. We can only specify several classes of summation methods which have this property.

1) Let H be a Hausdorff-method [6], i.e. let $H = (a_{nk})$ with

$$\begin{aligned} a_{nk} &= \binom{n}{k} \int_0^1 x^k(1-x)^{n-k}\, d\mu(x) \quad &&\text{for} \quad k \leqq n \\ &= 0 &&\text{otherwise,} \end{aligned}$$

where μ is a positive measure in $[0, 1)$, $\mu([0, 1)) = 1$ and $\mu(\{0\}) = 0$. Then for each irrational number ϑ the sequence $\{n\vartheta\}$ is H-uniformly distributed mod 1.

For in this case we have

[5] This remark is due to J. H. B. Kemperman.

[6] Cf. e.g. [4], [11].

$$\sum_k a_{nk} e^{2\pi iqk\vartheta} = \sum_{k=0}^{n} \binom{n}{k} \int_0^1 (xe^{2\pi iq\vartheta})^k (1-x)^{n-k} d\mu(x)$$
$$= \int_0^1 \left(\sum_{k=0}^{n} \binom{n}{k} (xe^{2\pi iq\vartheta})^k (1-x)^{n-k} \right) d\mu(x)$$
$$= \int_0^1 (xe^{2\pi iq\vartheta}+1-x)^n d\mu(x).$$

Now

$$\lim_{n\to\infty} (xe^{2\pi iq\vartheta}+1-x)^n = 0 \quad \text{for} \quad 0 < x < 1$$
$$= 1 \quad \text{for} \quad x = 0.$$

Therefore by Lebesgue's theorem on dominated convergence we get $\lim_{n\to\infty} \int_0^1 (xe^{2\pi iq\vartheta}+1-x)^n d\mu(x) = 0$ $(q = 1, 2, 3, \ldots)$ and this is what we wanted to prove.

2) The same method of proof applies to the so-called Henrikssen-means [7], defined by

$$a_{nk} = \int_0^\infty \frac{(nx)^k}{k!} e^{-nx} d\mu(x),$$

where μ is a positive normed measure on $[0, \infty)$ with $\mu(\{0\}) = 0$.

3) Let $A = (a_{nk})$ be a strongly regular Toeplitz-matrix, i.e. a Toeplitz-matrix which fulfills

$$(*) \qquad \lim_{n\to\infty} \sum_k |a_{nk} - a_{n,k+1}| = 0.$$

Also in this case the sequences $\{n\vartheta\}$ for irrational ϑ are A-uniformly distributed.

We have to show that for each $q = 1, 2, 3, \ldots$

$$\lim_{n\to\infty} \sum_k a_{nk} e^{2\pi iqk\vartheta} = 0.$$

Suppose this is not true. Then there exists a q and an increasing sequence n_l of integers such that

$$\lim_{l\to\infty} \sum_k a_{n_l k} e^{2\pi iqk\vartheta} = s \neq 0.$$

Now it follows from (*) that also

$$\lim_{l\to\infty} \sum_k a_{n_l k} e^{2\pi iq(k+1)\vartheta} = s \neq 0.$$

From (1) and (2) we conclude therefore $s = e^{2\pi iq\vartheta} s$ which is

[7] Cf. [11].

clearly impossible if $s \neq 0$. This is a contradiction to our assumption and therefore our theorem is proved [8].

This theorem can be generalized to sequences of powers of elements of a compact group X with countable base. As is well-known one says that a sequence $\{x_n\}$, $x_n \in X$, has the distribution measure μ if for each continuous function $f(x)$ on X

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n\leq N} f(x_n) = \int_X f(x) d\mu(x). \tag{3}$$

If $\mu = \lambda$ the Haar measure on X, then we call the sequence $\{x_n\}$ uniformly distributed on X. If the limit on the left hand side in (3) does not exist for all continuous functions f on X, then it is always possible to choose a subsequence N_l for which the limit exists for all continuous f. The right hand side of (3) is then of the form $\int_X f(x)\, d\nu(x)$. In this case we say that ν is a limit distribution measure of the given sequence. Obviously a sequence $\{x_n\}$ has a distribution measure if and only if all limit distribution measures coincide.

(A similar terminology applies if the arithmetic mean is replaced by other summability methods). Weyl's criterion remains true for sequences in a compact group and may be formulated in the following way: The sequence $\{x_n\}$ has the distribution measure μ if and only if (3) holds for all continuous characters χ of X (in the Abelian case) resp. for all continuous irreducible unitary representations $R(g)$ of X. [9])

The announced theorem may now be formulated in the following way:

Let X be a compact abelian group and g an element of X with the property that $\chi(g) \neq 1$ for each continuous character $\chi \not\equiv 1$ of X. Then the sequence $\{g^n\}$ is A-uniformly distributed in X. [10])

The proof is the same as in the case of numbers mod 1, for all one has to show is that for each $\chi \not\equiv 1$ $\lim_{n\to\infty} \sum_k a_{nk}\chi(g^k) = 0$. There is also a generalization to nonabelian compact topological groups which has some connections with a theorem of G. Helmberg [5]. In this case we have to consider multiple sequences. Let $A_1 = (a^1_{nk}), \ldots$ $A_l = (a^l_{nk})$ be strongly regular Toeplitz-matrices. Let A be the summability "matrix" with general

[8] By another method this theorem has been proved by Petersen [9].

[9] Cf. [8], [6].

[10] This theorem has first been proved by B. Eckmann [3] in the case of the arithmetic mean.

element $a^1_{n_1k_1} \dots a^l_{n_lk_l}$. A multiple sequence $s_{k_1k_2\dots k_l}$ is said to be A-summable to the limits s, if

$$\lim_{\min n_i \to \infty} \sum_{k_1=1}^{\infty} \dots \sum_{k_l=1}^{\infty} a^1_{n_1k_1} \dots a^l_{n_lk_l} s_{k_1k_2\dots k_l} = s.$$

Now the following theorem holds:

Let A fulfill the above assumptions and let g_i $(i = 1, 2 \dots, l)$ be elements of X (a compact, not necessarily abelian, topological group with countable base) such that for each nontrivial continuous irreducible unitary representation $R(x)$ there exists at least one g_j with det $(R(g_j)-I) \neq 0$. *(I denotes the identity matrix). Then the sequence $\{g_1^{k_1} g_2^{k_2} \dots g_l^{k_l}\}$ is A-uniformly distributed in X.*

Proof: It is sufficient to show, that for each nontrivial continuous irreducible unitary representation R

$$\lim_{\min n_i \to \infty} \sum_{k_1} \dots \sum_{k_l} a^1_{n_1k_1} \dots a^l_{n_lk_l} R(g_1^{k_1} \dots g_l^{k_l}) = 0 = \int_X R(x) d\lambda \tag{4}$$

holds. (λ = Haar measure on X).

Now we have $R(g_1^{k_1} \dots g_l^{k_l}) = R(g_1)^{k_1} \dots R(g_l)^{k_l}$, therefore the left hand side of (4) may be written in the form

$$\lim_{\min n_i \to \infty} \prod_{i=1}^{l} \sum_{k_i=1}^{\infty} a_{n_ik_i} R(g_i)^{k_i}.$$

(Of course the factors are to be taken in the same order as in the left hand side of (4)).

Let ν be an arbitrary limit distribution measure of the sequence $\{g_1^{k_1} \dots g_l^{k_l}\}$. Then there exist limit distribution measures ν_i of the sequences $\{g_i^{k_i}\}$ such that $\nu = \nu_1 \dots \nu_l$. (Here the product is understood in the sense of convolution.) By assumption (and using the proof of the foregoing theorem) for at least one j the sequence $\{g_j^{k_j}\}$ has a limit distribution measure ν_j such that $\int_X R(x) d\nu_j(x) = 0$. Therefore also $\int_X R(x) d\nu(x) = 0$. Since $R(x)$ was an arbitrary representation and ν an arbitrary limit distribution measure we conclude therefore that $\nu = \lambda$, q.e.d.

There are many other questions which arise in considering general summability matrices in the theory of uniform distribution. For these topics the reader may consult E. Hlawka [7], [8].

BIBLIOGRAPHY

J. Cigler

[1] Asymptotische Verteilung reeller Zahlen mod 1. Monatsh. Math. 64, 201–225 (1960).

J. Cigler

[2] Über eine Verallgemeinerung des Hauptsatzes der Theorie der Gleichverteilung. J. reine angew. Math. 210, 141—147 (1962).

B. Eckmann

[3] Über monothetische Gruppen. Commentarii math. Helvet. 16, 249—293 (1943/44).

G. H. Hardy

[4] Divergent Series, Oxford 1949.

G. Helmberg

[5] A theorem on equidistribution in compact groups. Pacific J. Math. 8, 227—241 (1958)

E. Hlawka

[6] Zur formalen Theorie der Gleichverteilung in kompakten Gruppen. Rend. Circ. mat. Palermo 4, 33—47 (1955).

E. Hlawka

[7] Folgen auf kompakten Räumen. Abh. math. Sem. Univ. Hamburg 20, 223—241 (1956).

E. Hlawka

[8] Folgen auf kompakten Râumen II. Math. Nachr. 18, 188—202 (1958).

G. M. Petersen

[9] Almost convergence and uniformly distributed sequences. Quart. J. Math., Oxford II ser. 7, 188—191 (1956).

G. Pólya und G. Szegö

[10] Aufgaben und Lehrsätze aus der Analysis, I. Berlin 1925.

A. Rényi

[11] Summation methods and probability theory. Magyar Tud. Akad. Mat. Kutató Int. Közl. 4, 389—399 (1959).

M. Tsuji

[12] On the uniform distribution of numbers mod. 1. J. math. Soc. Japan 4, 313—322 (1952).

(Oblatum 29-5-63)