COMPOSITIO MATHEMATICA

A. J. STAM

Derived stochastic processes

Compositio Mathematica, tome 17 (1965-1966), p. 102-140

<http://www.numdam.org/item?id=CM_1965-1966__17__102_0>

Derived Stochastic Processes [1]

by

A. J. Stam

1. Introduction

Let ${}_1P(t)$, $t \geqq 0$, be the transition matrix of a continuous parameter Markov process with stationary transition probabilities. For every $s \geqq 0$ let $b(s, \cdot)$ be an infinitely divisible distribution function, satisfying

$$(1.1) \qquad b(s, t) = 0, \quad t \leqq 0,$$

$$(1.2) \qquad b(0, t) = U(t), \quad -\infty < t < \infty,$$

where $U(t) = 0$, $t \leqq 0$, $U(t) = 1$, $t > 0$,

$$(1.3) \qquad \lim_{s \to 0^+} b(s, t) = U(t), \ -\infty < t < \infty,$$

$$(1.4) \qquad b(s, \cdot) * b(\sigma, \cdot) = b(s+\sigma, \cdot\ )\ s \geqq 0, \sigma \geqq 0,$$

where $*$ denotes the convolution operation. Then the matrix ${}_2P(\cdot)$, defined by

$$(1.5) \qquad {}_2P(s) \overset{\text{df}}{=} \int_{[0,\infty)} {}_1P(t) d_t b(s, t),\ s \geqq 0,$$

is the transition matrix of a Markov process with stationary transition probabilities. Cohen, in his papers [3] and [4], called it the derived transition matrix of ${}_1P(\cdot)$ by the deriving distribution $b(s, \cdot)$ and investigated its properties.

For an earlier definition of the same concept we refer to Bochner [1], Ch. 4.4. and 4.5., where the term "subordinate Markov process" is used.

In [3] and [4] definitions analogous to (5) are given for the case that one or both of the time parameters t and s are discrete. By a generalisation of (1.5) it is possible to derive a nonstationary transition matrix from a stationary transition matrix (see [4] and

[1] This paper describes the results of a study made by Prof. Dr. Ir. J. W. Cohen and the author at the Mathematical Institute of the Technological University Delft.

section 5 below). The large class of nonstationary transition matrices obtained in this way has many properties in common with stationary transition matrices. This is a consequence of the fact that there exists a strong connection between the properties of the original and the derived matrix.

For some applications of derived Markov matrices to queueing theory we refer to [5].

So far, the concept of derived transition matrix was not related to the underlying stochastic processes. Now $b(s, \cdot)$ may be considered as the distribution function of the random variable τ_s in a stochastic process $\{\tau_s, s \geqq 0\}$ with independent nonnegative stationary increments. Let $\{{}_1x_t, t \geqq 0\}$ be a Markov process with transition matrix ${}_1P(\cdot)$ and assume that the processes $\{{}_1x_t, t \geqq 0\}$ and $\{\tau_s, s \geqq 0\}$ are independent. Then it is not difficult to see (it will be proved rigorously in section 5 below), that the stochastic process $\{{}_2x_s, s \geqq 0\}$ defined by

$$(1.6) \qquad {}_2x_s \overset{\text{df}}{=} {}_1x_{\tau_s}, \quad s \geqq 0,$$

is a Markov process having the transition matrix ${}_2P(\cdot)$ given by (1.5). The definition (1.6) means that the process $\{{}_1x_t, t \geqq 0\}$ is sampled at random times τ_s.

Stochastic processes of the type defined in (1.6) will be called derived processes. Sections 3 and 4 of this paper deal with derived processes in general, i.e. the assumptions that $\{{}_1x_t, t \geqq 0\}$ is a Markov process and that $\{\tau_s, s \geqq 0\}$ has independent increments, is dropped. In sections 5 and 6 derived Markov processes of the type considered above are studied.

2. Notations

The following definitions and notations apply throughout the paper.

$({}_1\Omega, {}_1\mathscr{A}, {}_1\mu)$ is a probability space. The points of ${}_1\Omega$ are denoted by ${}_1\omega$. On ${}_1\Omega$ a stochastic process $\{{}_1x_t, t \in T)$ is defined. The parameter set T is a Borel set of the real line and we begin with admitting that the ${}_1x_t, t \in T$, are abstract valued random variables, i.e. measurable transformations on $({}_1\Omega, {}_1\mathscr{A})$ into $(X, \mathscr{X})$, where $\mathscr{X}$ is any σ-field of subsets of the state space X.

Further $(\Gamma, \mathscr{C}, Q)$, with points γ, is a probability space on which a real valued stochastic process $\{\tau_s, s \in S\}$ is defined. The parameter set S is a Borel set of the real line and the state space N of the process is assumed to be a subset of T.

Let $({}_2\Omega, {}_2\mathscr{A}, {}_2\mu)$, with points ${}_2\omega = ({}_1\omega, \gamma)$, be the Cartesian product of the probability spaces $({}_1\Omega, {}_1\mathscr{A}, {}_1\mu)$ and $(\Gamma, \mathscr{C}, Q)$, so ${}_2\mu$ is the product measure of ${}_1\mu$ and Q. The random variables ${}_1\boldsymbol{x}_t$, $t \in T$, and $\boldsymbol{\tau}_s$, $s \in S$, may be considered as random variables on $({}_2\Omega, {}_2\mathscr{A}, {}_2\mu)$, where ${}_1\boldsymbol{x}_t$ depends on ${}_1\omega$ only and $\boldsymbol{\tau}_s$ on γ only.

On ${}_2\Omega$ we define the functions ${}_2\boldsymbol{x}_s$, $s \in S$, by

$$(2.1) \qquad {}_2\boldsymbol{x}_s({}_2\omega) \equiv {}_2\boldsymbol{x}_s({}_1\omega, \gamma) \stackrel{\text{df}}{=} {}_1\boldsymbol{x}_{\boldsymbol{\tau}_s}(\gamma)({}_1\omega), \quad s \in S.$$

In section 3 conditions will be given under which $\{{}_2\boldsymbol{x}_s, s \in S\}$ is a stochastic process, i.e. conditions under which ${}_2\boldsymbol{x}_s$ for every $s \in S$ is a measurable transformation on $({}_2\Omega, {}_2\mathscr{A})$ into $(X, \mathscr{X})$.

The process $\{{}_1\boldsymbol{x}_t, t \in T\}$ will be called the original process, $\{\boldsymbol{\tau}_s, s \in S\}$ the deriving process and $\{{}_2\boldsymbol{x}_s, s \in S\}$ the derived process, viz. the process derived from $\{{}_1\boldsymbol{x}_t, t \in T\}$ by the deriving process $\{\boldsymbol{\tau}_s, s \in S\}$.

We assumed $N \subset T$ in order that $\boldsymbol{\tau}_s(\gamma) \in T$ for every $\gamma \in \Gamma$, so that the definition (2.1) makes sense for every $\gamma \in \Gamma$. The probability space $({}_2\Omega, {}_2\mathscr{A}, {}_2\mu)$ is defined as the cartesian product of $({}_1\Omega, {}_1\mathscr{A}, {}_1\mu)$ and $(\Gamma, \mathscr{C}, Q)$, since this is the simplest way to introduce the independence of the original and the deriving process.

We further denote by

${}_2E, {}_1E$: expectation with respect to ${}_2\mu$ and ${}_1\mu$, respectively.
$\mathscr{T}$: the class of Borel subsets of T.
$\mathscr{S}$: the class of Borel subsets of S.
$A^n, \mathscr{B}^n$: cartesian products of identical spaces and σ-fields, respectively (cf. [7], § 33).

3. General theorems

In this section some theorems on the general definition (2.1) of a derived process will be proved. For a more detailed treatment of these theorems we refer to [11].

THEOREM 3.1. If the state space N of the deriving process is countable, or if the ${}_1\boldsymbol{x}_t$-process is ${}_1\mathscr{A} \times \mathscr{T}$-measurable, then ${}_2\boldsymbol{x}_s$ for every $s \in S$ is an abstract valued random variable on $({}_2\Omega, {}_2\mathscr{A})$, in fact, ${}_2\boldsymbol{x}_s$ is measurable with respect to the σ-field ${}_1\mathscr{A} \times \mathscr{C}_s$, where $\mathscr{C}_s$ is the σ-field of γ sets induced by $\boldsymbol{\tau}_s$.

By ${}_1\mathscr{A} \times \mathscr{T}$-measurability of the ${}_1\boldsymbol{x}_t$-process is meant that the function ${}_1\boldsymbol{x}_t({}_1\omega)$ is measurable in its pair of arguments $({}_1\omega, t)$, more precisely: ${}_1\boldsymbol{x}_{(\cdot)}(\cdot)$ is a measurable transformation on $({}_1\Omega \times T, {}_1\mathscr{A} \times \mathscr{T})$ into $(X, \mathscr{X})$. Cf. Doob [6], § II, 2, Loève [9], section 35. Throughout this paper it will be assumed that the original process satisfies one of the conditions of theorem 3.1.

Proof. First assume that $N = \{n_1, n_2, \ldots\}$. Then the assertion of the theorem follows from the relation

$$\{{}_2\omega : {}_2x_s \in E\} = \bigcup_{k=1}^{\infty} \{{}_2\omega : {}_1x_{n_k} \in E, \tau_s = n_k\}.$$

In the second case define the transformation M on ${}_2\Omega$ into ${}_1\Omega \times T$ by

$$M({}_2\omega) \equiv M({}_1\omega, \gamma) \overset{\text{df}}{=} ({}_1\omega, \tau_s(\gamma)).$$

It is easily seen that M is a measurable transformation on $({}_2\Omega, {}_1\mathscr{A} \times \mathscr{C}_s)$ into $({}_1\Omega \times T, {}_1\mathscr{A} \times \mathscr{T})$. Now consider the transformation L on ${}_1\Omega \times T$ into X, defined by

$$L({}_1\omega, t) \overset{\text{df}}{=} {}_1x_t({}_1\omega).$$

By assumption L is a measurable transformation on $({}_1\Omega \times T, {}_1\mathscr{A} \times \mathscr{T})$ into $(X, \mathscr{X})$. Since ${}_2x_s({}_2\omega) = L(M({}_2\omega))$ and the product of two measurable transformations is measurable, our assertion follows.

We need the following lemmas on measurability of stochastic processes. Let Ω be any space, $\mathscr{A}$ be a σ-field of ω sets and $\{x_t, t \in T\}$ be a stochastic process on $(\Omega, \mathscr{A})$, i.e. x_t for every $t \in T$ is a measurable transformation on $(\Omega, \mathscr{A})$ into $(X, \mathscr{X})$. Further $\mathscr{T}$ denotes the σ-field of Borel subsets of T. Then we have

Lemma 3.1. If the process $\{x_t, t \in T\}$ is $\mathscr{A} \times \mathscr{T}$-measurable, then K defined by

$$K(\omega, t_1, \ldots, t_n) \overset{\text{df}}{=} (x_{t_1}(\omega), x_{t_2}(\omega), \ldots, x_{t_n}(\omega)) \tag{3.1}$$

is a measurable transformation on $(\Omega \times T^n, \mathscr{A} \times \mathscr{T}^n)$ into $(X^n, \mathscr{X}^n)$, i.e.

$$\{(\omega, t_1, \ldots, t_n) : K(\omega, t_1, \ldots, t_n) \in F\} \in \mathscr{A} \times \mathscr{T}^n \tag{3.2}$$

for every $F \in \mathscr{X}^n$.

Proof. Let $\mathscr{L}$ be the class of all $F \in \mathscr{X}^n$ that satisfy (3.2). Now $\mathscr{L}$ contains the class $\mathscr{G}$ of all rectangles $A_1 \times \ldots \times A_n$ with $A_i \in \mathscr{X}$, $i = 1, \ldots, n$, for we have

$$K^{-1}(A_1 \times \ldots \times A_n) = \bigcap_{r=1}^{n} C_r,$$

where by assumption of the lemma the cylinder

$$C_r \overset{\text{df}}{=} \{(\omega, t_1, \ldots, t_n) : x_{t_r}(\omega) \in A_r\}$$

has a measurable base in the product of Ω and the r^{th} factor space of T^n. Since K^{-1} preserves set operations, $\mathscr{L}$ is a σ-field. So $\mathscr{L}$ contains the minimal σ-field over $\mathscr{G}$, which is $\mathscr{X}^n$.

LEMMA 3.2. If the process $\{x_t, t \in T\}$ is $\mathscr{A} \times \mathscr{T}$-measurable, and $f(., \ldots, .)$ is an extended real valued $\mathscr{X}^n \times \mathscr{T}^n$-measurable function on $X^n \times T^n$, then $f(x_{t_1}, \ldots, x_{t_n}, t_1, \ldots, t_n)$ is an $\mathscr{A} \times \mathscr{T}^n$-measurable function on $\Omega \times T^n$. If moreover μ is a probability measure on $(\Omega, \mathscr{A})$, such that for every $(t_1, \ldots, t_n) \in T^n$ the expectation $Ef(x_{t_1}, \ldots, x_{t_n}, t_1, \ldots, t_n)$ with respect to μ exists, then $Ef(x_{t_1}, \ldots, x_{t_n}, t_1, \ldots, t_n)$ is a $\mathscr{T}^n$-measurable function of $(t_1, \ldots, t_n)$.

In particular $Eg(x_{t_1}, \ldots, x_{t_n})$ and $\mu\{(x_{t_1}, \ldots, x_{t_n}) \in B\}$ are $\mathscr{T}^n$-measurable functions of $(t_1, \cdots, t_n)$ if g is $\mathscr{X}^n$-measurable and the expectation exists, and if $B \in \mathscr{X}^n$.

PROOF. By lemma 3.1. it is easily seen that the transformation H on $\Omega \times T^n$ into $X^n \times T^n$:

$$H(\omega, t_1, \ldots, t_n) \overset{\text{df}}{=} (x_{t_1}(\omega), \ldots, x_{t_n}(\omega), t_1, \ldots, t_n)$$

is a measurable transformation on $(\Omega \times T^n, \mathscr{A} \times \mathscr{T}^n)$ into $(X^n \times T^n, \mathscr{X}^n \times \mathscr{T}^n)$. From this fact it follows that $f(x_{t_1}, \ldots, x_{t_n}, t_1, \ldots, t_n)$ is $\mathscr{A} \times \mathscr{T}^n$-measurable. The second assertion is proved by Fubini's theorem (cf. [7], § 35, theorem A, and § 36).

The following theorem connects the finite-dimensional distributions of the derived and original processes by means of the finite-dimensional distributions of the deriving process.

THEOREM 3.2. Let f be a nonnegative extended real valued $\mathscr{X}^n \times \mathscr{T}^n$-measurable function on $X^n \times T^n$. Then for any $(s_1, \ldots, s_n) \in S^n$ we have under the conditions of theorem 3.1.:

$$(3.3) \quad {}_2Ef({}_2x_{s_1}, \ldots, {}_2x_{s_n}, \tau_{s_1}, \ldots, \tau_{s_n}) = \int_{T^n} {}_1Ef({}_1x_{t_1}, \ldots, {}_1x_{t_n}, t_1, ., . .t_n) dQ_{s_1, \ldots, s_n}(t_1, \ldots t_n)$$

Here $Q_{s_1, \ldots, s_n}$ denotes the probability measure induced on $(T^n, \mathscr{T}^n)$ by $(\tau_{s_1}, \ldots, \tau_{s_n})$. In particular, for any nonnegative extended real valued $\mathscr{X}^n$-measurable function g and $B \in \mathscr{X}^n$

$$(3.4) \quad {}_2Eg({}_2x_{s_1}, \ldots, {}_2x_{s_n}) = \int_{T^n} {}_1Eg({}_1x_{t_1}, \ldots, {}_1x_{t_n}) dQ_{s_1, \ldots, s_n}(t_1, \ldots, t_n)$$

$$(3.5) \quad {}_2\mu\{({}_2x_{s_1}, \ldots, {}_2x_{s_n}) \in B\} = \int_{T^n} {}_1\mu\{({}_1x_{t_1}, \ldots, {}_1x_{t_n}) \in B\} dQ_{s_1, \ldots, s_n}(t_1, \ldots, t_n).$$

Note that the integrals in (3.3), (3.4) and (3.5) with respect to $Q_{s_1,\ldots,s_n}$ are defined. If N is countable, these integrals actually are sums, whereas, if the original process is measurable, the integrands are measurable functions of $t_1, \ldots, t_n$ by lemma 3.2.

If negative values for f are allowed, the theorem may be applied by decomposing f into its positive and negative parts.

Proof. Theorem 3.1. implies that $f({}_2x_{s_1}, \ldots, {}_2x_{s_n}, \ldots, \tau_{s_n})$ is a measurable function on $({}_2\Omega, {}_2\mathscr{A})$. So by Fubini's theorem

$$\begin{aligned}&{}_2Ef({}_2x_{s_1}, \ldots, {}_2x_{s_n}, \tau_{s_1}, \ldots, \tau_{s_n})\\ &= \int_\Gamma dQ(\gamma)\int_{{}_1\Omega} f({}_1x_{\tau_{s_1}(\gamma)}({}_1\omega), \ldots, {}_1x_{\tau_{s_n}(\gamma)}({}_1\omega),\\ &\qquad\qquad \tau_{s_1}(\gamma), \ldots, \tau_{s_n}(\gamma))d_1\mu({}_1\omega).\end{aligned}$$

If N is countable, the integration with respect to Q is a summation and (3.3) follows. If the ${}_1x_t$-process is ${}_1\mathscr{A}\times\mathscr{T}$-measurable, then the integral over ${}_1\Omega$ is a $\mathscr{T}^n$-measurable function of $\tau_{s_1}, \ldots, \tau_{s_n}$ by lemma 3.2 and (3.3) follows by a well known theorem of measure theory (Halmos [7], § 39, theorem C).

A derived process may be derived again. Let $\{\sigma_w, w \in W\}$ be a real valued stochastic process with state space $N'' \subset S$ and let $(\Gamma'', \mathscr{C}'', Q'')$ be the underlying probability space. Deriving the ${}_2x_s$-process by the σ_w-process gives the process $\{{}_3x_w, w \in W\}$ on ${}_2\Omega\times\Gamma'' = {}_1\Omega\times\Gamma\times\Gamma''$, defined by

$$(3.6)\qquad {}_3x_w({}_2\omega, \gamma'') \equiv {}_3x_w({}_1\omega, \gamma, \gamma'') \overset{\text{df}}{=} {}_2x_{\sigma_w(\gamma'')}({}_2\omega),\ w \in W.$$

Repeated derivation has an "associative" property. Deriving the τ_s-process by the σ_w-process gives the proces ${}_2\tau_w$ on $\Gamma\times\Gamma''$:

$$(3.7)\qquad {}_2\tau_w(\gamma, \gamma'') \overset{\text{df}}{=} \tau_{\sigma_w(\gamma'')}(\gamma), \quad w \in W,$$

and one has the relation

$$(3.8)\qquad \begin{aligned}{}_3x_w({}_1\omega, \gamma, \gamma'') &= {}_1x_{{}_2\tau_w(\gamma, \gamma'')}({}_1\omega),\\ &{}_1\omega \in {}_1\Omega,\ \gamma \in \Gamma,\ \gamma'' \in \Gamma'',\ w \in W.\end{aligned}$$

That (3.8) holds, is easily seen by (3.6), (2.1) and (3.7). So the process $\{{}_3x_w, w \in W\}$ is obtained by deriving the original process by the process $\{{}_2\tau_w, w \in W\}$ that is derived from $\{\tau_s, s \in S\}$ by the σ_w-process.

The relation (3.8) holds whether the ${}_2x_s$, ${}_2\tau_w$ and ${}_3x_w$ are random variables or not. If the τ_s-process is $\mathscr{C}\times\mathscr{S}$-measurable, the process $\{{}_2\tau_w, w \in W\}$ consists of random variables on $(\Gamma\times\Gamma'', \mathscr{C}\times\mathscr{C}'')$, as is seen by applying theorem 3.1 to the deriving operation (3.7).

So by (3.8) and theorem 3.1 the process $\{{}_3x_w, w \in W\}$ consists of random variables on $({}_1\Omega \times \Gamma \times \Gamma'', {}_1\mathscr{A} \times \mathscr{C} \times \mathscr{C}'')$. The latter conclusion also may be obtained from (3.6) and the fact that the ${}_2x_s$-process is measurable if the ${}_1x_t$- and τ_s-processes are measurable:

Theorem 3.3. If the ${}_1x_t$-process satisfies one of the conditions of theorem 3.1 and the τ_s-process is $\mathscr{C} \times \mathscr{S}$-measurable, then the ${}_2x_s$-process is ${}_2\mathscr{A} \times \mathscr{S}$-measurable.

Proof. It is sufficient to consider ${}_1\mathscr{A} \times \mathscr{T}$-measurability of the ${}_1x_t$-process, since if N is countable the ${}_1x_t$-process becomes ${}_1\mathscr{A} \times \mathscr{T}$-measurable if we take $T = N$, which does not change the ${}_2x_s$-process.

Let $\mathscr{N}$ be the intersection of N with the class of Borel sets of the real line. Define the transformation H on ${}_2\Omega \times S = {}_1\Omega \times \Gamma \times S$ into ${}_1\Omega \times N$ by

$$H({}_2\omega, s) \equiv H({}_1\omega, \gamma, s) \stackrel{\text{df}}{=} \big({}_1\omega, \tau_s(\gamma)\big).$$

By the $\mathscr{C} \times \mathscr{S}$-measurability of the τ_s-process it follows that H is a measurable transformation on $({}_2\Omega \times S, {}_2\mathscr{A} \times \mathscr{S}) \equiv ({}_1\Omega \times \Gamma \times S, {}_1\mathscr{A} \times \mathscr{C} \times \mathscr{S})$ into $({}_1\Omega \times N, {}_1\mathscr{A} \times \mathscr{N})$. Now define the transformation M on ${}_1\Omega \times N$ into X by

$$M({}_1\omega, t) \stackrel{\text{df}}{=} {}_1x_t({}_1\omega).$$

Since the ${}_1x_t$-process is ${}_1\mathscr{A} \times \mathscr{T}$-measurable M is a measurable transformation on $({}_1\Omega \times N, {}_1\mathscr{A} \times \mathscr{N})$ into $(X, \mathscr{X})$.

Since

$${}_2x_s({}_1\omega, \gamma) = {}_1x_{\tau_s(\gamma)}({}_1\omega) = M\big({}_1\omega, \tau_s(\gamma)\big) = MH({}_1\omega, \gamma, s),$$

and the product of two measurable transformations is measurable, we have

$$\{({}_1\omega, \gamma, s) : {}_2x_s({}_1\omega, \gamma) \in F\} \in {}_1\mathscr{A} \times \mathscr{C} \times \mathscr{S},$$

for every $F \in \mathscr{X}$, which proves the theorem.

The assumptions in the theorems of this section may be weakened to $\overline{{}_1\mathscr{A} \times \mathscr{T}}$-measurability of the original process. Here $\overline{{}_1\mathscr{A} \times \mathscr{T}}$ is the completion of ${}_1\mathscr{A} \times \mathscr{T}$ with respect to a measure ${}_1\mu \times \lambda_0$ on ${}_1\mathscr{A} \times \mathscr{T}$, where λ_0 is any measure on $\mathscr{T}$ having the following property: for $n = 1, 2, \ldots$ and every $(s_1, \ldots, s_n) \in S^n$, the distribution induced on $(T^n, \mathscr{T}^n)$ by $(\tau_{s_1}, \ldots, \tau_{s_n})$ is absolutely continuous with respect to λ_0^n.

Under this assumption ${}_2x_s$ for every $s \in S$ is an $\overline{{}_1\mathscr{A} \times \mathscr{C}}$-measurable function on ${}_1\Omega \times \Gamma$ into $(X, \mathscr{X})$. Here $\overline{{}_1\mathscr{A} \times \mathscr{C}}$ is the completion of ${}_1\mathscr{A} \times \mathscr{C}$ with respect to ${}_1\mu \times Q$. A similar modification of theorem 3.3 may be proved. We refer to [11], I.

If the measurability condition on the original process is replaced by the condition that ${}_1\mu\{{}_1\omega\colon ({}_1x_{t_1}, \ldots, x_{t_n}) \in E\}$ for $n = 1, 2, \ldots$ and for every $E \in \mathscr{X}^n$ is a $\mathscr{T}^n$-measurable function of $t_1, \ldots, t_n$, then it can be shown that there is a σ-field $\mathscr{A}^*$ of subsets of ${}_1\Omega \times \Gamma$, such that the ${}_2x_s$ are $\mathscr{A}^*$-measurable functions on ${}_1\Omega \times \Gamma$. Moreover, the measure ${}_2\mu$ on ${}_2\mathscr{A}$ may be extended to ${}_2\mathscr{A}^*$ in such a way that the finite dimensional distributions of the ${}_2x_s$-process are still given by (3.5). The σ-field ${}_2\mathscr{A}^*$, however, is finer than ${}_2\mathscr{A}$ or ${}_2\overline{\mathscr{A}}$, the completion of ${}_2\mathscr{A}$ by ${}_2\mu$, so the considerable advantage of measurability with respect to the product σ-field ${}_2\mathscr{A} = {}_1\mathscr{A} \times \mathscr{C}$ is lost. Proofs are given in [11], II.

In the study of continuous parameter processes often the assumptions of measurability and separability are needed. If the original and deriving process are measurable, then the derived process is measurable by theorem 3.3. With separability the situation is different. If the original and the deriving process are separable, the derived process may not be separable. The finite-dimensional distributions of the ${}_1x_t$-process and the τ_s-process even may be such that to every countable dense subset Σ of $[0, \infty)$ there corresponds a ${}_2\mu$-null event that contains all Σ-separable sample functions of the ${}_2x_s$-process defined by (2.1). A trivial example of such distributions is given by $P\{{}_1x_t = 0\} = 1$, $t \geqq 0$, $t \neq t_0 > 0$, $P\{{}_1x_{t_0} = 1\} = 1$, $P\{\tau_s - \tau_0 = s\} = 1$, $s \geqq 0$, and τ_0 having a continuous distribution restricted to $(0, t_0)$. This difficulty may be overcome by replacing ${}_2x_s$ by a separable standard modification. The finite dimensional distributions of this standard modification are still given by (3.5), so the standard modification provides an interpretation of the derived distributions in terms of random variables. The relation (2.1) however now holds outside a ${}_2\mu$-null set that depends on s.

4. Convergence properties

If $\lim_{s\to\infty} \tau_s(\gamma) = +\infty$, then ${}_2x_s({}_2\omega) = {}_1x_{\tau_s(\gamma)}({}_1\omega)$, considered as a function of s, is a "subsequence" of $\{{}_1x_t({}_1\omega),\ 0 \leqq t < \infty\}$. Therefore, if the original process converges in some way for $t \to \infty$, it is to be expected that the derived process has the same limiting behaviour for $s \to \infty$, if $\tau_s \to +\infty$ in a suitable way. For almost sure convergence the argument suggested above gives:

THEOREM 4.1. Let $+\infty$ be a limit point of T and S, and let $f(\cdot)$ be a real valued function on X. If $\lim_{s\to\infty} \tau_s(\gamma) = +\infty$ for $\gamma \in F$, then for $\gamma \in F$ one has

$$\liminf_{t\to\infty} f({}_1x_t) \leqq \liminf_{s\to\infty} f({}_2x_s) \leqq \limsup_{s\to\infty} f({}_2x_s) \leqq \limsup_{t\to\infty} f({}_1x_t). \tag{4.1}$$

COROLLARY. If $\lim_{t\to\infty} f({}_1x_t) = y$, a.s. $[{}_1\mu]$, and $\lim_{s\to\infty} \tau_s = +\infty$, a.s. $[Q]$, then $\lim_{s\to\infty} f({}_2x_s) = y$, a.s. $[{}_2\mu]$.
The relation (4.1) holds irrespective of measurability considerations.

For convergence in law and convergence in probability we have

THEOREM 4.2. Let $+\infty$ be a limit point of T and S.
If for a fixed $E \in \mathscr{X}$

$$\lim_{t\to\infty} {}_1\mu\{{}_1x_t \in E\} = p, \tag{4.2}$$

and if $\tau_s \overset{Q}{\to} +\infty$ for $s \to \infty$, then

$$\lim_{s\to\infty} {}_2\mu\{{}_2x_s \in E\} = p. \tag{4.3}$$

PROOF. The theorem follows by a continuous version of the Toeplitz-Schur theorem or by the following relation obtained from (3.5):

$$|p - {}_2\mu\{{}_2x_s \in E\}| = \left|\int_T (p - {}_1\mu\{{}_1x_t \in E\}) dQ_s(t)\right|$$
$$\leqq \int_{(-\infty, A]T} |p - {}_1\mu\{{}_1x_t \in E\}| dQ_s(t) + \int_{(A,\infty)T} |p - {}_1\mu\{{}_1x_t \in E\}| dQ_s(t)$$
$$\leqq Q\{\tau_s \leqq A\} + \sup_{t \geqq A} |p - {}_1\mu\{{}_1x_t \in E\}|$$

where $\lim_{s\to\infty} Q\{\tau_s \leqq A\} = 0$ since $\tau_s \overset{Q}{\to} +\infty$.

THEOREM 4.3. Let $+\infty$ be a limit point of T and S and let $f(\cdot)$ be a real valued measurable function on $(X, \mathscr{X})$. If $f({}_1x_t) \overset{{}_1\mu}{\to} y$ for $t \to \infty$ and if $\tau_s \overset{Q}{\to} +\infty$ for $s \to \infty$, then $f({}_2x_s) \overset{{}_2\mu}{\to} y$.

PROOF. Apply theorem 4.2 to the original process ${}_1z_t \overset{\text{df}}{=} f({}_1x_t) - y$, $t \in T$, and the derived process ${}_2z_s \overset{\text{df}}{=} {}_1z_{\tau_s} = f({}_2x_s) - y$, $s \in S$, where $\lim_{t\to\infty} {}_1\mu\{|{}_1z_t| \geqq \varepsilon\} = 0$.

In special cases theorem 4.1 may have a converse as is shown by

THEOREM 4.4. Let $T = \{\ldots, -2, -1, 0, 1, \ldots\}$ or $T = \{0, 1, 2, \ldots\}$ and $S = \{0, 1, 2, \ldots\}$ or $S = [0, \infty)$. If the τ_s-process has independent stationary aperiodic increments and if $E\{|\tau_{s+1}-\tau_s|\} < \infty$, $E\{\tau_{s+1}-\tau_s\} > 0$, then for every real valued measurable function $f(\cdot)$:

$$\liminf_{s\to\infty} f({}_2x_s) = \liminf_{t\to\infty} f({}_1x_t), \text{ a.s. } [{}_2\mu], \tag{4.4}$$

$$\limsup_{s\to\infty} f({}_2x_s) = \limsup_{t\to\infty} f({}_1x_t), \text{ a.s. } [{}_2\mu]. \tag{4.5}$$

In particular, under the conditions of this theorem a.s. convergence of the derived process and a.s. convergence of the original process are equivalent.

The increments of the τ_s-process are called aperiodic if 1 is the greatest common divisor of all k for which $P\{\tau_{s+1}-\tau_s = k\} > 0$, $k = 1, 2, \ldots$. Note that the τ_s must be integer valued since $\tau_s \in T$.

PROOF. First assume that $S = \{0, 1, 2, \ldots\}$. Then both sides of (4.4) and (4.5) are measurable functions on $({}_2\Omega, {}_2\mathscr{A})$, so

$$G \overset{\text{df}}{=} \{\liminf_{s\to\infty} f({}_2x_s) \neq \liminf_{t\to\infty} f({}_1x_t)\} \in {}_2\mathscr{A} = {}_1\mathscr{A} \times \mathscr{C}.$$

Take ${}_1\omega$ fixed. There is an increasing sequence $k_1, k_2, \ldots$, with $k_n \to \infty$, depending on ${}_1\omega$, such that $\liminf_{t\to\infty} f({}_1x_t({}_1\omega)) = \lim_{n\to\infty} f({}_1x_{k_n}({}_1\omega))$. A set $W \in \mathscr{C}$ exists, with $Q(W) = 0$, such that $\lim_{s\to\infty} \tau_s(\gamma) = +\infty$ for $\gamma \notin W$. Moreover, by a theorem of Chung and Derman [2], there is $V_{{}_1\omega} \in \mathscr{C}$ with $Q(V_{{}_1\omega}) = 0$, such that $\tau_s(\gamma) \in \{k_1, k_2, \ldots\}$ for infinitely many values of s, if $\gamma \notin V_{{}_1\omega}$. So, for $\gamma \notin W \cup V_{{}_1\omega}$, we have

$$\liminf_{s\to\infty} f({}_2x_s({}_1\omega, \gamma)) \leqq \lim_{n\to\infty} f({}_1x_{k_n}({}_1\omega)) = \liminf_{t\to\infty} f({}_1x_t({}_1\omega)).$$

By theorem 4.1, for $\gamma \notin W$:

$$\liminf_{s\to\infty} f({}_2x_s({}_1\omega, \gamma)) \geqq \liminf_{t\to\infty} f({}_1x_t({}_1\omega)).$$

So every ${}_1\omega$-section of G is Q-null, which implies that ${}_2\mu(G) = 0$, since $G \in {}_1\mathscr{A} \times \mathscr{C}$.

If $S = [0, \infty)$, then (4.4) follows by theorem 4.1 and what has been proved above, since, if $\Sigma \overset{\text{df}}{=} \{0, 1, 2, \ldots\}$, we have for every ${}_2\omega$:

$$\liminf_{s\in S} f({}_2x_s({}_2\omega)) \leqq \liminf_{s\in\Sigma} f({}_2x_s({}_2\omega)).$$

The proof of (4.5) proceeds in the same way.

Theorem 4.4 does not hold if the original process has a continuous parameter, not even under fairly restrictive assumptions on the continuity of the ${}_1x_t$-process, as is shown by the following example: $\{{}_1x_t,\ t \geqq 0\}$ is a Markov process with stationary transition probabilities, having the natural numbers as states, the Q-matrix of the process being given by

$$\begin{array}{lll} q_{ij} = 0, & j \neq i, & j \neq i+1, \\ q_{2n\ 2n+1} = \lambda_n, & q_{2n,2n} = -\lambda_n, & n = 1, 2, \ldots, \\ q_{2n+1,\ 2n+2} = a > 0, & q_{2n+1,2n+1} = -a, & n = 0, 1, 2, \ldots. \end{array}$$

If $\sum_{n=1}^{\infty} \lambda_n^{-1} < \infty$, the Lebesgue measure of the t set $E({}_1\omega) \stackrel{\text{df}}{=} \{t : {}_1x_t({}_1\omega) \text{ even}\}$ is finite with probability 1, which for a large class of τ_s-processes of the type considered in theorem 4.4 implies that $\tau_s(\gamma) \in E({}_1\omega)$ only for finitely many s-values.

Several theorems on derived Markov processes that were proved in [3] and [4], turn out to be special cases of the theorems above. That a derived Markov process has the same limiting distribution as the original Markov process follows by theorem 4.2; that a transient state of the original process is a transient state of the derived process, is a consequence of theorem 4.1; and that a recurrent state of the original process is a recurrent state of the derived process if $\int_{-\infty}^{+\infty} t d_t b(s, t) < \infty$, is immediate by (4.5) if the original process has a discrete parameter.

In the theorem below ${}_1\mathscr{B}[t, \infty)$, $\mathscr{C}[s, \infty)$ and ${}_2\mathscr{B}[s, \infty)$ respectively denote the σ-fields of subsets of ${}_1\Omega$, Γ and ${}_2\Omega$ generated by $\{{}_1x_\tau,\ \tau \in T,\ \tau \geqq t\}$, $\{\tau_\sigma,\ \sigma \in S,\ \sigma \geqq s\}$ and $\{{}_2x_\sigma,\ \sigma \in S,\ \sigma \geqq s\}$. Furthermore, ${}_1\mathscr{B}_\infty \stackrel{\text{df}}{=} \bigcap_t {}_1\mathscr{B}[t, \infty)$, $C_\infty \stackrel{\text{df}}{=} \bigcap_t \mathscr{C}[s, \infty)$, ${}_2\mathscr{B}_\infty \stackrel{\text{df}}{=} \bigcap_s {}_2\mathscr{B}[s, \infty)$.

Theorem 4.5. If the τ_s- process satisfies the zero-one law, i.e. if every $\mathscr{C}_\infty$-measurable γ function is constant a.s. $[Q]$, then every ${}_2\mathscr{B}_\infty$-measurable function is equal a.s. $[{}_2\mu]$ to an ${}_1\mathscr{A}$-measurable function.

If both the original and deriving process satisfy the zero-one law and $\tau_s \to +\infty$, a.s. $[Q]$, then the derived process satisfies the zero-one law.

Proof. Let f be a ${}_2\mathscr{B}_\infty$-measurable function on ${}_2\Omega$. It is no restriction to assume that f is bounded. By theorem 3.1 f is ${}_1\mathscr{A} \times \mathscr{C}[s, \infty)$-measurable for every $s \in S$. So any ${}_1\omega$-section $f({}_1\omega, \cdot)$ is $\mathscr{C}[s, \infty)$-measurable for every s, and therefore C_∞-

measurable. Since the deriving process satisfies the zero-one law, there is a function φ of ${}_1\omega$ such that

$$Q\{\gamma : f({}_1\omega, \gamma) \neq \varphi({}_1\omega)\} = 0,\ {}_1\omega \in {}_1\Omega. \tag{4.6}$$

By Fubini's theorem, since f is ${}_1\mathscr{A} \times \mathscr{C}$-measurable, $\int f({}_1\omega, \gamma)dQ(\gamma)$ is an ${}_1\mathscr{A}$-measurable function of ${}_1\omega$. But by (4.6)

$$\int f({}_1\omega, \gamma)dQ(\gamma) = \varphi({}_1\omega),\ {}_1\omega \in {}_1\Omega.$$

So φ is ${}_1\mathscr{A}$-measurable. Therefore the ${}_2\omega$ set: $\{({}_1\omega, \gamma) : f({}_1\omega, \gamma) \neq \varphi({}_1\omega)\}$ is ${}_1\mathscr{A} \times \mathscr{C}$-measurable. Since by (4.6) all its ${}_1\omega$-sections are Q-null, we have

$$f({}_1\omega, \gamma) = \varphi({}_1\omega), \quad \text{a.s. } [{}_2\mu] \tag{4.7}$$

which proves the first assertion of the theorem.

To prove the second assertion we note that there is $F \in \mathscr{C}$ with $Q(F) = 0$, so that for every $\gamma \notin F$ we have $\lim_{s\to\infty} \tau_s(\gamma) = +\infty$ and ${}_1\mu\{{}_1\omega : f({}_1\omega, \gamma) \neq \varphi({}_1\omega)\} = 0$ by (4.7). Take γ_0 fixed, $\gamma_0 \notin F$. Then it can be shown that the γ_0-section $f(\cdot, \gamma_0)$ is ${}_1\mathscr{B}[t, \infty)$-measurable for every $t \in T$ and therefore ${}_1\mathscr{B}_\infty$-measurable. So $f(\cdot, \gamma_0)$ is equal to a constant a.s. $[{}_1\mu]$, and therefore φ must be equal to a constant a.s. $[{}_1\mu]$, which completes the proof.

The following theorems are concerned with laws of large numbers and central convergence.

THEOREM 4.6. Let $+\infty$ be a limit point of T and S and let X be the real line. If

$$\lim_{t\to\infty} \frac{{}_1x_t}{t} = a, \text{ a.s. } [{}_1\mu],$$

$$\lim_{s\to\infty} \frac{\tau_s}{s} = b, \text{ a.s. } [Q],$$

and $\lim_{s\to\infty} \tau_s = +\infty$, a.s. $[Q]$, then

$$\lim_{s\to\infty} \frac{{}_2x_s}{s} = ab, \text{ a.s. } [{}_2\mu].$$

PROOF. The theorem is immediate by the relation

$$\frac{{}_2x_s}{s} = \frac{{}_1x_{\tau_s}}{\tau_s} \frac{\tau_s}{s}.$$

THEOREM 4.7. Let $T = \{0, 1, 2, \ldots\}$ and $S = [0, \infty)$ or $S = \{0, 1, 2, \ldots\}$ and let X be the real line. If the τ_s-process has stationary nonnegative aperiodic increments with

$$0 < \mu \overset{\text{df}}{=} E\{\tau_{s+1} - \tau_s\} < \infty \text{ and if}$$

$$\lim_{s\to\infty} \frac{{}_2x_s}{s} = c, \text{ a.s. } [{}_2\mu],$$

then

$$\lim_{t\to\infty} \frac{{}_1x_t}{t} = \frac{1}{\mu}\, c, \text{ a.s. } [{}_1\mu].$$

PROOF. Since $\lim_{s\to\infty} \tau_s/s = \mu$, a.s. $[Q]$, we have

$$\lim_{s\to\infty} \frac{{}_1x_{\tau_s}}{\tau_s} = \frac{1}{\mu}\, c, \text{ a.s. } [{}_2\mu].$$

By an argument similar to the proof of theorem 4.4 it is shown that

$$\lim_{t\to\infty} \frac{{}_1x_t}{t} = \lim_{s\to\infty} \frac{{}_1x_{\tau_s}}{\tau_s} = \frac{1}{\mu}\, c.$$

THEOREM 4.8. Let $+\infty$ be a limit point of T and S and let X be the real line. Let

$$\text{(4.8)} \quad \lim_{t\to\infty} {}_1E \exp\left[iu \frac{{}_1x_t - at}{{}_1b(t)}\right] = g(u), \; -\infty < u < \infty,$$

$$\text{(4.9)} \quad \lim_{s\to\infty} E \exp\left[iu \frac{\tau_s - \alpha(s)}{\beta(s)}\right] = h(u), \; -\infty < u < \infty,$$

$$\text{(4.10)} \quad \tau_s \overset{Q}{\to} +\infty \quad \text{for} \quad s \to \infty,$$

$$\text{(4.11)} \quad \frac{{}_1b(\tau_s)}{{}_2b(s)} \overset{Q}{\to} c \quad \text{for} \quad s \to \infty,$$

$$\text{(4.12)} \quad \frac{\beta(s)}{{}_2b(s)} \to \lambda \quad \text{for} \quad s \to \infty,$$

where $g(\cdot)$ and $h(\cdot)$ are characteristic functions, ${}_1b(\cdot)$, ${}_2b(\cdot)$ and $\beta(\cdot)$ are nonnegative increasing functions converging to $+\infty$, whereas c and λ are finite constants. Then we have

$$\text{(4.13)} \quad \lim_{s\to\infty} {}_2E \exp\left[iu \frac{{}_2x_s - a\alpha(s)}{{}_2b(s)}\right] = g(cu)h(a\lambda u), \; -\infty < u < \infty,$$

If $a = 0$, conditions (4.9) and (4.12) may be omitted.

In terms of derived distributions (cf. section 7 below) the theorem may be stated roughly as follows: if a distribution and a deriving distribution are attracted by certain types of probability laws, then the derived distribution under certain conditions is attracted by a convolution of distributions belonging to the attracting types. Whether or not both components actually are present in this convolution depends on the relative limiting behaviour of the norming sequences for the original and deriving distribution. The component arising from the deriving distribution is present only if the attraction of the original distribution requires centering.

For the proof of theorem 4.8 we need the following lemma.

LEMMA 4.1. For every s let $f_s(\cdot)$ and $\varphi(_s\cdot)$ be real valued measurable functions on Γ, satisfying

$$|f_s(\gamma)| \leqq M, \quad |\varphi_s(\gamma)| \leqq M, \text{ a.s. } [Q],$$
$$\varphi_s(\gamma) \overset{Q}{\rightarrow} \mu \quad \text{for} \quad s \rightarrow \infty,$$
$$\int f_s(\gamma) dQ(\gamma) \rightarrow \nu \quad \text{for} \quad s \rightarrow \infty,$$

where μ and ν are constants. Then

$$\int f_s(\gamma)\varphi_s(\gamma)dQ(\gamma) \rightarrow \mu\nu.$$

PROOF. We have

$$\int f_s(\gamma)\varphi_s(\gamma)dQ = \mu \int f_s(\gamma)dQ + \int f_s(\gamma)\{\varphi_s(\gamma)-\mu\}dQ. \tag{4.14}$$

Since $(\varphi_s-\mu)f_s \overset{Q}{\rightarrow} 0$ and

$$|f_s(\gamma)\{\varphi_s(\gamma)-\mu\} \leqq M^2+M\mu,$$

the last term in (4.14) converges to zero for $s \rightarrow \infty$.

Proof of theorem 4.8. By Fubini's theorem

$$_2E \exp\left[iu \frac{_2x_s - a\alpha(s)}{_2b(s)}\right] = \int f_s(\gamma)\varphi_s(\gamma)dQ(\gamma), \tag{4.15}$$

where

$$f_s(\gamma) = \exp\left[iua \frac{\beta(s)}{_2b(s)} \frac{\tau_s - \alpha(s)}{\beta(s)}\right],$$
$$\varphi_s(\gamma) = {_1E} \exp\left[iu \frac{_1b(\tau_s(\gamma))}{_2b(s)} \frac{_1x_{\tau_s}(\gamma) - a\tau_s(\gamma)}{_1b(\tau_s(\gamma))}\right].$$

By (4.9), (4.12) and a well known theorem on characteristic functions (cf. Loève [9], p. 192, corollary 1) it follows that

$$\lim_{s\rightarrow\infty} \int f_s(\gamma)dQ(\gamma) = h(a\lambda u). \tag{4.16}$$

Moreover

(4.17) $$\varphi_s(\cdot) \overset{Q}{\rightarrow} g(cu).$$

To prove (4.17) let $\{s_n\}$ be any sequence in S with $s_n \to +\infty$. By (4.10) and (4.11) there is a subsequence $\{\sigma_n\}$ of $\{s_n\}$ such that

(4.18) $$\tau_{\sigma_n}(\gamma) \to +\infty, \quad \gamma \notin F,$$

(4.19) $$\frac{{}_1b(\tau_{\sigma_n}(\gamma))}{{}_2b(\sigma_n)} \to c, \ \gamma \notin F,$$

with $Q(F) = 0$, the set F depending on $\{\sigma_n\}$. By (4.18), (4.19), (4.8) and the theorem on characteristic functions mentioned above, it follows that $\varphi_{\sigma_n}(\gamma) \to g(cu)$ for every $\gamma \in F$. So every sequence $\{s_n\} \in S$ with $s_n \to +\infty$ contains a subsequence $\{\sigma_n\}$ such that $\varphi_{\sigma_n}(\cdot) \to g(cu)$, a.s. $[Q]$, which proves (4.17).

The theorem now follows from (4.16), (4.17) and lemma 4.1.

5. Derived Markov processes

Here it will be shown that if the original process is a Markov process with stationary transition probabilities and the deriving process has nonnegative independent increments, then the derived process is a Markov process.

Throughout this section the parameter sets T and S will be restricted to be either $[0, \infty)$ or $\{0, 1, 2, \ldots\}$. Then the difference set $T_\Delta \overset{\text{df}}{=} \{t'-t'': t' \in T, t'' \in T\}$ is identical with T and $S_\Delta = S$, which simplifies the formulation of the conditions below. We assume that the ${}_1x_t$-process is a Markov process with stationary transition probabilities, by which is meant here that the following condition holds:

A. For every $E \in \mathscr{X}$ there exists a function ${}_1p(\cdot, E, \cdot)$ on $X \times T$ with the following properties:

A_1 For every $\tau \in T$ the function ${}_1p(\cdot, E, \tau)$ is an $\mathscr{X}$-measurable function on X.

A_2 For every $E \in \mathscr{E}$, $n = 0, 1, 2, \ldots$ and every $t_1, \ldots, t_n, t, t+\tau$ in T with $t_1 \leqq \ldots \leqq t_n \leqq t \leqq t+\tau$ we have that ${}_1p({}_1x_t, E, \tau)$ is a version of the conditional probability ${}_1\mu\{{}_1x_{t+\tau} \in E | {}_1x_{t_1}, \ldots, {}_1x_{t_n}, {}_1x_t\}$.

In general we will have to require that the function ${}_1p$ satisfies the following condition:

B. For every $E \in \mathscr{X}$ the function ${}_1p(\cdot, E, \cdot)$ is an $\mathscr{X} \times \mathscr{T}$-measurable function on $X \times T$.

For easy reference two further important conditions on the original process are listed below:

C. The function ${}_1p(\cdot, \cdot, \cdot)$ defines regular versions of the transition probabilities, i.e. for every $x \in X$ and $\tau \in T$ the set function ${}_1p(x, \cdot, \tau)$ is a probability measure on $\mathscr{X}$.

D. The Chapman-Kolmogorov equation is satisfied without the exception of null sets, i.e.

$$ {}_1p(x, E, \tau_1+\tau_2) = \int_{\xi \in X} {}_1p(x, d\xi, \tau_1)\, {}_1p(\xi, E, \tau_2), $$
$$ x \in X,\ E \in \mathscr{X},\ \tau_1 \in T,\ \tau_2 \in T. $$

It is essential that ${}_1p(E, x, t)$ for $t = 0$ is defined as 1 if $x \in E$ and 0 if $x \notin E$, whether ${}_1p(x, E, t)$ is continuous in $t = 0$ or not:

$$ (5.1) \qquad {}_1p(x, E, 0) \stackrel{\text{df}}{=} I_E(x),\ x \in X,\ E \in \mathscr{X}. $$

We now have

Theorem 5.1. If the original process is an ${}_1\mathscr{A} \times \mathscr{T}$-measurable Markov process satisfying conditions A and B, and the deriving process has nonnegative independent increments, then the derived process is a Markov process whose transition probabilities from time s to $s+\sigma$ are given by

$$ (5.2) \quad {}_2p(x, F, s, s+\sigma) \stackrel{\text{df}}{=} \int_T {}_1p(x, F, \tau) dG_{s, s+\sigma}(\tau), $$
$$ F \in \mathscr{X},\ x \in X,\ s \in S,\ s+\sigma \in S,\ \sigma \geqq 0. $$

Here $G_{s, s+\sigma}(\cdot)$ denotes the distribution function of $\boldsymbol{\tau}_{s+\sigma} - \boldsymbol{\tau}_s$.

If moreover the deriving process has stationary increments, the derived process has stationary transition probabilities given by

$$ (5.3) \quad {}_2p(x, F, \sigma) \stackrel{\text{df}}{=} \int_T {}_1p(x, F, \tau) d_\tau b(\sigma, \tau),\ F \in \mathscr{X},\ x \in X,\ \sigma \in S, $$

where $b(\sigma, \cdot)$ is the distribution function of $\boldsymbol{\tau}_{s+\sigma} - \boldsymbol{\tau}_s$.

If the original process satisfies condition C, then (5.2) or (5.3) define regular versions of the transition probabilities of the derived process.

Proof. By theorem 3.1 the ${}_2\boldsymbol{x}_s$-process is a stochastic process. By condition B and Fubini's theorem the integrals in (5.2) and (5.3) exist and are $\mathscr{X}$-measurable functions of x.

Now take $s_1 \leqq \ldots \leqq s_n \leqq s \leqq s+\sigma$ in S and $E \in \mathscr{X}^{n+1}$, $H \in \mathscr{X}$. Then by theorem 3.2 and the fact that the $\boldsymbol{\tau}_s$-process has independent increments:

(5.4) $${}_2\mu\{({}_2x_s, \ldots, {}_2x_{s_n}, {}_2x_s) \in E, {}_2x_{s+\sigma} \in H\}$$
$$= \int_{T^{n+2}} {}_1\mu\{({}_1x_{t_1}, \ldots, {}_1x_{t_n}, {}_1x_t) \in E, {}_1x_{t+\tau} \in H\} \, dQ_{s_1,\ldots,s_n,s}(t_1, \ldots, t_n, t) dG_{s,s+\sigma}(\tau)$$
$$= \int_{T^{n+1}} \left[\int_T {}_1\mu\{({}_1x_{t_1}, \ldots, {}_1x_{t_n}, {}_1x_t) \in E, {}_1x_{t+\tau} \in H\} dG_{s,s+\sigma}(\tau) \right] dQ_{s_1,\ldots,s_n,s}(t_1, \ldots, t_n, t),$$

changing the order of integration being justified by Fubini's theorem since the integrand is a nonnegative measurable function of $(t_1, \ldots, t_n, t, t+\tau)$ by lemma 3.2 and the measurability of the ${}_1x_t$-process.

Since the τ_s-process has nonnegative increments, the integration with respect to $t_1, \ldots, t_n, t$ may be restricted to the domain $t_1 \leqq \ldots \leqq t_n \leqq t$. So by condition A and the definition of conditional probability, we have for $\tau \geqq 0$, taking into account (5.1):

$$ {}_1\mu\{({}_1x_{t_1}, \ldots, {}_1x_{t_n}, {}_1x_t) \in E, {}_1x_{t+\tau} \in H\} = \int_{{}_1\Omega} I\{({}_1x_{t_1}, \ldots, {}_1x_{t_n}, {}_1x_t) \in E\} {}_1p({}_1x_t, H, \tau) d_1\mu,$$

where $I\{A\}$ denotes the indicator function of the event A. For $t_1, \ldots, t_n$ and t fixed, the integrations in (5.4) with respect to $d_1\mu$ and $dG_{s,s+\sigma}(\tau)$ may be interchanged since $I\{({}_1x_{t_1}, \ldots, {}_1x_{t_n}, {}_1x_t) \in E\}{}_1p({}_1x_t, H, \tau)$ is a nonnegative ${}_1\mathscr{A} \times \mathscr{T}$-measurable function of ${}_1\omega$ and τ by condition B and the fact that $E \in \mathscr{X}^{n+1}$. So

(5.5) $${}_2\mu\{({}_2x_{s_1}, \ldots, {}_2x_{s_n}, {}_2x_s) \in E, {}_2x_{s+\sigma} \in H\}$$
$$= \int_{T^{n+1}} \left[\int_{{}_1\Omega} I\{({}_1x_{t_1}, \ldots, {}_1x_{t_n}, {}_1x_t) \in E\} {}_2p({}_1x_t, H, s, s+\sigma) d_1\mu \right] dQ_{s_1,\ldots,s_n,s}(t_1, \ldots, t_n, t),$$

where ${}_2p(x, H, s, s+\sigma)$ is given by (5.2). Since the integrand in the right-hand side of (5.5) is a bounded $\mathscr{E}^{n+1}$-measurable function ${}_1x_{t_1}, \ldots, {}_1x_{t_n}, {}_1x_t$, it follows by (3.4) that

(5.6) $${}_2\mu\{({}_2x_{s_1}, \ldots, {}_2x_{s_n}, {}_2x_s) \in E, {}_2x_{s+\sigma} \in H\} = \int_{{}_2\Omega} I\{({}_2x_{s_1}, \ldots, {}_2x_{s_n}, {}_2x_s) \in E\} {}_2p({}_2x_s, H, s, s+\sigma) d_2\mu.$$

From (5.6) it follows by definition of conditional probability that the ${}_2x_s$-process is a Markov process and that ${}_2p({}_2x_s, H, s, s+\sigma)$ is a version of ${}_2\mu\{{}_2x_{s+\sigma} \in H | {}_2x_s\}$.

The second assertion of the theorem is immediate, since if $G_{s,s+\sigma}(\cdot)$ does not depend on s, the same is true of ${}_2p(x, H, s, s+\sigma)$.

Finally, if the original process satisfies condition C, then it is easily seen from (5.2) or (5.3) that ${}_2p(x, \cdot, s, s+\sigma)$ or ${}_2p(x, \cdot, \sigma)$

for fixed x, s and σ determines a probability measure on $\mathscr{X}$.

THEOREM 5.2. If the state space N of the $\boldsymbol{\tau}_s$-process is countable, the conclusions of theorem 5.1 hold without the assumption that the original process is ${}_1\mathscr{A}\times\mathscr{T}$-measurable or satisfies condition B.

The proof proceeds in the same way as the proof of theorem 5.1, no measurability conditions being involved since the integrals in (5.2)-(5.5) reduce to sums. That ${}_2p(\cdot, E, s, s+\sigma)$ is an $\mathscr{X}$-measurable function, follows by condition A.

It is noted that (5.2) and (5.3) may be written as

$$ {}_2p(x, F, s, s+\sigma) = E_1 p(x, F, \boldsymbol{\tau}_{s+\sigma}-\boldsymbol{\tau}_s). \tag{5.7}$$

If $\boldsymbol{\tau}_0 = 0$, we may write (5.3) as

$$ {}_2p(x, F, \sigma) = E_1 p(x, F, \boldsymbol{\tau}_\sigma). \tag{5.8}$$

Theorems 5.1 and 5.2 show that under certain conditions the interpretation in terms of random variables given in section 1 for the concept of derived transition matrix as defined in [3] and [4], is correct. It is seen that by deriving a stationary Markov process by a process with nonstationary independent nonnegative increments a Markov process with nonstationary transition probabilities may be obtained.

A remark similar to the one made at the end of section 3 applies to the theorems of this section: the assumptions may be weakened to $\overline{{}_1\mathscr{A}\times\mathscr{T}}$-measurability of the original process and $\overline{\mathscr{X}\times\mathscr{T}}$-measurability of the transition probabilities. We refer to [11], I.

In section 3 we mentioned the fact that measurability of the original process may be replaced by a measurability condition on its finite-dimensional distributions if it is merely required that the ${}_2\boldsymbol{x}_s$ are random variables measurable with respect to *some* σ-field of ${}_2\omega$ sets and theorem 3.2 holds. In [11], II it is shown that if the original process is a Markov process, condition B is sufficient to guarantee the required measurability in $(t_1, \ldots, t_n)$ of ${}_1\mu\{({}_1\boldsymbol{x}_{t_1}, \ldots, {}_1\boldsymbol{x}_{t_n}) \in E\}$. So theorem 5.1 actually holds under condition B alone, except that the ${}_2\boldsymbol{x}_s$ may be measurable with respect to a σ-field ${}_2\mathscr{A}^* \supset {}_2\mathscr{A}$.

That condition B is central in the theory of derived Markov processes, also follows from the fact that it is essential in the proof of the following theorem which states that under condition B the derived transition probabilities satisfy the Chapman-Kolmogorov equation, if the original transition probabilities satisfy this equation.

THEOREM 5.3. If the function ${}_1p(\cdot, \cdot, \cdot)$ satisfies conditions B, C and D, then (5.2) and (5.3) for fixed x, s and σ define probability measures on $\mathscr{X}$ and we have

$$(5.9)\quad {}_2p(x, E, s_1, s_3) = \int_{\xi \in X} {}_2p(x, d\xi, s_1, s_2)\,{}_2p(\xi, E, s_2, s_3),$$
$$x \in X,\ E \in \mathscr{X},\ s_1 \leqq s_2 \leqq s_3,\ s_1, s_2, s_3 \in S.$$

For the proof we make use of

LEMMA 5.1. For every $\tau \in T$ let ν_τ be a measure on $(X, \mathscr{X})$, such that $\nu_\tau(E)$ for every $E \in \mathscr{X}$ is a Borel function of τ. Let $G(\cdot)$ be the distribution function of a measure on T and

$$(5.10)\qquad \nu'(E) \overset{\text{df}}{=} \int_T \nu_\tau(E)\,dG(\tau), \quad E \in \mathscr{X}.$$

Then ν' is a measure on $(X, \mathscr{X})$. If $f(\cdot)$ is a nonnegative $\mathscr{X}$-measurable function on X, then $\int_X f(x)d\nu_\tau(x)$ is an extended real valued Borel function of τ and

$$(5.11)\qquad \int_X f(x)d\nu'(x) = \int_T dG(\tau)\left\{\int_X f(x)\,d\nu_\tau(x)\right\}.$$

If $f(\cdot)$ is allowed to assume positive and negative values, the conclusions continue to hold if $\int_X |f(x)|d\nu'(x) < \infty$.

That ν' is a measure is immediate. The assertions on $f(\cdot)$ follow in the usual way by first taking for $f(\cdot)$ indicators and simple functions, then applying the monotone convergence theorem and finally decomposing $f(\cdot)$ into its positive and negative parts.

Proof of theorem 5.3. It is easily seen that (5.2) and (5.3) define probability measures on $\mathscr{X}$. By applying first (5.2) and (5.11), then (5.2), then condition B and Fubini's theorem, and finally condition D, we have

$$\begin{aligned}
&\int_X {}_2p(x, d\xi, s_1, s_2)\,{}_2p(\xi, E, s_2, s_3)\\
&= \int_T dG_{s_1,s_2}(\tau)\int_X {}_1p(x, d\xi, \tau)\,{}_2p(\xi, E, s_2, s_3)\\
&= \int_T dG_{s_1,s_2}(\tau)\int_X {}_1p(x, d\xi, \tau)\int_T dG_{s_2,s_3}(\tau')\,{}_1p(\xi, E, \tau')\\
&= \int_T dG_{s_1,s_2}(\tau)\int_T dG_{s_2,s_3}(\tau')\int_X {}_1p(x, d\xi, \tau)\,{}_1p(\xi, E, \tau')\\
&= \int_T dG_{s_1,s_2}(\tau)\int_T dG_{s_2,s_3}(\tau')\,{}_1p(x, E, \tau+\tau').
\end{aligned}$$

Since $G_{s_1,s_3}(\cdot)$ is the convolution of $G_{s_1,s_2}(\cdot)$ and $G_{s_2,s_3}(\cdot)$, the last expression reduces to $\int_T dG_{s_1,s_3}(t)\,{}_1p(x, E, t) = {}_2p(x, E, s_1, s_3)$, which proves (5.9).

As before, if the state space N of the deriving process is countable, the conclusions of theorem 5.3 are valid without the assumption that condition B holds.

The conditions that are usually imposed on a family of transition probabilities, are sufficiently strong to guarantee that the assumptions of the theorems of this section are satisfied. To illustrate this point we assume that X is the real line and $T = [0, \infty)$.

First assume that the function ${}_1p(\cdot, \cdot, \cdot)$ satisfies condition A and C, and that for every $x \in X$ the distribution function of the probability measure ${}_1p(x, \cdot, t)$ converges completely to the unit step function with jump at x if $t \to 0^+$. Then it is easily seen that the ${}_1\boldsymbol{x}_t$-process is continuous in probability from the right. So there exists a measurable standard modification of the ${}_1\boldsymbol{x}_t$-process (see Loève [9], p. 512, 513). Since this standard modification has the same finite-dimensional distributions as the ${}_1\boldsymbol{x}_t$-process, there is no restriction in assuming a priori that the ${}_1\boldsymbol{x}_t$-process is ${}_1\mathscr{A} \times \mathscr{T}$-measurable.

If moreover the function ${}_1p(\cdot, \cdot, \cdot)$ satisfies condition D, then it is shown by a simple argument that for every $x \in X$ the distribution function of the probability measure ${}_1p(x, \cdot, t+h)$ converges completely to the distribution function of ${}_1p(x, \cdot, t)$ if $h \to 0^+$. And this is sufficient to conclude that condition B is fulfilled, as is seen by the following lemma:

Lemma 5.2. Let X be the real line and $\mathscr{X}$ be the class of Borel sets of X. Let the function ${}_1p(\cdot, \cdot, \cdot)$ be subject to the following conditions:

(i) For every interval $[a, b)$ and every $t \in T$ the function ${}_1p(\cdot, [a, b), t)$ is a Borel function on X.

(ii) For every $x \in X$ and $t \in T$ the set function ${}_1p(x, \cdot, t)$ is a probability measure on $\mathscr{X}$.

(iii) If $F_x(\cdot, t)$ denotes the distribution function of the probability measure ${}_1p(x, \cdot, t)$, then $F_x(\cdot, t+h) \overset{c}{\to} F_x(\cdot, t)$ if $h \to 0^+$, for every $x \in X$ and $t > 0$.

Then or every $E \in \mathscr{X}$ the function ${}_1p(\cdot, E, \cdot)$ on $X \times T$ is $\mathscr{X} \times \mathscr{T}$-measurable.

Proof. Let

$$g(u, x, t) \overset{\text{df}}{=} \int_{-\infty}^{+\infty} e^{iu\xi}\, {}_1p(x, d\xi, t),\ x \in X,\ t \in T,\ -\infty < u < \infty.$$

By (i) and the fact that $g(u, x, t)$ for every u, x, t is a limit of Rieman-Stieltjes sums, $g(u, \cdot, t)$ for every u and t is an $\mathscr{X}$-measurable function on X. By (iii) and the continuity theorem on characteristic functions $g(u, x, t)$ is continuous from the right in t for $t > 0$. From these facts it follows that for every u the function $g(u, \cdot, \cdot)$ is $\mathscr{X}\times\mathscr{T}$-measurable. For, if

$$g_m(u, x, t) \overset{\text{df}}{=} g\left(u, x, \frac{k}{m}\right) \quad \text{if} \quad \frac{k-1}{m} < t \leqq \frac{k}{m},$$
$$k = 0, 1, \ldots, m = 1, 2, \ldots,$$

then $g_m(u, \cdot, \cdot)$ for every m is $\mathscr{X}\times\mathscr{T}$-measurable and $g(u, x, t) = \lim_{m\to\infty} g_m(u, x, t)$ every u, x and $t \geqq 0$.

Now let

$$\hat{F}_x(\xi, t) \overset{\text{df}}{=} \tfrac{1}{2}F_x(\xi^-, t) + \tfrac{1}{2}F_x(\xi^+, t),\ x \in X,\ t \in T,\ -\infty < \xi < \infty.$$

Since $g(u, x, t)$ is continuous in u, we have that for every $U > 0$ and $a \leqq b$

$$J(U, a, b, x, t) \overset{\text{df}}{=} \frac{1}{2\pi}\int_{-U}^{U} \frac{e^{-iua} - e^{-iub}}{iu} g(u, x, t)du,$$

as a limit of Riemann sums, is an $\mathscr{X}\times\mathscr{T}$-measurable function of (x, t). So the same is true of

$$\hat{F}_x(b, t) - \hat{F}_x(a, t) = \lim_{U\to\infty} J(U, a, b, x, t).$$

Since $F_x(\xi, t) = \lim_{n\to\infty} \hat{F}_x(\xi - 1/n, t)$, we have that ${}_1p(x, [a, b), t)$ for every finite a and b is an $\mathscr{X}\times\mathscr{T}$-measurable function of (x, t).

Finally let $\mathscr{L}$ be the class of all sets E for which the assertion of the theorem is true. By what has been shown above, $\mathscr{L}$ contains the ring $\mathscr{R}$ of all finite disjoint unions of bounded intervals of the form $[a, b)$. Since it is easily shown by (ii) that $\mathscr{L}$ is a monotone class, $\mathscr{L}$ contains the minimal σ-field over $\mathscr{R}$, which is $\mathscr{X}$.

6. Continuity and Q-matrices

This section contains some theorems on the continuity of derived transition matrices and on their Q-matrices and transition law derivatives. Throughout this section the following assumptions hold:

The parameter sets S and T are taken to be the interval $[0, \infty)$ except in the remark at the end on the case $S = [0, \infty)$, $T = \{0, 1, 2, \ldots\}$.

The ${}_1x_t$-process is a Markov process with stationary transition probabilities, its transition matrix ${}_1p(\cdot, \cdot, \cdot)$ satisfying conditions A, B, C and D of section 5.

The τ_s-process has nonnegative independent increments and is continuous in probability. Then the distribution of $\tau_{s+\sigma}-\tau_s$ is infinitely divisible (Loève [9], p. 545). Since the τ_s-process has nonnegative increments, the Laplace-Stieltjes transform of the distribution function $G_{s,s+\sigma}(\cdot)$ of $\tau_{s+\sigma}-\tau_s$ is of the following form (cf. Phillips [10], Hille and Phillips [8], section 23.15):

$$(6.1)\quad E\,e^{-\lambda(\tau_{s+\sigma}-\tau_s)} = \int_{[0,\infty)} e^{-\lambda\tau}\,dG_{s,s+\sigma}(\tau)$$
$$= \exp\left[-\lambda m(s, s+\sigma) + \int_{(0,\infty)} (e^{-\lambda x}-1)d\Psi_{s,s+\sigma}(x)\right],$$
$$\text{Re}\,\lambda \geqq 0,\ s \geqq 0,\ \sigma \geqq 0,$$

where $m(s, s+\sigma) \geqq 0$ and $\Psi_{s,s+\sigma}(\cdot)$ is a nonpositive nondecreasing function on $(0, \infty)$, continuous from the left and satisfying

$$(6.2)\quad \int_0^1 t\,d\Psi_{s,s+\sigma}(t) < \infty, \quad s \geqq 0, \sigma \geqq 0,$$

$$(6.3)\quad \lim_{x\to\infty} \Psi_{s,s+\sigma}(x) = 0, \quad s \geqq 0, \sigma \geqq 0,$$

$$(6.4)\quad \Psi_{s,s+\sigma}(0^+) \overset{\text{df}}{=} \lim_{x\to 0^+} \Psi_{s,s+\sigma}(x) \geqq -\infty, s \geqq 0, \sigma \geqq 0.$$

Moreover, we must have

$$(6.5)\quad m(s_1, s_3) = m(s_1, s_2) + m(s_2, s_3), 0 \leqq s_1 \leqq s_2 \leqq s_3,$$

$$(6.6)\quad \Psi_{s_1,s_3}(x) = \Psi_{s_1,s_2}(x) + \Psi_{s_2,s_3}(x), -\infty < x < \infty,$$
$$0 \leqq s_1 \leqq s_2 \leqq s_3.$$

$$(6.7)\quad m_{s,s} = 0, \Psi_{s,s}(x) = 0, -\infty < x < \infty, s \geqq 0.$$

Since the τ_s-process is continuous in probability, the following relations hold

$$(6.8)\quad \lim_{\sigma\to 0^+} m(s, s+\sigma) = 0,$$

$$(6.9)\quad \lim_{\sigma\to 0^+} \Psi_{s,s+\sigma}(x) = 0, \quad 0 < x < \infty.$$

If the τ_s-process has stationary increments, then $G_{s,s+\sigma}(x) = b(\sigma, x)$, $s \geqq 0, \sigma \geqq 0, -\infty < x < \infty$, and

$$(6.10)\quad m(s, s+\sigma) = \sigma m, s \geqq 0, \sigma \geqq 0,$$

$$(6.11)\quad \Psi_{s,s+\sigma}(x) = \sigma\Psi(x), s \geqq 0, \sigma \geqq 0, x > 0,$$

where

$$m \geqq 0, \int_0^1 t d\Psi(t) < \infty, \lim_{x\to\infty} \Psi(x) = 0,$$

so that

$$(6.12)\quad E\,e^{-\lambda(\tau_{s+\sigma}-\tau_s)} = \int_{[0,\infty)} e^{-\lambda\tau}\, d_\tau b(\sigma, \tau)$$
$$= \exp\left[-\lambda m\sigma - \sigma\int_{(0,\infty)} (1-e^{-\lambda x})\, d\Psi(x)\right],\ \mathrm{Re}\,\lambda \geqq 0,\ s \geqq 0,\ \sigma \geqq 0.$$

To obtain a slight simplification in formulation we assume that

$$(6.13)\qquad \tau_0 \equiv 0.$$

First, a continuity condition of the following type will be considered:

$$(6.14)\qquad \lim_{t\to 0^+} p(x, E, t) = I_E(x),\ x \in X,\ E \in \mathscr{X}.$$

(Cf. Loève [9], § 39). If $\{x\} \in \mathscr{X}$, we may write (6.14) as

$$(6.15)\qquad \lim_{t\to 0^+} p(x, \{x\}, t) = 1.$$

If the original process satisfies a continuity condition of this type, then the same is true of the derived process:

THEOREM 6.1. If

$$(6.16)\qquad \lim_{t\to 0^+} {}_1p(x, E \cdot t) = I_E(x),$$

then

$$(6.17)\qquad \lim_{\sigma - 0^+} {}_2p(x, E, s, s+\sigma) = I_E(x),\ s \geqq 0.$$

The proof is immediate by the relation

$${}_2p(x, E, s, s+\sigma) = \int_{[0,\varepsilon)} {}_1p(x, E, t) dG_{s,s+\sigma}(t) + \int_{[\varepsilon,\infty)} {}_1p(x, E, t) dG_{s,s+\sigma}(t),$$

where the second term on the right is bounded by $\int_{[\varepsilon,\infty)} dG_{s,s+\sigma}(\tau)$ which for every $\varepsilon > 0$ converges to zero if $\sigma \to 0^+$, since the τ_s-process is continuous in probability.

There exists a large class of deriving processes for which (6.17) holds irrespective of the nature of the original transition matrix. In fact, the only condition ${}_1p(x, E, t)$ has to satisfy in this case, is measurability in t in order that the integral in (5.2) has a meaning.

THEOREM 6.2. If the distributions of the deriving process are such that in (6.1):

$$(6.18)\qquad m(s, s+\sigma) = 0,\ \sigma \geqq 0$$

(6.19) $$\lim_{\sigma\to 0^+} \Psi_{s,s+\sigma}(0^+) = 0,$$

then

(6.20) $$\lim_{\sigma\to 0^+} {}_2p(x, E, s, s+\sigma) = I_E(x),\ x\in X,\ E\in\mathscr{X}.$$

NOTE. If the τ_s-process has stationary increments, then for (6.18) and (6.19) to hold, it is sufficient that $m = 0$, $\Psi(0^+) > -\infty$ in (6.10) and (6.11).

PROOF. By (5.2) and (5.1)

$${}_2p(x, E, s, s+\sigma) = I_E(x)G_{s,s+\sigma}(0^+) + \int_{(0,\infty)} {}_1p(x, E, t)dG_{s,s+\sigma}(t).$$

Since

$$G_{s,s+\sigma}(0^+) = \lim_{\lambda\to+\infty}\int_{[0,\infty)} e^{-\lambda t}\, dG_{s,s+\sigma}(t),$$

it follows from (6.1) and (6.18) that

$$G_{s,s+\sigma}(0^+) = \exp\left[-\int_{(0,\infty)} d\Psi_{s,s+\sigma}(x)\right] = \exp\,[\Psi_{s,s+\sigma}(0^+)],$$

so $\lim_{\sigma\to 0^+} G_{s,s+\sigma}(0^+) = 1$ by (6.19), and (6.20) follows since

$$\int_{(0,\infty)} {}_1p(x\ \ E\ \ t)dG_{s,s+\sigma}(t) \leqq \int_{(0,\infty)} dG_{s,s+\sigma}(t) = 1-G_{s,s+\sigma}(0^+).$$

If the state space X is countable and $\lim_{t\to 0^+} {}_1p_{ij}(t) \neq \delta_{ij}$, then (6.18) and (6.19) are necessary in order that $\lim_{\sigma\to 0^+} {}_2p_{ij}(s, s+\sigma) = \delta_{ij}$. This is shown in [12].

Under the continuity condition (6.14) the Q-matrix of the transition matrix $p(\cdot,\cdot,\cdot)$ exists. (cf. Loève [9], § 39). The theorems below deal with the Q-matrix of the derived process. We restrict the discussion to the case that the deriving process has stationary increments. Similar results may be obtained in the general case. Let

$${}_iq(x) \overset{\text{df}}{=} \lim_{h\to 0^+} \frac{1-{}_ip(x, \{x\}, h)}{h} \leqq \infty, \quad i = 1, 2,$$

$${}_iq(x, E) \overset{\text{df}}{=} \lim_{h\to 0^+} \frac{1}{h}\,{}_ip(x, E, h) \leqq \infty,\ E \not\ni x,\ i = 1, 2,$$

whenever the limits exist.

THEOREM 6.3. Let the distributions of the τ_s-process be given by (6.12).

(a) $\lim_{t\to 0^+} {}_1p(x, \{x\}, t) = 1$, $x\in X$, then ${}_1q(x)$ and ${}_2q(x)$ exist and

(6.21) $${}_2q(x) = m\,{}_1q(x) + \int_{(0,\infty)} [1-{}_1p(x, \{x\}, t)]d\Psi(t) \leqq \infty,\ x\in X.$$

Here $m\,{}_1q(x) \overset{\text{df}}{=} 0$ if $m = 0$ and ${}_1q(x) = +\infty$.

(b) If $m = 0$ and ${}_1p(x, \{x\}, t)$ is continuous in t for $t > 0$, then ${}_2q(x)$ exists and

$$(6.22) \qquad {}_2q(x) = \int_{(0,\infty)} [1 - {}_1p(x, \{x\}, t)] d\Psi(t) \leqq \infty.$$

PROOF. The proof will not be written out in full, since it follows the same lines as the proof given in [12] for the case that the state space X is discrete.

If $\lim_{t\to 0+} {}_1p(x, E, t) = I_E(x)$, $x \in X$, by theorem 5.2 we have that $\lim_{t \to 0+} {}_2p(x, E, s) = I_E(x)$, $x \in X$, and the existence of ${}_1q(x)$ and ${}_2q(x)$ follows by a theorem on transition matrices (Loève [9], p. 586). Let

$$(6.23) \qquad c_s(t) \overset{\mathrm{df}}{=} 0,\ t \leqq 0,\ c_s(t) \overset{\mathrm{df}}{=} \int_{[0,t)} \tau d_\tau b(s, \tau),\ t > 0.$$

$$(6.24) \qquad \Phi(t) \overset{\mathrm{df}}{=} 0,\ t \leqq 0,\ \Phi(t) \overset{\mathrm{df}}{=} \int_{(0,t)} \tau d\Psi(\tau),\ t > 0.$$

Then, as is shown in [12], it follows easily from (6.12) that

$$(6.25) \qquad \frac{1}{s} c_s(\cdot) = mb(s, \cdot) + b(s, \cdot) * \Phi(\cdot),\ s > 0,$$

where $*$ denotes convolution. From the relation

$$1 - {}_2p(x, \{x\}, h) = \int_{(0,\infty)} t^{-1}[1 - {}_1p(x, \{x\}, t)] dc_h(t),\ h > 0,$$

it may be derived by (6.25) and (6.13) that

$$(6.26) \qquad \frac{1}{h}[1 - {}_2p(x, \{x\}, h)] = mE\tau_h^{-1}[1 - {}_1p(x, \{x\}, \tau_h)]$$
$$+ \int_{(0,\infty)\times\Gamma} (t + \tau_h)^{-1}[1 - {}_1p(x, \{x\}, t + \tau_h)] d(\Phi \times Q),\ h > 0.$$

Here $\Phi \times Q$ denotes the product of the measure on $(0, \infty)$ determined by the distribution function Φ and the probability measure Q on $(\Gamma, \mathscr{C})$. It is noted that the first term on the right in (6.26) is present only if $m > 0$ and then $\tau_h > 0$ a.s. $[Q]$ if $h > 0$.

Since $\tau_0 = 0$ and the τ_s-process is continuous, we have (cf. Loève [9], p. 545) that $\lim_{h\to 0+} \tau_h = 0$, a.s. $[Q]$. So

$$(6.27) \qquad \lim_{h\to 0+} (t + \tau_h)^{-1}[1 - {}_1p(x, \{x\}, t + \tau_h)]$$
$$= t^{-1}[1 - {}_1p(x, \{x\}, t],\ \text{a.s. } [\Phi \times Q],\ t > 0,$$

since ${}_1p(x, \{x\}, t)$ is continuous in t for $t > 0$, under the assumptions of (a) as well as (b). Moreover, if ${}_1q(x)$ exists,

$$(6.28) \qquad \lim_{h\to 0+} \tau_h^{-1}[1 - {}_1p(x, \{x\}, \tau_h)] = {}_1q(x),\ \text{a.s. } [Q].$$

If ${}_1q(x) < \infty$ or if $m = 0$ and $\Psi(0^+) > -\infty$, the assertion of the theorem follows by (6.26), (6.27), (6.28) and the Lebesgue dominated convergence theorem.

By Fatou's lemma we have

$$(6.28') \quad \liminf_{h\to 0^+} \frac{1}{h}[1-{}_2p(x, \{x\}, h)] \geqq m_1q(x) + \int_{(0,\infty)} [1-{}_1p(x, \{x\}, t)]d\Psi(t), h > 0,$$

which disposes of the case $m > 0$, $q(x) = +\infty$ and the case that the integral in (6.22) diverges.

To prove the only case left, viz. $m = 0$, $q_1(x) = +\infty$, whereas the integral in (6.22) converges, we take resort to the following inequality, holding for $m = 0$:

$$(6.29) \quad {}_2p(x, \{x\}, h) \geqq \exp\,[-h\int_{(0,\infty)}\{1-{}_1p(x, \{x\}, t)\}d\Psi(t), h > 0,$$

which implies

$$(6.30) \quad \limsup_{h\to 0^+} \frac{1}{h}[1-{}_1p(x, \{x\}, h)] \leqq \int_{(0,\infty)} [1-{}_1p(x, \{x\}, t)]d\Psi(t).$$

Then (6.22) follows by (6.28′) and (6.30). The proof of (6.29) is the same as the proof of lemma 1 in [12], for in the more general case treated here we still have the relations

$${}_2p(x, \{x\}, h) \geqq \exp\,\{-h_2q(x)\},$$

(cf. Loève [9], p. 586) and

$${}_1p(x, \{x\}, t'+t'') \geqq {}_1p(x, \{x\}, t')_1p(x, \{x\}, t''), \quad t' \geqq 0, t'' \geqq 0.$$

From (6.22) it is seen that a state may be instantaneous in the original process and steady in the derived process. A derived process with $m = 0$ and $\Psi(0^+) > -\infty$ has no instantaneous states.

THEOREM 6.4. Let the distributions of the τ_s process be given by (6.12).

(a) If $\lim_{t\to 0^+} {}_1p(x, \{x\}, t) = 1$, $x \in X$, then ${}_1q(x, U)$ and ${}_2q(x, U)$ exist for every uniform continuity state set [1] $U \not\ni x$ and

$$(6.31) \quad {}_2q(x, U) = m_1q(x, U) + \int_{(0,\infty)} {}_1p(x, U, t)d\Psi(t) < \infty, x \in X.$$

(b) If $m = 0$ and ${}_1p(E, x, t)$ is continuous in t for $t > 0$ and every $E \in \mathscr{X}$, and if

[1] cf. Loève [9], p. 587.

$$(6.32)\qquad \int_{(0,\infty)} [1-{}_1p(x, \{x\}, t)]d\Psi(t) < \infty,$$

then ${}_2q(x, E)$ exists for every $E \in \mathscr{X}$ with $E \not\ni x$, we have

$$(6.33)\qquad {}_2q(x, E) = \int_{(0,\infty)} {}_1p(x, E, t)d\Psi(t) < \infty,\ E \not\ni x,$$

and the set function ${}_2q(x, \cdot)$ on $\mathscr{X}-\{x\}$ is a finite measure with ${}_2q(x, X-\{x\}) = {}_2q(x)$ given by (6.22).

Proof of (a). By the same method that led to (6.26) we find

$$(6.34)\qquad \frac{1}{h}\,{}_2p(x, U, h) = mE\tau_h^{-1}\,{}_1p(x, U, \tau_h) + \int_{(0,\infty)\times\Gamma} (t+\tau_h)^{-1}\,{}_1p(x, U, t+\tau_h)d(\Phi\times Q),\ h > 0.$$

If $U \not\ni x$ is a uniform continuity state set, $\lim_{t\to 0^+} t^{-1}{}_1p(x, U, t) = {}_1q(x, U) < \infty$ by a theorem on transition matrices (Loève [9], p. 587). From this fact it is easily seen that the Lebesgue dominated convergence theorem may be applied to (6.34), and our assertions follow.

Proof of (b). If $m = 0$, (6.34) is replaced by

$$(6.35)\qquad \frac{1}{h}\,{}_2p(x, E, h) = \int_{(0,\infty)\times\Gamma} (t+\tau_h)^{-1}\,{}_1p(x, E, t+\tau_h)d(\Phi\times Q).$$

Since $\lim_{h\to 0^+} \tau_h = 0$, a.s. $[Q]$, and ${}_1p(x, E, t)$ is continuous in t for $t > 0$, we have

$$(6.36)\qquad \lim_{h\to 0^+} (t+\tau_h)^{-1}{}_1p(x, E, t+\tau_h) = t^{-1}\,{}_1p(x, E, t),\ \text{a.s. } [\Phi\times Q],\ t > 0.$$

In theorem 6.3 it has been shown that under (6.32), if $m = 0$:

$$(6.37)\qquad \lim_{h\to 0^+} \int_{(0,\infty)\times\Gamma} (t+\tau_h)^{-1}[1-{}_1p(x, \{x\}, t+\tau_h]d(\Phi\times Q) = \int_{(0,\infty)\times\Gamma} t^{-1}[1-{}_1p(x, \{x\}, t)]d(\Phi\times Q) < \infty,$$

whereas

$$(6.38)\qquad \lim_{h\to 0^+} (t+\tau_h)^{-1}[1-{}_1p(x, \{x\}, t+\tau_h] = t^{-1}[1-{}_1p(x, \{x\}, t)],\ \text{a.s. } [\Phi\times Q].$$

From (6.37) and (6.38) and a theorem of measure theory (Loève [9], p. 140, ex. 16, 17) it follows that the convergence in (6.38) is also in absolute $\Phi\times Q$-mean. Since for every $h \geqq 0$ and $t > 0$

$$(t+\tau_h)^{-1}{}_1p(x, E, t+\tau_h) \leqq (t+\tau_h)^{-1}[1-{}_1p(x, \{x\}, t+\tau_h)],$$

the convergence in (6.36) also must be in absolute $\Phi \times Q$-mean (cf. Halmos [7], § 26, theorem C). This proves (6.33).

The last assertions of the theorem follow by (6.33) and the assumption that ${}_1p(\cdot, \cdot, \cdot)$ satisfies condition C of section 5.

For the type of continuity considered next we have to assume that X is the real line. Let

$$(6.39) \quad {}_1F_x(\xi, t) \overset{\text{df}}{=} {}_1p(x, (-\infty, \xi), t) = {}_1\mu\{{}_1\mathbf{x}_t < \xi | {}_1\mathbf{x}_0 = x\}, \quad x \in X, t \geqq 0, -\infty < \xi < \infty,$$

$$(6.40) \quad {}_1\bar{F}_x(\xi, t) \overset{\text{df}}{=} {}_1p(x, (-\infty, x+\xi), t) = {}_1\mu\{{}_1\mathbf{x}_t - x < \xi | {}_1\mathbf{x}_0 = x\}, \quad x \in X, t \geqq 0, -\infty < \xi < \infty,$$

$$(6.41) \quad {}_1f_x(u, t) \overset{\text{df}}{=} \int_{-\infty}^{+\infty} e^{iu\xi} d\, {}_1F_x(\xi, t) = E\{e^{iu_1\mathbf{x}_t} | {}_1\mathbf{x}_0 = x\}, \quad x \in X, t \geqq 0, -\infty < u < \infty,$$

$$(6.42) \quad {}_1\bar{f}_x(u, t) \overset{\text{df}}{=} \int_{-\infty}^{+\infty} e^{iu\xi} d\, {}_1\bar{F}_x(\xi, t) = E\{e^{iu({}_1\mathbf{x}_t - x)} | {}_1\mathbf{x}_0 = x\}, \quad x \in X, t \geqq 0, -\infty < u < \infty.$$

The corresponding distribution and characteristic functions for the derived process will be denoted by ${}_2F_x(\xi, s, s+\sigma)$, ${}_2\bar{F}_x(\xi, s, s+\sigma)$, ${}_2f_x(u, s, s+\sigma)$, ${}_2\bar{f}_x(u, s, s+\sigma)$. If the derived process has stationary transition probabilities, the argument s will be omitted.

We now consider the following continuity condition:

$$(6.43) \qquad {}_1\bar{F}_x(\cdot, t) \overset{c}{\to} U(\cdot) \quad \text{for} \quad t \to 0^+, x \in X,$$

where $U(\cdot)$ denotes the unit stepfunction at zero. Equivalently:

$$(6.44) \qquad \lim_{t \to 0^+} {}_1\bar{f}_x(u, t) = 1, \; -\infty < u < \infty, x \in X.$$

THEOREM 6.5. If the original transition probabilities satisfy (6.43), and the deriving process is continuous in probability, then

$$(6.45) \quad {}_2\bar{F}_x(\cdot, s, s+\sigma) \overset{c}{\to} U(\cdot) \quad \text{for} \quad \sigma \to 0^+, x \in X, s \geqq 0.$$

PROOF. By (5.11) we have

$$\begin{aligned} {}_2\bar{f}_x(u, s, s+\sigma) &= \int_{-\infty}^{+\infty} e^{iu(y-x)} {}_2p(x, dy, s, s+\sigma) \\ &= \int_{[0, \infty)} dG_{s, s+\sigma}(\tau) \int_{-\infty}^{+\infty} e^{iu(y-x)} {}_1p(x, dy, \tau) \\ &= \int_{[0, \infty)} {}_1\bar{f}_x(u, \tau) dG_{s, s+\sigma}(\tau). \end{aligned}$$

In the same way as (6.17) follows from (6.16), it may be derived from (6.44) that

$$\lim_{\sigma\to 0^+} {}_2\bar{f}_x(u, s, s+\sigma) = 1, \ -\infty < x < \infty, s \in S,$$

which proves the theorem.

The existence of a transition law derivative stands in a similar relation to the continuity condition (6.44) as the existence of a Q-matrix to the continuity condition (6.16). The ${}_1x_t$-process is said to have the transition law derivative ${}_1f_x(u)$, if ${}_1f_x(u)$ is a characteristic function and

$$\text{(6.46)} \quad \lim_{t\to 0^+} \{{}_1\bar{f}_x(u, t)\}^{\frac{1}{t}} = {}_1f_x(u), \ -\infty < u < \infty, x \in X.$$

We refer to Loève [9], p. 572. The function ${}_1f_x(\cdot)$ must be the characteristic function of an infinitely divisible distribution. Its Lévy representation will be denoted by

$$\text{(6.47)} \quad \log {}_1f_x(u) = iu\,{}_1\alpha_x - {}_1\beta_x u^2 + \int_{-\infty}^{+\infty} \left(e^{iuy} - 1 - \frac{iuy}{1+y^2}\right) d\,{}_1L_x(y), \ -\infty < u < \infty, x \in X.$$

A set of necessary and sufficient conditions for (6.46) and (6.47) is

$$\text{(6.48a)} \qquad \lim_{t\to 0^+} \frac{1}{t}\int_{-\infty}^{+\infty} \frac{y}{1+y^2} d\,{}_1\bar{F}_x(y) = {}_1\alpha_x, \quad x \in X,$$

$$\text{(6.48b)} \qquad \frac{1}{t}\int_{-\infty}^{\eta} \frac{y^2}{1+y^2} d\,{}_1\bar{F}_x(y) \overset{c}{\to} \int_{-\infty}^{\eta} \frac{y^2}{1+y^2} d\,{}_1L_x(y),$$

as $t \to 0^+$, $x \in X$.

It is easily seen that (6.46) is equivalent with

$$\text{(6.49)} \quad \lim_{t\to 0^-} \frac{1}{t}\{{}_1\bar{f}_x(u, t) - 1\} = \log {}_1f_x(u), \ -\infty < u < \infty, x \in X.$$

Moreover, it is obvious that (6.43) is a necessary condition for (6.46).

Theorem 6.6. Let the distributions of the τ_s-process be given by (6.12).

If the original process has the transition law derivative given by (6.46) and (6.47), the derived process has the transition law derivative ${}_2f_x(u)$ with

$$\text{(6.50)} \quad \log {}_2f_x(u) = m \log {}_1f_x(u) + \int_{(0,\infty)} \{{}_1\bar{f}_x(u, t) - 1\} d\Psi(t), \ -\infty < u < \infty, x \in X.$$

Its Lévy representation is determined by

$$(6.51)\qquad {}_2\alpha_x = m_1\alpha_x + \int_{(0,\infty)} \left\{\int_{-\infty}^{+\infty} \frac{\xi}{1+\xi^2} d\,{}_1F_x(\xi, t)\right\} d\Psi(t),$$

$$(6.52)\qquad {}_2\beta_x = m\,{}_1\beta_x,$$

$$(6.53)\qquad {}_2L_x(\xi) = m_1 L_x(\xi) + \int_{(0,\infty)} {}_1F_x(\xi, t)d\Psi(t),\ \xi < 0,$$

$$(6.54)\qquad {}_2L_x(\xi) = m\,{}_1L_x(\xi) + \int_{(0,\infty)} \{{}_1F_x(\xi, t)-1\}d\Psi(t),\ \xi > 0.$$

If $m = 0$ and $\Psi(0^+) > -\infty$, the same conclusions hold whenever ${}_1\bar{f}_x(u, t)$ is continuous in t for $t > 0$, the transition law derivative ${}_2f_x(u)$ being given by (6.50)—(6.54) with the terms in m vanishing.

Proof. By (5.3), (5.11), (6.42) and (5.1):

$$\frac{1}{h}\{{}_2\bar{f}_x(u, h)-1\} = \frac{1}{h}\int_{-\infty}^{+\infty} \{e^{iu(\xi-x)}-1\}{}_2p(x, d\xi, h)$$

$$= \frac{1}{h}\int_{[0,\infty)} db_\tau(h, \tau)\int_{-\infty}^{+\infty} \{e^{iu(\xi-x)}-1\}{}_1p(x, d\xi, \tau)$$

$$= \frac{1}{h}\int_{[0,\infty)} \{{}_1\bar{f}_x(u, \tau)-1\}db_\tau(h, \tau)$$

$$= \frac{1}{h}\int_{(0,\infty)} \{{}_1\bar{f}_x(u, \tau)-1\}\, db_\tau(h, \tau),\ h > 0,\ -\infty < u < \infty,\ x \in X.$$

In the same way as in the proof of theorem 6.3 this is transformed into

$$(6.55)\qquad \frac{1}{h}\{{}_2\bar{f}_x(u, h)-1\} = mE\tau_h^{-1}\{{}_1\bar{f}_x(u, \tau_h)-1\}$$
$$+ \int_{(0,\infty)\times\Gamma} (t+\tau_h)^{-1}\{{}_1\bar{f}_x(u, t+\tau_h)-1\}d(\Phi\times Q).$$

If (6.46) holds, we have by (6.49), since $\lim_{h\to 0^+}\tau_h = 0$, a.s. $[Q]$:

$$\lim_{h\to 0^+} \tau_h^{-1}\{{}_1\bar{f}_x(u, \tau_h)-1\} = \log {}_1f_x(u),$$

and (6.50) follows by applying the Lebesgue dominated convergence theorem to (6.55).

If $m = 0$, the first term in the right-hand side of (6.55) vanishes. If moreover $\Psi(0^+) > -\infty$ and ${}_1\bar{f}_x(u, t)$ is continuous in t for $t > 0$, the Lebesgue dominated convergence theorem may be applied to the second term and (6.50) follows.

To prove (6.51)—(6.54) we write in (6.50)

$$(6.56)\quad {}_1f_x(u,t)-1=\int_{-\infty}^{+\infty}(e^{iu\xi}-1)d_1F_x(\xi,t)$$
$$=iu\int_{-\infty}^{+\infty}\frac{\xi}{1+\xi^2}d_1F_x(\xi,t)+\int_{(-\infty,0)}\left(e^{iu\xi}-1-\frac{iu\xi}{1+\xi^2}\right)d_1F_x(\xi,t)$$
$$+\int_{(0,\infty)}\left(e^{iu\xi}-1-\frac{iu\xi}{1+\xi^2}\right)d[{}_1F_x(\xi,t)-1].$$

Let

$$K_x(\xi)\overset{\text{df}}{=}\int_{(0,\infty)}{}_1F_x(\xi,t)d\Psi(t),\ \xi<0,$$
$$K_x(\xi)\overset{\text{df}}{=}\int_{(0,\infty)}\{{}_1F_x(\xi,t)-1\}d\Psi(t),\ \xi>0.$$

If $\Psi(0^+)>-\infty$, these integrals converge and the same is true by (6.2) if (6.46) holds, for then it follows from (6.48b) that for $t\to0^+$:

$${}_1F_x(\xi,t)=0(t),\ \xi<0,\ {}_1F_x(\xi,t)-1=0(t),\ \xi>0.$$

If (6.46) is given, then from (6.48a) and (6.2) it is seen that

$$(6.57)\qquad \int_{(0,\infty)}d\Psi(t)\left|\int_{-\infty}^{+\infty}\frac{\xi}{1+\xi^2}d_1F_x(\xi,t)\right|<\infty,$$

and from (6.48b) and (6.2) that

$$(6.58)\quad \int_{(0,\infty)}d\Psi(t)\int_{(-\infty,0)}\left|e^{iu\xi}-1-\frac{iu\xi}{1+\xi^2}\right||d_1F_x(\xi,t)<\infty,$$

$$(6.59)\int_{(0,\infty)}d\Psi(t)\int_{(0,\infty)}\left|e^{iu\xi}-1-\frac{iu\xi}{1+\xi^2}\right|d[{}_1F_x(\xi,t)-1)]<\infty.$$

The same conclusions hold if $\Psi(0^+)>-\infty$. From (6.50), (6.56) and (6.57) it follows, by applying (5.11), which is justified by (6.58) and (6.59), that

$$\log{}_2f_x(u)=m\log{}_1f_x(u)+\int_{-\infty}^{+\infty}\left(e^{iu\xi}-1-\frac{iu\xi}{1+\xi^2}\right)dK_x(\xi)$$
$$+iu\int_{(0,\infty)}\left\{\int_{-\infty}^{+\infty}\frac{\xi}{1+\xi^2}d_1F_x(\xi,t)\right\}d\Psi(t),$$

which is connection with (6.47) shows that (6.51)—(6.54) hold.

It is possible that $T=\{0,1,2,\ldots\}$ and $S=[0,\infty)$, i.e. it is possible to derive a continuous parameter Markov process from a discrete parameter Markov process. For the derived process then the questions of continuity and existence of the Q-matrix arise, which for the original process have no meaning. Since this case has been discussed in [3], a short remark will be sufficient.

We assume that the deriving process has stationary increments, so that the distributions of the τ_s-process are determined by (6.12). As $\tau_s \in T$, the increments of the τ_s-process must be integer valued, so $m = 0$ and $\Psi(\cdot)$ is a step function with jumps at the natural numbers. Putting.

$$\lambda \stackrel{\text{df}}{=} -\Psi(0+), \quad c_0 \stackrel{\text{df}}{=} 0,$$
$$c_n \stackrel{\text{df}}{=} \frac{1}{\lambda}\Psi(n^+) - \frac{1}{\lambda}\Psi(n), \; n = 1, 2, \ldots,$$

we have

$$P\{\tau_s = k\} = \sum_{r=0}^{\infty} \frac{(\lambda s)^r}{r!} e^{-\lambda s} c_k^{(r)}, \quad k = 0, 1, 2, \ldots, s \geqq 0,$$

where $\{c_k^{(r)}, k = 0, 1, \ldots\}$ is the r-fold convolution of the sequence $\{c_0, c_1, \ldots\}$, and $c_n^{(0)} \stackrel{\text{df}}{=} \delta_{0,n}$.

Let the one-step transition matrix of the original process be denoted by ${}_1P(\cdot, \cdot)$, i.e.

$${}_1P(x, E) \stackrel{\text{df}}{=} {}_1p(x, E, 1) = {}_1\mu\{{}_1x_1 \in E | {}_1x_0 = x\}, x \in X, E \in \mathscr{X}.$$

Then (5.3) gives for the transition matrix of the derived process:

$${}_2p(x, E, \sigma) = \sum_{k=0}^{\infty} \sum_{r=0}^{\infty} \frac{(\lambda\sigma)^r}{r!} e^{-\lambda\sigma} c_k^{(r)} {}_1P^{(k)}(x, E),$$

where ${}_1P^{(0)}(x, E) \stackrel{\text{df}}{=} I_E(x)$.

Since $c_0 = 0$, we may write

$$\begin{aligned} {}_2p(x, E,) &= \sum_{k=0}^{\infty} \sum_{r=0}^{k} \frac{(\lambda\sigma)^r}{r!} e^{-\lambda\sigma} c_k^{(r)} {}_1P^{(k)}(x, E) \\ &= \sum_{r=0}^{\infty} \frac{(\lambda\sigma)^r}{r!} e^{-\lambda\sigma} \sum_{k=r}^{\infty} c_k^{(r)} {}_1P^{(k)}(x, E). \end{aligned}$$

From this relation it is easily seen that $\lim_{\sigma\to 0+} {}_2p(x, E, \sigma) = I_E(x)$ and that

$$\lim_{h\to 0+} \frac{1}{h} \{{}_2p(x, E, h) - I_E(x)\} = \lambda \sum_{k=1}^{\infty} c_k \{{}_1P^{(k)}(x, E) - I_E(x)\}.$$

7. Processes with independent increments

In this section it is assumed that the original process has stationary independent increments. The deriving process is subject to the same restrictions as in section 5.

Since a continuous parameter process with stationary independent increments is a.s. continuous, it is no restriction to assume that $\{{}_1x_t, t \in T\}$ is an ${}_1\mathscr{A} \times \mathscr{T}$-measurable stochastic process. If we denote by $F(\cdot, \tau)$ the distribution function of ${}_1x_{t+\tau} - {}_1x_t$, the transition matrix of the Markov process $\{{}_1x_t, t \geqq 0\}$ is determined by

$$ {}_1p(x, (-\infty, \xi), t) = F(\xi - x, t), $$

and therefore is measurable in x. So if $T = [0, \infty)$, the conditions of lemma 5.2 are satisfied.

By theorem 5.1 it now follows that the derived process is a Markov process. Since ${}_1p(x, (-\infty, \xi), t)$ depends only on $\xi - x$, the same is true of ${}_2p(x, (-\infty, \xi), s, s+\sigma)$ by (5.2), which shows that the derived process has independent increments.

This result, obtained before by Bochner [1] and Zolotarev [13], [14], also may be proved directly by a simplified version of the proof of theorem 5.1. If the τ_s-process has stationary increments, then by (5.3) the same is true of the ${}_2x_s$-process.

First we consider the case that $T = [0, \infty)$, $S = [0, \infty)$, under the assumption that the deriving process is continuous in probability, so that the distributions of its increments are determined by (6.1)—(6.9). The characteristic functions

$$ (7.1) \qquad {}_1\bar{f}(u, \tau) \overset{\text{df}}{=} {}_1E\, e^{iu({}_1x_{t+\tau} - {}_1x_t)} $$

of the increments of the original process may be written in the form

$$ (7.2) \quad \log {}_1\bar{f}(u, \tau) = iu\, {}_1\alpha t - {}_1\beta \tau u^2 $$
$$ \tau \int_{-\infty}^{+\infty} \left(e^{iu\xi} - 1 - \frac{\xi^2}{1+\xi^2} \right) d\, {}_1L(\xi), \; -\infty < u < \infty, \tau \geqq 0, $$

or

$$ (7.3) \qquad {}_1\bar{f}(u, \tau) = e^{\tau \log {}_1f(u)}, \; -\infty < u < \infty, \tau \geqq 0, $$

where

$$ (7.4) \qquad {}_1f(u) \overset{\text{df}}{=} {}_1\bar{f}(u, 1), \; -\infty < u < \infty, $$

may be considered as the transition law derivation of the ${}_1x_t$-process.

For the characteristic functions

$$ (7.5) \qquad {}_2\bar{f}(u, s, s+\sigma) \overset{\text{df}}{=} {}_2E\, e^{iu({}_2x_{s+\sigma} - {}_2x_s)} $$

of the increments of the derived process we find by (3.4) and the fact that the τ_s-process has independent increments:

$$
{}_2\bar{f}(u,\, s,\, s+\sigma) = \iint {}_1E\, e^{iu({}_1x_{t+\tau}-{}_1x_t)}\, dQ_s(t) dG_{s,\,s+\sigma}(\tau)
$$
$$
= \int_{[0,\,\infty)} {}_1\bar{f}(u,\, \tau)\, dG_{s,\,s+\sigma}(\tau).
$$

So by (6.1) and (7.3), since $\operatorname{Re}\log {}_1f(u) \leqq 0$:

(7.6) ${}_2\bar{f}(u,\, s,\, s+\sigma)$
$$
= \exp\left[m(s,\, s+\sigma)\log {}_1f(u) + \int_{(0,\,\infty)} \{{}_1\bar{f}(u,\, t)-1\} d\Psi_{s,\,s+\sigma}(t)\right],
$$
$$
-\infty < u < \infty,\ s \geqq 0,\ \sigma \geqq 0.
$$

If we denote by ${}_1\bar{F}(\cdot,\, \tau)$ the distribution function of ${}_1x_{t+\tau}-{}_1x_t$, then it follows from (7.6), in the same way as (6.51)—(6.54) were derived from (6.50), that

(7.7)
$$
\log {}_2\bar{f}(u,\, s,\, s+\sigma) = iu\, {}_2\alpha(s,\, s+\sigma) - u^2\,{}_2\beta(s,\, s+\sigma)
$$
$$
+ \int_{-\infty}^{+\infty} \left(e^{iu\xi}-1-\frac{iu\xi}{1+\xi^2}\right) d\,{}_2L_{s,\,s+\sigma}(\xi),\ -\infty < u < \infty,\ s \geqq 0,\ \sigma \geqq 0,
$$

where

(7.8)
$$
{}_2\alpha(s,\, s+\sigma) = {}_1\alpha\, m(s,\, s+\sigma)
$$
$$
+ \int_{(0,\,\infty)} \left\{ \int_{-\infty}^{+\infty} \frac{\xi}{1+\xi^2}\, d\,{}_1\bar{F}(\xi,\, t) \right\} d\Psi_{s,\,s+\sigma}(t),
$$

(7.9)
$$
{}_2\beta(s,\, s+\sigma) = {}_1\beta m(s,\, s+\sigma),
$$

(7.10)
$$
{}_2L_{s,\,s+\sigma}(\xi) = m(s,\, s+\sigma)\,{}_1L(\xi) + \int_{(0,\,\infty)} {}_1\bar{F}(\xi,\, t) d\Psi_{s,\,s+\sigma}(t),\ \xi < 0,
$$

(7.11) ${}_2L_{s,\,s+\sigma}(\xi)$
$$
= m(s,\, s+\sigma)\,{}_1L(\xi) + \int_{(0,\,\infty)} \{{}_1\bar{F}(\xi,\, t)-1\} d\Psi_{s,\,s+\sigma}(t),\ \xi > 0.
$$

If the $\mathfrak{r}_s$-process has stationary increments, with distributions given by (6.12), the relations (7.5)—(7.11) reduce to

(7.12)
$$
{}_2\bar{f}(u,\, \sigma) \overset{\text{df}}{=} {}_2E\, e^{iu({}_2x_{s+\sigma}-{}_2x_s)},
$$

(7.13)
$$
\log {}_2\bar{f}(u,\, \sigma) = \sigma m \log {}_1f(u) + \sigma \int_{(0,\,\infty)} \{{}_1\bar{f}(u,\, t)-1\} d\Psi(t),
$$
$$
-\infty < u < \infty,\ \sigma \geqq 0,
$$

(7.14)
$$
\log {}_2\bar{f}(u,\, \sigma) = iu\,{}_2\alpha\sigma - {}_2\beta\sigma u^2
$$
$$
+ \sigma \int_{-\infty}^{+\infty} \left(e^{iu\xi}-1-\frac{iu\xi}{1+\xi^2}\right) d\,{}_2L(\xi),\ -\infty < u < \infty,\ \sigma \geqq 0,
$$

(7.15)
$$
{}_2\alpha = {}_1\alpha m + \int_{(0,\,\infty)} \left\{ \int_{-\infty}^{+\infty} \frac{\xi}{1+\xi^2}\, d\,{}_1\bar{F}(\xi,\, t) \right\} d\Psi(t),
$$

$$(7.16) \qquad {}_2\beta = {}_1\beta m,$$

$$(7.17) \qquad {}_2L(\xi) = m_1L(\xi) + \int_{(0,\infty)} {}_1F(\xi, t)d\Psi(t),\ \xi < 0,$$

$$(7.18) \qquad {}_2L(\xi) = m_1L(\xi) + \int_{(0,\infty)} \{{}_1F(\xi, t) - 1\}d\Psi(t),\ \xi > 0.$$

These relations also follow as a special case of theorem 6.6.

If the τ_s-process is considered as the sum of two or more independent processes with independent increments, ${}_2f(u, s, s+\sigma)$ is the product of the characteristic functions of the corresponding derived processes, as is seen from (7.6). This suggests that such splitting up of the deriving process is equivalent to writing the derived process as the sum of independent derived processes, each having independent increments. That this is true is easily established from the relation

$${}_1x_{\tau_s'+\tau_s''} = {}_1x_{\tau_s'} + ({}_1x_{\tau_s'+\tau_s''} - {}_1x_{\tau_s'}),$$

by making use of the independence of τ_s' and τ_s'' and the independence and stationarity of the increments of the ${}_1x_t$-process.

In this connection it is noted that the independence of the increments of the ${}_2x_s$-process may be proved by a similar direct appeal to the definition (2.1).

In the same way as with Markov processes in general, if $[0, \infty)$ and $\{0, 1, 2, \ldots\}$ are admitted for T and S, four combinations for the character of the time parameters of the original and derived processes arise. E.g., if the original process is a sequence of sums of independent random variables having the same distribution function ${}_1F(\cdot)$, and the deriving process is a sequence of sums of independent nonnegative integer valued random variables with $Q(\tau_{n+1} - \tau_n = k) = a_k$, $k = 0, 1, \ldots, n = 0, 1, \ldots$, then the derived process is a sequence of sums of independent random variables with common distribution function

$$(7.19) \qquad {}_2F(\xi) = \sum_{k=0}^{\infty} a_k F^{(k)}(\xi),\ -\infty < \xi < \infty,$$

where $F^{(k)}(\cdot)$ is the k-fold convolution of ${}_1F(\cdot)$, $k = 1, 2, \ldots$, and $F^{(0)}(\cdot)$ is the unit stepfunction. Distributions of a form like (7.13) or (7.19), that may be considered as the transition distribution of a derived process with independent increments, were called derived distributions by Cohen (see [3] and [4]).

An interesting problem, not studied so far, is to find conditions under which a distribution is a derived distribution in a nontrivial way, or conditions under which it is a derived distribution from a given distribution or from a distribution belonging to a given class, either in the sense of continuous parameter or discrete parameter processes.

In this connection we mention the following elementary facts on derived distributions of the form (7.13), i.e. we assume that $S = T = [0, \infty)$. From (7.15)—(7.18) it is seen that the normal distribution is not a derived distribution. Furthermore, if the derived distribution is a lattice distribution with period c, then ${}_1L(\cdot)$ must be a pure step function with jumps at integral multiples of c, and we must have ${}_1\beta = 0$. Moreover,

$$\tau_s \left\{ \alpha - \int \frac{x^2}{1+x^2} d\,{}_1L(x) \right\}$$

must be a lattice variable with period equal to an integral multiple of c. Finally, an infinitely divisible distribution with ${}_2L(x) = 0$ for $|x| \geqq A$ for some constant A, cannot be a derived distribution of the form (7.13). This follows drom (7.17) and (7.18), since the variation of an infinitely divisible distribution function cannot be restricted to a bounded interval. In particular a Poisson process is not the derived process of a continuous parameter process with independent increments. In these remarks we exclude the trivial case $\Psi(t) = 0, t > 0$.

An interesting example of derived distributions is provided by the stable distributions. Let us assume that $S = T = [0, \infty)$ and take

$$m = 0,\ \Psi(t) = -ct^{-\theta},\ t > 0, \tag{7.20}$$

where $c > 0$, i.e. the increments of the deriving process have a one-sided stable distribution of order θ. From (6.2) and (6.3) it follows that necessarily $0 < \theta < 1$.

By (7.3) and (7.13) we find for the derived distribution

$$\log {}_2\bar{f}(u, \sigma) = -\sigma c A_\theta(-\log {}_1f(u))^\theta,\ -\infty < u < \infty, \tag{7.21}$$

where

$$A_\theta \overset{\mathrm{df}}{=} \int_0^\infty (1-e^{-\tau})\tau^{-1-\theta}\,d\tau, \tag{7.22}$$

which shows that (7.21) determines the characteristic function of an infinitely divisible distribution.

Now assume that the increments of the original process have stable distributions of order γ, i.e. ${}_1f(u)$ has one of the following forms:

$$\log {}_1f(u) = i_1\alpha u - {}_1\beta u^2, \quad {}_1\beta > 0, \tag{7.23}$$

$$\log {}_1f(u) = i_1\alpha u - {}_1\beta |u|^\gamma \left\{1 + i\,{}_1b \frac{u}{|u|} \tan \frac{\pi}{2}\gamma\right\}, \tag{7.24}$$

$${}_1\beta > 0,\ 0 < \gamma < 2,\ \gamma \neq 1,\ |{}_1b| \leqq 1,$$

$$\log {}_1f(u) = i_1\alpha u - {}_1\beta |u| \left\{1 + i_1 b \frac{u}{|u|} \frac{2}{\pi} \log |u|\right\}, \tag{7.25}$$

$${}_1\beta > 0,\ |{}_1b| \leqq 1.$$

Then in the following cases (7.21) determines the characteristic function of a stable distribution:

(i) $\gamma = 2$, ${}_1\alpha = 0$. Then by (7.23) and (7.21):

$$\log {}_2\bar{f}(u, \sigma) = -\sigma c\, {}_1\beta^\theta A_\theta |u|^{2\theta}, \tag{7.26}$$

which shows that every symmetric stable distribution may be considered as derived from a normal distribution with zero first moment.

(ii) $0 < \gamma < 2$, $\gamma \neq 1$, ${}_1\alpha = 0$. Then (7.21) and (7.24) give

$$\log {}_2\bar{f}(u, \sigma) = -\sigma c_1\beta^\theta A_\theta |u|^{\theta\gamma} \left\{1 + i\,{}_1b \frac{u}{|u|} \tan \frac{\pi}{2}\gamma\right\}^\theta. \tag{7.27}$$

It is easily seen that $\{1 + i\,{}_1bu/|u| \tan \frac{1}{2}\pi\gamma\}^\theta$ may be written in the form $a\{1 + i\,{}_2bu/|u| \tan \frac{1}{2}\pi\theta\gamma\}$, with $|{}_2b| \leqq 1$, so (7.27) determines the characteristic function of a stable distribution of order $\theta\gamma$. In particular, every symmetric stable distribution may be considered as derived from any symmetric stable distribution of higher order. If $\theta\gamma = 1$, (7.27) reduces to

$$\log {}_2\bar{f}(u, \sigma) = -\sigma c_1\beta^\theta A_\theta |u| \left\{1 + i_1 b \frac{u}{|u|} \tan \frac{\pi}{2}\gamma\right\}^\theta, \tag{7.28}$$

which may be written in the form

$$\log {}_2\bar{f}(u, \sigma) = i\,{}_2\alpha\sigma u - {}_2\beta\sigma |u|. \tag{7.29}$$

So if $\theta\gamma = 1$, the derived distribution is a Cauchy distribution, not necessarily centered at zero. Any Cauchy distribution may be considered as derived from a stable distribution of order γ with $1 < \gamma < 2$; which values of γ are possible, depends on the ratio ${}_2\alpha/{}_2\beta$.

(iii) $\gamma = 1$, ${}_1b = 0$. Then

$$\log {}_2\bar{f}(u, \sigma) = -\sigma c_1\beta^\theta A_\theta |u|^\theta \left\{1 - i \frac{{}_1\alpha}{{}_1\beta} \frac{u}{|u|}\right\}^\theta, \tag{7.30}$$

which shows that the derived distribution is stable. From (7.30) it is also seen that any stable distribution of order $\theta < 1$ with $|{}_2b| < 1$ may be considered as derived from a nondegenerate Cauchy distribution.

These results on stable distributions have been obtained before by Bochner [1] and Zolotarev [13], [14]. The interpretation (2.1) of derived distributions in terms of random variables offers the following illustration of these facts. It is well known that in a stochastic process $\{x_t, t \geqq 0\}$ with stationary independent increments and $x_0 \overset{\text{df}}{=} 0$ the random variables x_t have a distribution of the type considered in (i), (ii), (iii) above, if and only if ${}_1x_{\lambda t}$ has the same distribution as $\lambda^{1/\gamma}{}_1x_t$, for every $t > 0$ and $\lambda > 0$. If the distributions of the τ_s-process are determined by (7.20), then $\tau_{\lambda s}$ has the same distribution as $\lambda^{1/\theta}\tau_s$, $s > 0$, $\lambda > 0$. Therefore ${}_2x_{\lambda s} = {}_1x_{\tau_{\lambda s}}$ has the same distribution as ${}_1x_{\lambda^{1/\theta}\tau_s}$. Since the conditional distribution given τ_s of ${}_1x_{\lambda^{1/\theta}\tau_s}$ is the same as the conditional distribution given τ_s of $\lambda^{1/\theta\gamma}{}_1x_{\tau_s}$, if the ${}_1x_t$-distributions are of the types (i), (ii) or (iii) considered above, it follows that in this case the distribution of ${}_2x_{\lambda s}$ is the same as the distribution of $\lambda^{1/\theta\gamma}{}_2x_s$, $\lambda > 0$, $s > 0$, which shows that the distribution of ${}_2x_s$ must be a stable distribution of order θ of one of the types (i), (ii) or (iii).

REFERENCES

S. Bochner,

[1] Harmonic Analysis and the Theory of Probability. Univ. of California Press, 1955.

K. L. Chung and C. Derman,

[2] Non-recurrent random walks. Pac. Journ. of Math. 6 (1956), 441–447.

J. W. Cohen,

[3] Derived Markov Chains. Proc. Kon. Ned. Ak. van Wetensch. A 65 (1962), 55–92 (Indag. Math. 24).

J. W. Cohen,

[4] On derived and nonstationary Markov chains. Theory of Probability and its Applications (Moscow), VII (1962), 410–432.

J. W. Cohen,

[5] Applications of derived Markov chains in queueing theory. Report Mathematical Institute, Techn. Univ. Delft. 1961.

J. L. Doob,

[6] Stochastic Processes. Wiley, 1953.

P. R. Halmos,

[7] Measure Theory. Van Nostrand, 1950.

E. Hille and R. S. Phillips,
[8] Functional Analysis and Semigroups. Amer. Math. Soc. Colloquium Publications XXXI.

M. Loève,
Probability Theory, Sec. Ed. Van Nostrand, 1960.

R. S. Phillips,
[10] On the generation of semigroups of linear operators. Pac. Journ. of Math. 2 (1952), 343—369.

A. J. Stam,
[11] On defining derived processes, I, II. Reports of the Mathematical Institute, Technological University Delft, Oct., Dec. 1961.

A. J. Stam,
[12] On the Q-matrix of a derived Markov chain. Proc. Kon. Ned. Ak. van Wetensch. A 65 (1962) 576—582. (Indag. Math. 24).

V. M. Zolotarev,
[13] Mellin-Stieltjes transformations in the theory of probability. Theory of Probability and its Applications, 2 (1957), 433—460.

V. M. Zolotarev,
[14] Distribution of the superposition of infinitely divisible processes. Theory of Probability and its Applications, 3 (1958), 185—188.

(Oblatum 21-11-63) Mathematical Institute, University of Groningen