COMPOSITIO MATHEMATICA

L. A. RUBEL

A relation between Fourier and Mellin averages

Compositio Mathematica, tome 17 (1965-1966), p. 146-148
<http://www.numdam.org/item?id=CM_1965-1966__17__146_0>

A relation between Fourier and Mellin averages

by

L. A. Rubel

We establish a relation here between additive and multiplicative convolution averages of a bounded function. The real numbers are a locally compact Abelian group, under the operation of addition, with Haar measure dt. The positive real numbers are a locally compact Abelian group, under the operation of multiplication, with Haar measure dt/t. Given a bounded Lebesgue measurable function g, with $g(x)$ defined for all real numbers x, we may study its behaviour for large values of x by forming certain averages. One kind is with respect to integrable functions on $(-\infty, \infty)$, the other with respect to integrable functions on $(0, \infty)$ where we use only the restriction of g to $(0, \infty)$. In each case, integrability is with respect to the appropriate measure, and the average depends only on the behaviour of g at $+\infty$. We call the first kind of average a Fourier average, the second kind a Mellin average, and we establish a connection between them. We shall assume that all our functions are Lebesgue measurable.

MAIN THEOREM. If g is bounded, $K \geqq 0$,

$$\int_{-\infty}^{\infty} K(t)dt = 1, H \geqq 0, \quad \text{and} \quad \int_{0}^{\infty} H(t)dt/t = 1,$$

then

$$\limsup_{x\to\infty} \int_{0}^{\infty} H(x/t)g(t)dt/t \leqq \limsup_{x\to\infty} \int_{-\infty}^{\infty} K(x-t)g(t)dt.$$

The next result follows from the main theorem on normalizing K so that $\int_{-\infty}^{\infty} K(t)dt = 1$ (i.e. replacing $K(t)$ by $K(t)/\int_{-\infty}^{\infty} K(s)ds$), and writing $H = H^+ - H^-$, where $H^+(t) = \max\big(H(t), 0\big)$ and $H^-(t) = -\min\big(H(t), 0\big)$. Now considering the normalizations of H^+ and H^-, the main theorem and the corresponding result for lim inf may be applied.

TAUBERIAN THEOREM. Suppose g is bounded,

$$K \geqq 0, \quad 0 < \int_{-\infty}^{\infty} K(t)dt < \infty,$$

and

$$\lim_{x\to\infty} \int_{-\infty}^{\infty} K(x-t)g(t)dt = 0.$$

If $$\int_0^\infty |H(t)|dt/t < \infty,$$

then $$\lim_{x\to\infty} \int_0^\infty H(x/t)g(t)dt = 0.$$

PROOF. We need the following result from [2, p. 1005].

LEMMA 1. Given a bounded function g and $0 < \xi < 1$, let

$$L(\xi) = \limsup_{x\to\infty} \frac{1}{x-\xi x}\int_{\xi x}^{x} g(t)dt,$$

and let $L(1) = \sup_{0<\xi<1} L(\xi)$. There exists a bounded function g^* such that $g^* \geqq g$ and $\lim_{x\to\infty} x^{-1}\int_0^x g^*(t)dt = L(1)$.

Let us write $A = \limsup \int_0^\infty H(x/t)g(t)dt/t$ and $B = \limsup \int_{-\infty}^\infty K(x-t)g(t)dt$. We must prove $A \leqq B$, which obviously follows from the next two lemmas.

LEMMA 2. $A \leqq L(1)$.

LEMMA 3. $L(1) \leqq B$.

We prove Lemma 2 via Lemma 1. Since $g \leqq g^*$ and $K \geqq 0$, we have $A \leqq \limsup \int_0^\infty H(x/t)g^*(t)dt$. But since $\lim x^{-1}\int_0^x g^*(t)dt = L(1)$ (i.e., the Cesaro limit of g^* is $L(1)$), we may apply the Mellin form of the Wiener Tauberian theorem [1, p. 296] to conclude that $\lim \int_0^\infty H(x/t)g^*(t)dt/t = L(1)$, and hence $A \leqq L(1)$. In more detail, we have $\lim x^{-1} \int_1^x g^*(t)dt = L(1)$, and we may write $x^{-1}\int_1^x g^*(t)dt = \int_0^\infty g^*(t)C(x/t)dt/t$, where $C(s) = 0$ for $0 < s < 1$, and $C(s) = s^{-1}$ for $s \geqq 1$. Denoting by $C^\wedge$ the Mellin transform of C, $C^\wedge(r) = \int_0^\infty t^{ir}C(t)dt/t$, we have $C^\wedge(r) = (1-ir)^{-1}$. Since $C^\wedge(r) \neq 0$ for real r, we obtain the conclusion.

To prove Lemma 3, it is enough to do it under the special hypothesis that for some N, $K(x) = 0$ for $|x| \geqq N$. The general case follows on letting

$$K_N(x) = \begin{cases} K(x)/\int_{-N}^{N} K(t)dt & \text{for} \quad |x| \leqq N \\ 0 & \text{for} \quad |x| > N, \end{cases}$$

and then letting $N \to \infty$. Let us write

$$(K * g)(x) = \int_{-\infty}^{\infty} K(x-t)g(t)dt.$$

We shall prove that for $\xi < 1$,

$$\int_{\xi x}^{x} (K * g)(y)dy = \int_{\xi x}^{x} g(t)dt + o(x). \tag{1}$$

If this is done, we get

$$L(\xi) \leqq \limsup_{x\to\infty} {}_{\xi x \leqq y \leqq x} (K * g)(y)$$

from which Lemma 3 follows directly. To prove (1), write

$$I(x) = \int_{\xi x}^{x} (K * g)(y)dy = \int_{-\infty}^{\infty} g(t) \int_{\xi x}^{x} K(y-t)dy\,dt.$$

But $\int_{\xi x}^{x} K(y-t)dy$ vanishes if $t < \xi x - N$ or $t > x+N$. And $\int_{\xi x}^{x} K(y-t)dy = \int_{\xi x-t}^{x-t} K(y)dy$. Hence

$$I(x) = \int_{\xi x-N}^{x+N} g(t) \int_{\xi x-t}^{x-t} K(y)dy\,dt.$$

We write $\int_{\xi x-N}^{x+N} = \int_{\xi x-N}^{\xi x+N} + \int_{\xi x+N}^{x-N} + \int_{x-N}^{x+N}$. For $\xi x+N < t < x-N$, $\int_{\xi x-t}^{x-t} K(y)dy = 1$, and for any a and b with $a < b$,

$$0 \leqq \int_{a}^{b} K(y)dy \leqq 1.$$

Hence

$$I(x) = \int_{\xi x+N}^{x-N} g(t)dt + 0(1) = \int_{\xi x}^{x} g(t)dt + 0(1),$$

and the proof is complete.

REFERENCES

G. H. Hardy

[1] Divergent Series, Oxford, 1949.

L. A. Rubel

[2] Maximal means and Tauberian theorems, Pacific J. Math., Vol. 10, No. 3, 1960, pp. 997—1007.

(Oblatum 25-3-63)

University of Illinois