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On the order and type of integral functions
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by

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1.

Let

\( f(z_1, z_2) = \sum_{m, n=0}^{\infty} a_{mn} z_1^m z_2^n, \)

be a function of two complex variables \( z_1 \) and \( z_2 \), where the coefficients \( a_{mn} \) are complex numbers. The series (1.1) represents an integral function of two variables \( z_1 \) and \( z_2 \), if it converges absolutely for all values of \( |z_1| < \infty \) and \( |z_2| < \infty \). M. M. Dzrbasyan (1, p. 1) has shown that the necessary and sufficient condition for the series (1.1) to represent an integral function of variables \( z_1 \) and \( z_2 \), is

\[ \lim_{m+n \to \infty} \left( |a_{mn}| \right)^{1/(m+n)} = 0. \]

Let \( \mathcal{G}_r \) be the family of closed polycircular domains in space \((z_1, z_2)\) dependent on parameter \( r > 0 \) and possess the property that \((z_1, z_2) \in \mathcal{G}_r\), if and only if \((z_1/r, z_2/r) \in \mathcal{G}_1\). The maximum modulus of the integral function \( f(z_1, z_2) \) is denoted by

\[ M_G(r, f) = \max_{(z_1, z_2) \in \mathcal{G}_r} |f(z_1, z_2)| \]

and the function will be called \( G \) — order and \( G \) — type respectively, if

\[ \rho_G = \lim_{r \to \infty} \sup \frac{\log \log M_G(r, f)}{\log r} \]

(1.3)

\[ T_G = \lim_{r \to \infty} \sup \frac{\log M_G(r, f)}{r^{\rho_G}}. \]

(1.4)

Denote

\[ \phi_G(m, n) = \max_{(z_1, z_2) \in \mathcal{G}_r} |z_1|^m |z_2|^n. \]

A. A. Goldberg (2, p. 146) has proved the following theorems.

1 For simplicity we consider only two variables, though the results can easily be extended to several complex variables.
THEOREM A. All orders $\rho_G$ be equal and

\begin{equation}
\rho = \rho_G = \limsup_{m+n \to \infty} \frac{(m+n) \log (m+n)}{-\log |a_{mn}|}.
\end{equation}

THEOREM B. G-type $T_G$ satisfies the correlation

\begin{equation}
(epT_G)^{1/\rho} = \limsup_{m+n \to \infty} [(m+n)^{1/\rho} \phi_G(m, n)|a_{mn}|^{1/(m+n)}].
\end{equation}

In this paper we shall obtain the relations between two or more integral functions and study the relations between the coefficients in the Taylor expansion of integral functions and their orders and types.

2.

THEOREM 1. Let $f_1(z_1, z_2) = \sum_{m,n=0}^{\infty} a_{mn} z_1^m z_2^n$ and $f_2(z_1, z_2) = \sum_{m,n=0}^{\infty} b_{mn} z_1^m z_2^n$ be two integral functions of non-zero finite orders $\rho_1$ and $\rho_2$ respectively. Then the function

\begin{equation}
f(z_1, z_2) = \sum_{m,n=0}^{\infty} c_{mn} z_1^m z_2^n,
\end{equation}

where

\begin{equation}
|c_{mn}| \sim |a_{mn}| |b_{mn}|
\end{equation}

is an integral function such that

\begin{equation}
1/\rho \geq 1/\rho_1 + 1/\rho_2,
\end{equation}

where $\rho$ is the order of $f(z_1, z_2)$.

PROOF: Since $f_1(z_1, z_2)$ and $f_2(z_1, z_2)$ are integral functions, therefore, using (1.2), we have

\begin{equation}
\limsup_{m+n \to \infty} |a_{mn}|^{1/(m+n)} = \limsup_{m+n \to \infty} |b_{mn}|^{1/(m+n)} = 0.
\end{equation}

Also, $|c_{mn}| \sim |a_{mn}| |b_{mn}|$, therefore,

\begin{equation}
\limsup_{m+n \to \infty} |c_{mn}|^{1/(m+n)} \leq \limsup_{m+n \to \infty} |a_{mn}|^{1/(m+n)} \limsup_{m+n \to \infty} |b_{mn}|^{1/(m+n)}.
\end{equation}

Hence $f(z_1, z_2)$ is an integral function.

Now using (1.5) for the functions $f_1(z_1, z_2)$ and $f_2(z_1, z_2)$, we have

\begin{equation}
\limsup_{m+n \to \infty} \frac{(m+n) \log (m+n)}{-\log |a_{mn}|} = \rho_1
\end{equation}

and

\begin{equation}
|c_{mn}| \sim |a_{mn}| |b_{mn}|, \text{ we mean } \lim_{m+n \to \infty} \{|c_{mn}|/|a_{mn}| |b_{mn}|\} = 1.
\end{equation}
Therefore, for an arbitrary $\epsilon > 0$, we get

\[ (2.1) \quad -\log |a_{mn}| > (1/\rho_1 - \epsilon/2)(m+n) \log (m+n), \text{ for } m+n > k_1 \]

and

\[ (2.2) \quad -\log |b_{mn}| > (1/\rho_2 - \epsilon/2)(m+n) \log (m+n), \text{ for } m+n > k_2. \]

Thus, for $(m+n) > k = \max. (k_1, k_2)$

\[ -\log (|a_{mn}| |b_{mn}|) > (1/\rho_1 + 1/\rho_2 - \epsilon)(m+n) \log (m+n) \]

or,

\[ \liminf_{m+n \to \infty} \frac{-\log (|a_{mn}| |b_{mn}|)}{(m+n) \log (m+n)} \geq 1/\rho_1 + 1/\rho_2. \]

Therefore, if $|c_{mn}| \sim |a_{mn}| |b_{mn}|$, we get

\[ \liminf_{m+n \to \infty} \frac{-\log |c_{mn}|}{(m+n) \log (m+n)} \geq 1/\rho_1 + 1/\rho_2. \]

Hence

\[ 1/\rho \geq 1/\rho_1 + 1/\rho_2. \]

**COROLLARY.** Let $f_s(z_1, z_2) = \sum_{m,n=0}^{\infty} a_{mn}^{(s)} z_1^m z_2^n$, where $s = 1, 2, \ldots, p$ be $p$ integral functions of non-zero finite orders $\rho_1, \rho_2, \ldots, \rho_p$ respectively. Then the function

\[ f(z_1, z_2) = \sum_{m,n=0}^{\infty} c_{mn} z_1^m z_2^n, \text{ where } |a_{mn}| \sim \prod_{s=1}^{p} |a_{mn}^{(s)}| \]

is an integral function such that

\[ 1/\rho \geq \sum_{s=1}^{p} 1/\rho_s, \]

where $\rho$ is the order of $f(z_1, z_2)$.

**THEOREM 2.** Let $f_1(z_1, z_2) = \sum_{m,n=0}^{\infty} a_{mn} z_1^m z_2^n$ and $f_2(z_1, z_2) = \sum_{m,n=0}^{\infty} b_{mn} z_1^m z_2^n$ be two integral functions of finite non-zero orders $\rho_1$ and $\rho_2$ respectively. Then the function $f(z_1, z_2) = \sum_{m,n=0}^{\infty} c_{mn} z_1^m z_2^n$, where

\[ \log (1/|c_{mn}|) \sim \{\log (1/|a_{mn}|) \log (1/|b_{mn}|)\}^{1/2} \]

is an integral function, such that

\[ \rho \leq (\rho_1 \rho_2)^{1/2}, \]

where $\rho$ is the order of $f(z_1, z_2)$. 
PROOF: Since \( f_1(z_1, z_2) \) and \( f_2(z_1, z_2) \) are integral functions, therefore, using (1.2), we have for an arbitrary \( \epsilon > 0 \) and large \( R \)

\[
1/|a_{mn}| > (R-\epsilon)^{m+n}, \text{ for } m+n > k_1
\]

and

\[
1/|b_{mn}| > (R-\epsilon)^{m+n}, \text{ for } m+n > k_2.
\]

Therefore, for \( m+n > k = \max(k_1, k_2) \)

\[
|\log(1/|a_{mn}|) \log(1/|b_{mn}|)|^\frac{1}{2} > (m+n) \log(R-\epsilon).
\]

Thus, if

\[
\log(1/|c_{mn}|) \sim |\log(1/|a_{mn}|) \log(1/|b_{mn}|)|^\frac{1}{2}
\]

then, for large \( m+n \)

\[
\log(1/|c_{mn}|) > (m+n) \log(R-\epsilon),
\]

or,

\[
\limsup_{m+n \to \infty} |c_{mn}|^{1/(m+n)} = 0.
\]

Hence \( f(z_1, z_2) \) is an integral function.

Now, from (2.1) and (2.2), we have for sufficiently large \( (m+n) \)

\[
|\log(1/|a_{mn}|) \log(1/|b_{mn}|)|^\frac{1}{2} \geq ((1/\rho_1-\epsilon/2)(1/\rho_2-\epsilon/2))\frac{1}{2}(m+n) \log(m+n),
\]

or,

\[
\liminf_{m+n \to \infty} \frac{|\log(1/|a_{mn}|) \log(1/|b_{mn}|)|^\frac{1}{2}}{(m+n) \log(m+n)} \geq \left(\frac{1}{\rho_1 \rho_2}\right)^\frac{1}{2}.
\]

Thus, if \( \log(1/|c_{mn}|) \sim |\log(1/|a_{mn}|) \log(1/|b_{mn}|)|^\frac{1}{2} \)

\[
1/\rho = \liminf_{m+n \to \infty} \frac{\log(1/|c_{mn}|)}{(m+n) \log(m+n)} \geq \left(\frac{1}{\rho_1 \rho_2}\right)^\frac{1}{2}
\]

or,

\[
\rho \leq (\rho_1 \rho_2)^\frac{1}{2}.
\]

COROLLARY: Let \( f_s(z_1, z_2) = \sum_{m, n=0}^{\infty} a^{(s)}_{mn} z_1^m z_2^n \), where \( s = 1, 2, \ldots, p \) be \( p \) integral functions of finite non-zero orders \( \rho_1, \rho_2, \ldots, \rho_p \) respectively. Then the function

\[
f(z_1, z_2) = \sum_{m, n=0}^{\infty} a_{mn} z_1^m z_2^n,
\]

where

\[
\log(1/|a_{mn}|) \sim \left| \left( \prod_{s=1}^{p} \log(1/|a^{(s)}_{mn}|) \right)^{1/p} \right|
\]

is an integral function such that
\[ p \leq \left( \prod_{s=1}^{p} \rho_s \right)^{1/p}, \]

where \( p \) is the order of \( f(z_1, z_2) \).

**Theorem 3.** Let \( f_1(z_1, z_2) = \sum_{m,n=0}^{\infty} a_{mn} z_1^m z_2^n \) and \( f_2(z_1, z_2) = \sum_{m,n=0}^{\infty} b_{mn} z_1^m z_2^n \) be two integral functions of finite non-zero orders \( \rho_1, \rho_2 \) and finite non-zero types \( T_1, T_2 \) respectively. Then the function

\[ f(z_1, z_2) = \sum_{m,n=0}^{\infty} c_{mn} z_1^m z_2^n, \]

where

\[ |c_{mn}| \sim \left| \left( |a_{mn}| |b_{mn}| \right)^{1/2} \right| \]

is an integral function such that

\[ (\rho T)^{2/\rho} \leq (\rho_1 T_1)^{1/\rho_1} (\rho_2 T_2)^{1/\rho_2}, \]

where \( \rho \) and \( T \) are the order and type of \( f(z_1, z_2) \) respectively and \( 2/\rho = 1/\rho_1 + 1/\rho_2 \).

**Proof:** We can prove, as in the proof of Theorem 1 that \( f(z_1, z_2) \) is an integral function, when

\[ |c_{mn}| \sim \left| \left( |a_{mn}| |b_{mn}| \right)^{1/2} \right|. \]

Further, using (1.6) for the functions \( f_1(z_1, z_2) \) and \( f_2(z_1, z_2) \), we have

\[ \limsup_{m+n \to \infty} \left[ (m+n)^{1/\rho_1} \{ \phi(m, n) |a_{mn}| \}^{1/(m+n)} \right] = (e^{\rho_1 T_1} \epsilon)^{1/\rho_1} \]

and

\[ \limsup_{m+n \to \infty} \left[ (m+n)^{1/\rho_2} \{ \phi(m, n) |b_{mn}| \}^{1/(m+n)} \right] = (e^{\rho_2 T_2} \epsilon)^{1/\rho_2}. \]

From (2.3) and (2.4), we get for an arbitrary \( \epsilon > 0 \)

\[ (m+n)^{1/\rho_1} \{ \phi(m, n) |a_{mn}| \}^{1/(m+n)} < \{ e^{\rho_1 (T_1 + \epsilon) \epsilon} \}^{1/\rho_1}, \]

for \( m+n > k_1 \) and

\[ (m+n)^{1/\rho_2} \{ \phi(m, n) |b_{mn}| \}^{1/(m+n)} < \{ e^{\rho_2 (T_2 + \epsilon) \epsilon} \}^{1/\rho_2}, \]

for \( m+n > k_2 \).

Thus, for \( m+n > k = \max (k_1, k_2) \) and \( 2/\rho = 1/\rho_1 + 1/\rho_2 \)

\[ \left[ (m+n)^{1/\rho} \{ \phi(m, n) \left( |a_{mn}| |b_{mn}| \right)^{1/2} \}^{1/(m+n)} \right]^2 \]

< \( \{ e^{\rho_1 (T_1 + \epsilon) \epsilon} \}^{1/\rho_1} \{ e^{\rho_2 (T_2 + \epsilon) \epsilon} \}^{1/\rho_2}. \)

* The types \( T_1 \) and \( T_2 \) correspond to the same family of closed polycircular domains \( \mathcal{C}_r \).
Therefore, if \(|c_{mn}| \sim |(a_{mn} | b_{mn})^{1/2}|\), we obtain

\[
\limsup_{m+n \to \infty} [(m+n)^{1/\rho} (\phi(m,n)|c_{mn}|^{1/(m+n)}]) \leq (e^{\alpha_1 T_1})^{1/(\rho_1)} (e^{\alpha_2 T_2})^{1/(\rho_2)}
\]
or,

\[
(e^{\rho T})^{1/\rho} \leq (e^{\alpha_1 T_1})^{1/(\rho_1)} (e^{\alpha_2 T_2})^{1/(\rho_2)},
\]

where \(\rho, T\) are respectively the order and type of \(f(z_1, z_2)\). Hence,

\[
(r T)^{p/\rho} \leq (\rho_1 T_1)^{1/\rho_1} (\rho_2 T_2)^{1/\rho_2}.
\]

**COROLLARY 1.** Let \(f_s(z_1, z_2) = \sum_{m,n=0}^{\infty} a(s)_{mn} z_1^m z_2^n\), where \(s = 1, 2, \ldots, p\) be \(p\) integral functions of finite non-zero orders \(\rho_1, \rho_2, \ldots, \rho_p\) and finite non-zero types \(T_1, T_2, \ldots, T_p\) respectively. Then the function \(f(z_1, z_2) = \sum_{m,n=0}^{\infty} a_{mn} z_1^m z_2^n\), where \(|c_{mn}| \sim |(\prod_{s=1}^{p} |a(s)_{mn}|^{1/p}|\) is an integral function such that

\[
(r T)^{p/\rho} \leq (\prod_{s=1}^{p} (\rho_s T_s))^{1/\rho_s},
\]

where \(r\) and \(T\) are the order and type of \(f(z_1, z_2)\) respectively and \(p/\rho = \sum_{s=1}^{p} 1/\rho_s\).

**COROLLARY 2.** If in the above theorem the functions \(f_1(z_1, z_2)\) and \(f_2(z_1, z_2)\) are of the same finite non-zero order, then

\[
T \leq (T_1 T_2)^{1/4}.
\]

**COROLLARY 3.** The result of the corollary 2 can be extended to \(p\) integral functions.

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**REFERENCES**

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