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A theorem on Riesz summability $(R, \omega, 2)$ on Banach space

by

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A series $\sum_{i=0}^{i=\infty} x_i$ is said to be summable $(R, \omega, 2)$, to sum s, if

$$R_\omega^2 = \sum_{i \leqq \omega} \left(1 - \frac{i}{\omega}\right)^2 x_i$$

tends to a limit s, as $\omega \to \infty$. R_ω^2 is called the Riesz mean of second order, $(R, \omega, 2)$, of $\sum x_i$. Similarily, we say that $\sum x_i/i^p$ is summable x^*, if

$$R_\omega^2(p) = \sum_{i \leqq \omega} \left(1 - \frac{i}{\omega}\right)^2 \frac{x_i}{i^p} \to x^*, \text{ as } \omega \to \infty.$$

On the other hand, Cèsaro summability, $(C, 2)$, of $\sum x_i$ or $\sum x_i/i^p$ is defined as follows:

$$\sigma_n^2 = \sum_{i=0}^{n} \left(1 - \frac{i}{n+1}\right)\left(1 - \frac{i}{n+2}\right) x_i$$

or

$$\sigma_n^2(p) = \sum_{i=1}^{n} \left(1 - \frac{i}{n+1}\right)\left(1 - \frac{i}{n+2}\right) \frac{x_i}{i^p}.$$

The relationship between these two summabilities in our case can be reduced to the following lemma:

Lemma.

$$\sigma_n^2 = \frac{2}{(n+1)(n+2)} \{2(n+\tfrac{1}{4})^2 R_{n+\frac{1}{4}}^2 - \tfrac{11}{2}(n+\tfrac{1}{2})^2 R_{n+\frac{1}{2}}^2 + 4(n+\tfrac{3}{4})^2 R_{n+\frac{3}{4}}^2\}, \tag{1}$$

$$R_\omega^2(p) = \frac{1}{2\omega^2} \{(n+1)(n+2)\lambda^2 \sigma_n^2(p) + n(n+1)(1+2\lambda-2\lambda^2) \sigma_{n-1}^2(p) + (n-1)n(1-\lambda)^2 \sigma_{n-2}^2(p)\} \tag{2}$$

where $\omega = n+\lambda$, $0 \leqq \lambda < 1$. Since its truth is trivial, we omit the prove here.

A normed linear space becomes a metric space if the distance $d(x, y)$ is defined as $||x-y||$, and it is called a Banach space if it is complete in this metric. Banach space is generally a complex space. However, it is assumed to be a real Banach space throughout this paper. But our conclusions can readily be generalized to complex case.

The purpose of this paper is to prove Theorem A.

THEOREM A. *Assume that* $0 \leqq q < 1$ *and* $0 < p < q$. *If there exists an element* $x \in X$, *where* X *is a Banach space, such that*

$$||R^2_\omega(q)-x|| = O(\omega^{-q})$$

then we can choose another element x^* *from* X *such that*

$$||R^2_\omega(-p)-x^*|| = O(\omega^{-q+p}).$$

PROOF. Our proof depends on the following relations. From (1) and hypothesis, we have

$$(3) \qquad ||\sigma^2_n-x|| = O(n^{-q}).$$

If, based on (1), we can choose $x^* \in X$ such that

$$(4) \qquad ||\sigma^2_n-x^*|| = 0(n^{-q+p}) = O(\omega^{-q+p}),$$

then,

$$||R^2_\omega(-p)-x^*|| = O(n^{-q+p}) = O(\omega^{-q+p})$$

follows. The crucial point in the proof is how to deduce (4) from (3). If we suppose $x = 0$, (3) is

$$||\sigma^2_n|| = 0(n^{-q})$$

Now,

$$\sigma^2_k(-p)-\sigma^2_{k-1}(-p) = \sum_{i=1}^{k}\left(1-\frac{i}{k+1}\right)\left(1-\frac{i}{k+2}\right)\frac{x_k}{i^{-p}} - \sum_{i=1}^{k-1}\left(1-\frac{i}{k+1}\right)\left(1-\frac{i}{k+2}\right)\frac{x_k}{i^{-p}}$$
$$= 2\left[\frac{\sum_{i=1}^{k} i^{1+p}x_i}{k(k+2)} - \frac{\sum_{i=1}^{k} i^{2+p}x_i}{k(k+1)(k+2)}\right].$$

Set $m > n$. Then,

$$\sigma^2_m(-p)-\sigma^2_n(-p) = \sum_{k=n+1}^{m}[\sigma^2_k(-p)-\sigma^2_{k-1}(-p)]$$
$$= 2\left[\sum_{k=n+1}^{m}\frac{\sum_{i=1}^{k} i^{1+p}x_i}{k(k+2)} - \sum_{k=n+1}^{m}\frac{\sum_{i=1}^{k} i^{2+p}x_i}{k(k+1)(k+2)}\right].$$

Set

$$Y_i = \sum_{j=1}^{i} x_j, \quad Y'_i = \sum_{j=1}^{i} Y_i, \quad \Delta_1(i) = i^{1+p}-(i+1)^{1+p},$$
$$\Delta_j(i) = \Delta_{j-1}(i)-\Delta_{j-1}(i+1).$$

Then,

$$Y'_i = \tfrac{1}{2}[(i+1)(i+2)\sigma_i^2 - i(i+1)\sigma_{i-1}^2].$$

By the "3rd Abel Transformation", we have

$$\sum_{i=1}^{k} i^{1+p}x_i = \tfrac{1}{2}\sum_{i=1}^{k} \Delta_3(i)\cdot(i+1)(i+2)\sigma_i^2 + \tfrac{1}{2}\Delta_2(k+1)\cdot(k+1)(k+2)\sigma_k^2$$
$$+\tfrac{1}{2}\Delta_1(k+1)\{(k+1)(k+2)\sigma_k^2 - k(k+1)\sigma_{k-1}^2\} + (k+1)^{1+p}Y_k$$
$$= \textstyle\sum'_{1k} + \sum'_{2k} + \sum'_{3k} + \sum'_{4k}.$$

From (5) and $\Delta_j(i) = O(i^{p+1-j})$,

$$\left\|\sum_{k=n+1}^{m} \frac{\sum'_{1k}}{k(k+2)}\right\| = \sum_{k=n+1}^{m} O\left(\frac{1}{k^2}\right)\left[\sum_{i=1}^{k} O(i^{p-2})O(i^2)O\left(\frac{1}{i^q}\right)\right.$$
$$= \sum_{k=n+1}^{m} O\left(\frac{1}{k^{-p+q+1}}\right) = O(n^{p-q}),$$

$$\left\|\sum_{k=n+1}^{m} \frac{\sum'_{2k}}{k(k+2)}\right\| = \sum_{k=n+1}^{m} O\left(\frac{1}{k^2}\right) O(k^{p-1}\cdot k^2\cdot k^{-q}) = O(n^{p-q}).$$

And

$$\sum_{k=n+1}^{m} \frac{\sum'_{3k}}{k(k+2)} = \frac{1}{2}\sum_{k=n+1}^{m}\left[\frac{(k+1)^{1+p}-(k+2)^{1+p}}{k(k+2)}\right.$$
$$\left. - \frac{(k+2)^{1+p}-(k+3)^{1+p}}{(k+1)(k+3)}\right](k+1)(k+2)\sigma_k^2$$
$$-\frac{1}{2}\frac{(n+2)^{1+p}-(n+3)^{1+p}}{(n+1)(n+3)}(n+1)(n+2)\sigma_n^2$$
$$+\frac{1}{2}\frac{(m+2)^{1+p}-(m+3)^{1+p}}{(m+1)(m+3)}(m+1)(m+2)\sigma_m^2,$$

$$\frac{(k+1)^{1+p}-(k+2)^{1+p}}{k(k+2)} - \frac{(k+2)^{1+p}-(k+3)^{1+p}}{(k+1)(k+3} = O(k^{p-3}).$$

Hence,

$$\left\|\sum_{k=n+1}^{m} \frac{\sum'_{3k}}{k(k+2)}\right\| = O\left(\sum_{k=n+1}^{m} k^{p-3}\cdot k^2\cdot k^{-q}\right) + O(n^{p-q}) + O(m^{p-q})$$
$$= O(n^{p-q}).$$

Finally,

$$\sum_{k=n+1}^{m} \frac{\sum'_{4k}}{k(k+2)} = \sum_{k=n+1}^{m} \frac{(k+1)^{1+p}}{k(k+2)} Y_k$$
$$= \sum_{k=n+1}^{m} \left[\frac{(k+1)^{1+p}}{k(k+2)} - \frac{(k+2)^{1+p}}{(k+1)(k+3)}\right] Y'_k$$
$$+ \frac{(m+2)^{1+p}}{(m+1)(m+3)} Y'_m - \frac{(n+2)^{1+k}}{(n+1)(n+3)} Y'_n.$$

Let $\sum_{mn}$ denote the first term of the above equality. And we employ the first Abel Transformation for it.

$$\sum_{mn} = \sum_{k=n+1}^{m} \left[\frac{(k+1)^{1+p}}{k(k+2)} - \frac{2(k+2)^{1+p}}{(k+1)(k+3)} + \frac{(k+3)^{1+p}}{(k+2)(k+4)}\right] (k+1)(k+2)\sigma_k^2$$
$$+ \left[\frac{(m+2)^{1+p}}{(m+1)(m+3)} - \frac{(m+3)^{1+p}}{(m+2)(m+4)}\right] (m+1)(m+2)\sigma_m^2$$
$$- \left[\frac{(n+2)^{1+p}}{(n+1)(n+3)} - \frac{(n+3)^{1+p}}{(n+2)(n+4)}\right] (n+1)(n+2)\sigma_n^2.$$

Hence,

$$||\sum_{mn}|| = O\left(\sum_{k=n=1}^{m} k^{-p-3} \cdot k^2 \cdot k^{-q}\right) + O(n^{p-q}) + O(m^{-p-q}) = O(n^{p-q}).$$

From the previous results, we have

$$\sum_{k=n+1}^{m} \frac{\sum_{i=1}^{k} i^{1+p} x_i}{k(k+2)} = \sum'_{mn} + \frac{(m+2)^{1+p}}{(m+1)(m+3)} Y'_m - \frac{(n+2)^{1+p}}{(n+1)(n+3)} Y'_n$$

where

$$||\sum'_{mn}|| = O(n^{p-q}).$$

Similarily, if we set

$$\Delta'_1(i) = i^{2+p} - (i+1)^{2+p}, \quad \Delta'_j(i) = \Delta'_{j-1}(i) - \Delta'_{j-1}(i+1),$$

then

$$\sum_{i=1}^{k} (i)^{2+p} x_i = \frac{1}{2} \sum_{i=1}^{k} \Delta'_3(i)(i+1)(i+2)\sigma_i^2 + \frac{1}{2}\Delta'_2(k+1) \cdot (k+1)(k+2)\sigma_k^2$$
$$+ \frac{1}{2}\Delta'_1(k+1)\{(k+1)(k+2)\sigma_k^2 - k(k+1)\sigma_{k-1}^2\} + (k+1)^{(2+p}Y_k$$
$$= \sum_{1k}^2 + \sum_{2k}^2 + \sum_{3k}^2 + \sum_{4k}^2.$$

It is ready to prove

$$\left\| \sum_{k=n+1}^{m} \frac{\Sigma_{jk}^{2}}{k(k+1)(k+2)} \right\| = O(n^{p-q}) \qquad (j = 1, 2, 3)$$

and

$$\sum_{k=n+1}^{m} \frac{\Sigma_{4k}^{2}}{k(k+1)(k+2)} = \Sigma_{mn}^{*} + \frac{(m+2)^{2+p}}{(m+1)(m+2)m+3)} Y'_m - \frac{(n+2)^{2+p}}{(n+1)(n+2)(n+3)} Y'_n$$

where

$$\|\Sigma_{mn}^{*}\| = O(n^{p-q}).$$

Hence

$$\sum_{k=n+1}^{m} \frac{\sum_{i=1}^{k} i^{2+p} x_i}{k(k+1)(k+2)} = \Sigma_{mn}^{2} + \frac{(m+2)^{1+p}}{(m+1)(m+3)} Y'_m - \frac{(n+1)^{1+p}}{(n+1)(n+3)} Y'_n,$$

where

$$\|\Sigma_{mn}^{2}\| = O(n^{p-q}).$$

Finally,

$$\|\sigma_m^2(-p) - \sigma_n^2(-p)\| = 2\|\Sigma_{mn}^{1} - \Sigma_{mn}^{2}\| = O(n^{p-q}).$$

Since X is a complete space, there exists $x^* \in X$ such that

$$\|\sigma_m^2(-p) - x^*\| \to O,$$

as $m \to \infty$. We have completed our proof.

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