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by

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Suppose that $f(x)$ is integrable in Lebesgue's sense and periodic with period 2π. Let its Fourier series be

$$f(x) \sim \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx),$$

and let

$$\sum_{n=1}^{\infty} (b_n \cos nx - a_n \sin nx) \equiv \sum_{n=1}^{\infty} \bar{A}_n(x)$$

be the conjugate series of the Fourier series of $f(x)$. Write

$$\bar{S}_n(x) = \sum_{\nu=1}^{n} (b_\nu \cos \nu x - a_\nu \sin \nu x) \equiv \sum_{\nu=1}^{n} \bar{A}_\nu(x).$$

Let $\bar{\sigma}_n^\alpha(x)$ be the n-th Cesàre mean of order α of the sequence $\{\bar{S}_n(x)\}$. O. Szàsz [3] has established the following

THEOREM A. *At a given point x, if there exists a number $D(x)$, such that*

(i)
$$\begin{aligned}\Psi(t) &= \int_0^t \psi(u)du \\ &\equiv \int_0^t \{f(x+u) - f(x-u) - D(x)\}du \\ &= o(t),\end{aligned}$$

(ii)
$$\Psi^*(t) = \int_0^t |\psi(u)|du = O(t)$$

as $t \to +0$, then

$$\lim_{n\to\infty} \{\bar{\sigma}_{2n}^1(x) - \bar{\sigma}_n^1(x)\} = \frac{1}{\pi} D(x) \log 2.$$

It should be noted that the corresponding conditions:

(iii)
$$\begin{aligned}\Phi(t) &= \int_0^t \varphi(u)du \\ &= \int_0^t \{f(x+u)+f(x-u)-2f(x)\}du \\ &= o(t),\end{aligned}$$

(iv)
$$\Phi^*(t) = \int_0^t |\varphi(u)|du = O(t)$$

give Lebesgue's $(C, 1)$ *summability criterion for the Fourier series of* $f(x)$ *at* x.

On the other hand, we have a (C, α) summability criterion due to Hahn [1].

THEOREM B. (iii) *is not sufficient for* $(C, 1)$ *summability of the Fourier series of* $f(x)$ *at* x, *though it implies* (C, α) *summability for every* $\alpha > 1$.

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In this note, by applying Hahn's condition to Szàsz's theorem, we extend Theorem A as follows.

THEOREM. *Let* $\lambda \geqq 2$ *be any positive integer. If* (i) *holds, then*

$$\lim_{n\to\infty} \{\bar{\sigma}^{\alpha}_{\lambda n}(x)-\bar{\sigma}^{\alpha}_{n}(x)\} = \frac{1}{\pi} D(x)\log\lambda$$

for every $\alpha > 1$.

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The following lemmas are used.

LEMMA 1 [2]. *If* $\alpha > -1$ *and* $\bar{\tau}^{\alpha}_{n}(x)$ *denotes the n-th Cesàro mean of order* α *of the sequence* $\{n\bar{A}_n(x)\}$, *then*

$$\begin{aligned}\bar{\tau}^{\alpha}_{n}(x) &= n\{\bar{\sigma}^{\alpha}_{n}(x)-\bar{\sigma}^{\alpha}_{n-1}(x)\}, \\ \bar{\tau}^{\alpha+1}_{n}(x) &= (\alpha+1)\{\bar{\sigma}^{\alpha}_{n}(x)-\bar{\sigma}^{\alpha+1}_{n}(x)\}.\end{aligned}$$

LEMMA 2 [2]. *If* $g^{\alpha}_{n}(t)$ *denotes the n-th Cesàro mean of order* α *of the sequence* $\{g_n(t)\}$, *where* $g_n(t) = \cos nt\, (n \geqq 1)$, $g_0(t) = \frac{1}{2}$, *then, for* $\alpha > 0$, $0 < t < \pi$, $k = 0, 1, 2, \ldots$,

$$\left|\left(\frac{d}{dt}\right)^k g^{\alpha}_{n}(t)\right| \leqq \begin{cases} An^k & (k \geqq 0), \\ An^{-2}t^{-k-2} & (k \leqq \alpha-2), \\ An^{k-\alpha}t^{-\alpha} & (k > \alpha-2). \end{cases}$$ [1]

[1] Through this paper, A denotes an absolute constant not necessarily the same at each occurrence.

This lemma can easily be proved with a similar argument used by Zygmund [4].

LEMMA 3. *If*

$$h_n^\alpha(t) = \sum_{\nu=n+1}^{\lambda n} \frac{1}{\nu} g_\nu^\alpha(t),$$

then, for $\alpha > 0$, $0 < t < \pi$, $k = 1, 2, \ldots,$

$$\left|\left(\frac{d}{dt}\right)^k h_n^\alpha(t)\right| \leqq \begin{cases} An^k & (k \geqq 0), \\ An^{-2}t^{-k-2} & (k \leqq \alpha-1), \\ An^{-k-\alpha-1}t^{-\alpha-1} & (k > \alpha-1). \end{cases}$$

By Lemma 1, with $g_n(t)$ in place of $\bar{S}_n(t)$, we have

$$(\alpha+1)\frac{1}{n} g_n^\alpha(t) = (\alpha+1)\frac{1}{n} g_n^{\alpha+1}(t) + g_n^{\alpha+1}(t) - g_{n-1}^{\alpha+1}(t).$$

Thus,

$$(\alpha+1)\left|\left(\frac{d}{dt}\right)^k h_n^\alpha(t)\right| \leqq (\alpha+1)\sum_{\nu=n+1}^{\lambda n}\frac{1}{\nu}\left|\left(\frac{d}{dt}\right)^k g_\nu^{\alpha+1}(t)\right| + \left|\left(\frac{d}{dt}\right)^k g_{\lambda n}^{\alpha+1}(t)\right| + \left|\left(\frac{d}{dt}\right)^k g_n^{\alpha+1}(t)\right|.$$

The lemma follows from Lemma 2.

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Now, we are in a position to prove the theorem. Since

$$n\bar{A}_n(x) = \frac{n}{\pi}\int_0^\pi \{f(x+t)-f(x-t)\}\sin nt\,dt$$
$$= -\frac{1}{\pi}\int_0^\pi \{f(x+t)-f(x-t)\}\frac{d}{dt}\cos nt\,dt,$$

from this, we get

$$\bar{\tau}_n^\alpha(x) = -\frac{1}{\pi}\int_0^\pi \{f(x+t)-f(x-t)\}\frac{d}{dt} g_n^\alpha(t)dt.$$

Now,

$$\bar{\sigma}_{\lambda n}^\alpha(x) - \bar{\sigma}_n^\alpha(x) = \sum_{\nu=n+1}^{\lambda n}\frac{1}{\nu}\bar{\tau}_\nu^\alpha(x)$$
$$= -\frac{1}{\pi}\int_0^\pi \{f(x+t)-f(x-t)\}\frac{d}{dt} h_n^\alpha(t)dt.$$

Denote

$$\omega_n = -\frac{1}{\pi}\int_0^\pi \frac{d}{dt} h_n^\alpha(t)dt.$$

Then

$$\begin{aligned}-\pi\{\bar\sigma_{\lambda n}^\alpha(x)-\bar\sigma_n^\alpha(x)-\omega_n D(x) &= \int_0^\pi \psi(t)\frac{d}{dt} h_n^\alpha(t)dt\\ &= \int_0^{l/n} + \int_{l/n}^\pi\\ &= I_1+I_2,\end{aligned}$$

say, where $0 < l < n$. Let $\beta = \min(\alpha, 2)$, then, by Lemma 3, we have

$$\begin{aligned}|I_2| &\leqq \left|\left\{\Psi(t)\frac{d}{dt} h_n^\alpha(t)\right\}_{l/n}^\pi\right| + \left|\int_{l/n}^\pi \Psi(t)\left(\frac{d}{dt}\right)^2 h_n^\alpha(t)dt\right|\\ &\leqq An^{-\beta}\left\{|\Psi(\pi)|\pi^{-\beta-1}+\left|\Psi\left(\frac{l}{n}\right)\right|\left(\frac{l}{n}\right)^{-\beta-1}\right\}+An^{-\beta+1}\int_{l/n}^\pi |\Psi(t)|t^{-\beta-1}dt\\ &\leqq An^{-\beta+1}+Al^{-\beta+1},\end{aligned}$$

and

$$\begin{aligned}I_1 &= \left\{\Psi(t)\frac{d}{dt} h_n^\alpha(t)\right\}_0^{l/n} - \int_0^{l/n} \Psi(t)\left(\frac{d}{dt}\right)^2 h_n^\alpha(t)dt\\ &= o\left(\frac{1}{n}\right)O(n)+o\left(n^2\int_0^{l/n} t\,dt\right)\\ &= o(1)\end{aligned}$$

as $n \to \infty$. Hence,

$$\limsup_{n\to\infty} \pi\,|\bar\sigma_{\lambda n}^\alpha(x)-\bar\sigma_n^\alpha(x)-\omega_n D(x)| \leqq Al^{-\beta-1}$$

for all $l > 0$. Since $\beta \geqq \alpha > 1$, it follows that

$$\lim_{n\to\infty} \{\bar\sigma_{\lambda n}^\alpha(x)-\bar\sigma_n^\alpha(x)-\omega_n D(x)\} = 0.$$

Now,

$$\begin{aligned}\omega_n &= -\frac{1}{\pi}\{h_n^\alpha(\pi)-h_n^\alpha(0)\}\\ &= -\frac{1}{\pi}\sum_{\nu=n+1}^{\lambda n}\frac{1}{\nu A_\nu^\alpha}\sum_{\mu=1}^{\nu} A_{\nu-\mu}^{\alpha-1}\{(-1)^\mu-1\},\end{aligned}$$

where

$$A_n^\alpha = \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)\Gamma(\alpha+1)}.$$

Since the sequence $\{(-1)^n-1\}$ is summable (C, α) to -1 for every $\alpha > 0$, we may write

$$\frac{1}{A_\nu^\alpha}\sum_{\mu=1}^{\nu} A_{\nu-\mu}^{\alpha-1}\{(-1)^\mu-1\} = -1+\varepsilon_\nu,$$

where $\varepsilon_\nu \to 0$ as $\nu \to \infty$. Therefore,

$$\omega_n = -\frac{1}{\pi}\sum_{\nu=n+1}^{\lambda n}\frac{-1+\varepsilon_\nu}{\nu} = \frac{1}{\pi}\log\lambda+o(1)$$

as $n \to \infty$. I.e.,

$$\lim_{n\to\infty}\{\bar{\sigma}_{\lambda n}^\alpha(x)-\bar{\sigma}_n^\alpha(x)\} = \frac{D(x)}{\pi}\log\lambda.$$

This completes the proof of the theorem.

REFERENCES

H. Hahn,

[1] Über Fejér's Summierung der Fourierschen Reihe, Jah. der Deutsch. Math.-Verein., 25 (1916), 359—366.

E. Kegebetliantz,

[2] Sommation des séries et intégrales divergentes par les moyennes arithmétiques et typiques, Mémorial des Sciences Math., 5 (1931), pp. 23 and 30.

O. Szàsz,

[3] The jump of a function and its Fourier coefficients, Duke Math. Journ., 4 (1938), 401—407.

A. Zygmund,

[4] Trigonometr[illegible] Cambridge, 1959, Vol. II, pp. 60—61.

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