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On the derivative of a G -function whose argument is a power of the variable

by

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In this paper we have established some formulae on the N -th order derivative of $G_{pq}^{ln}(\beta x^r |_{bs}^{aj})$. The Mellin-Barnes type integral [2. p. 207] which we have employed is

$$(1.1) \quad G_{pq}^{ln} \left(x \left| \begin{matrix} a_1 \dots a_p \\ b_1 \dots b_q \end{matrix} \right. \right) = \frac{1}{2\pi i} \int_L \frac{\prod_{j=1}^l \Gamma(b_j - s) \prod_{j=1}^n \Gamma(1 - a_j + s)}{\prod_{j=1}^q \Gamma(1 - b_j + s) \prod_{j=n+1}^p \Gamma(a_j - s)} x^s ds$$

where an empty product is interpreted as 1, $0 \leq l \leq q$, $0 \leq n \leq p$ and the path L of integration runs from $-i\infty$ to $+i\infty$ so that all the poles of $\Gamma(b_j - s)$, $j = 1, 2, \dots, l$ are to the right and all the poles of $\Gamma(1 - a_j + s)$, $j = 1, 2, \dots, n$ to the left of L . The formula is valid for $p + q < 2(1 + n)$ and $|\arg x| < (l + n - \frac{1}{2}p - \frac{1}{2}q)\pi$. $a_j - b_h \neq 1, 2, \dots$ for $j = 1, \dots, n$ and $h = 1, \dots, l$. In the formulae (2.1), (2.2), (3.1), (4.1), (4.3)—(4.5) the conditions mentioned as (1.1) are tacitly supposed to be fulfilled. Although the well known technique is employed, the final result depends on the fact that in the formula

$$(1.2) \quad \Gamma(mz) = (2\pi)^{(1-m)/2} m^{mz - \frac{1}{2}} \prod_{R=0}^{m-1} \Gamma \left(z + \frac{R}{m} \right) \quad m = 2, 3 \dots$$

$z, z + 1/m, z + 2/m, \dots$ are in Arithmetical Progression. The other formulae used are

$$(1.3) \quad z(z-1) \dots (z - \overline{N-1}) = \frac{\Gamma(z+1)}{\Gamma(z - \overline{N-1})},$$

$$(1.4) \quad z(z+1) \dots (z + N - 1) = \frac{\Gamma(z+N)}{\Gamma(z)}.$$

2

The first formula to be proved is

$$(2.1) \quad \frac{d^N}{dx^N} x^{r(a_1-1)} G_{pa}^{ln} \left(\frac{\beta}{x^r} \middle| a_1 \dots a_p \right) \\ = (-n)^N x^{r(a_1-1)-N} G_{pa}^{ln} \left(\frac{\beta}{x^r} \middle| a_1 - N/r, \dots a_r - N/r, a_{r+1}, \dots a_p \right)$$

provided $r < n$ and the parameters $a_1, a_2, \dots a_r$ are in A.P. with common difference $-1/r$.

PROOF:

Using (1.1) the L.H.S. of (2.1)

$$= \frac{1}{2\pi i} \int_L \frac{\prod_{j=1}^j \Gamma(b_j - s) \prod_{j=r+1}^n \Gamma(1 - a_j + s)}{\prod_{j=l+1}^q \Gamma(1 - b_j + s) \prod_{j=n+1}^p \Gamma(a_j - s)} \\ \cdot \beta^s \prod_{j=1}^r \Gamma(1 - a_j + s) \frac{d^N}{dx^N} x^{(a_1-1-s)} ds$$

using (1.4) and (1.2) we get

$$= (-r)^N x^{r(a_1-1)-N} \\ \frac{1}{2\pi i} \int_L \frac{\prod_{j=1}^l \Gamma(b_j - s) \prod_{j=1}^r \Gamma(1 - \overline{a_j - N/r} + s) \prod_{j=r+1}^n \Gamma(1 - a_j + s)}{\prod_{j=l+1}^q \Gamma(1 - b_j + s) \prod_{j=n+1}^p \Gamma(a_j - s)} \left(\frac{\beta}{x^r} \right)^s ds \\ = (-r)^N x^{(a_1-1)-N} G_{pa}^{ln} \left(\frac{\beta}{x^r} \middle| a_1 - N/r, \dots a_r - N/r, a_{r+1}, \dots a_p \right)$$

provided $r < n$ and the parameters $a_1, a_2, \dots a_r$ are in A.P. with common difference $-1/r$.

Putting $N = 1$ and $s = 1/x$ we get

$$(2.2) \quad x \frac{d}{dx} G_{pa}^{ln} \left(\beta x^r \middle| a_1 \dots a_p \right) = r G_{pa}^{ln} \left(\beta x^r \middle| a_1 - 1/r, \dots a_r - 1/r, a_{r+1}, \dots a_p \right) \\ + r(a_1 - 1) G_{pa}^{ln} \left(\beta x^r \middle| a_1 \dots a_p \right)$$

where $a_1, a_2, \dots a_r$ are in A.P. with common difference $-1/r$.

Putting $r = 1$ in (2.1) a result of Bhise [1] follows.

Putting $r = 1$ in (2.2) we get a known result (2. p. 210).

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The second formula to be established is

$$\begin{aligned}
 (3.1) \quad & \frac{d^N}{dx^N} x^{-rb_1} G_{pq}^{in} \left(\beta x^r \left| \begin{matrix} a_1 \dots a_p \\ b_1 \dots b_q \end{matrix} \right. \right) \\
 & = (-r)^N x^{-rb_1-N} G_{pq}^{in} \left(\beta x^r \left| \begin{matrix} a_1 \dots a_p \\ b_1+N/r, \dots, b_r+N/r, b_{r+1}, \dots, b_q \end{matrix} \right. \right)
 \end{aligned}$$

provided $r < l$ and the parameters b_1, b_2, \dots, b_r are in A.P. with common difference $1/r$.

This formula can be derived from (2.1) by using the well-known property

$$G_{pq}^{in} \left(x \left| \begin{matrix} a_j \\ b_s \end{matrix} \right. \right) = G_{qp}^{ni} \left(\frac{1}{x} \left| \begin{matrix} 1-b_s \\ 1-a_j \end{matrix} \right. \right).$$

Putting $N = 1$ in (3.1) we get

$$\begin{aligned}
 (3.2) \quad & x \frac{d}{dx} G_{pq}^{in} \left(\beta x^r \left| \begin{matrix} a_1 \dots a_p \\ b_1 \dots b_q \end{matrix} \right. \right) = rb_1 G_{pq}^{in} \left(\beta x^r \left| \begin{matrix} a_1 \dots a_p \\ b_1 \dots b_q \end{matrix} \right. \right) \\
 & - r G_{pq}^{in} \left(\beta x^r \left| \begin{matrix} a_1 \dots a_p \\ b_1+1/r, \dots, b_r+1/r, b_{r+1} \dots b_q \end{matrix} \right. \right)
 \end{aligned}$$

where b_1, b_2, \dots, b_r are in A.P. with common difference $1/r$. Putting $r = 1$ in (3.1) and (3.2) two results of Bhise [1] follow.

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The third formula sought to be established is

$$\begin{aligned}
 (4.1) \quad & \frac{d^N}{dx^N} x^{r(a_{p-r+1}-1/r)} G_{pq}^{in} \left(\frac{\beta}{x^r} \left| \begin{matrix} a_1 \dots a_p \\ b_1 \dots b_q \end{matrix} \right. \right) \\
 & = r^N x^{r(a_{p-r+1}-1/r)-N} G_{pq}^{in} \left(\frac{\beta}{x^r} \left| \begin{matrix} a_1 \dots a_{p-r}, a_{p-r+1}-N/r, \dots, a_p-N/r \\ b_1 \dots b_q \end{matrix} \right. \right)
 \end{aligned}$$

provided $p-r+1 > n$ and the parameters a_{p-r+1}, \dots, a_p are in A.P. with common difference $1/r$.

PROOF: Using (1.1) the L.H.S. of (4.1) becomes

$$\begin{aligned}
 (4.2) \quad &= \frac{1}{2\pi i} \int_L \frac{\prod_{j=1}^l \Gamma(b_j - s) \prod_{j=1}^n \Gamma(1 - a_j + s)}{\prod_{j=l+1}^q \Gamma(1 - b_j + s) \prod_{j=n+1}^{p-r} \Gamma(a_j - s) \prod_{j=p-r+1}^p \Gamma(a_j - s)} \\
 &\quad \cdot \beta^s \frac{d^N}{dx^N} x^{r(a_{p-r+1} - 1/r - s)} ds.
 \end{aligned}$$

Using (1.3) and (1.2) we get after little simplification (4.2) to be

$$= r^N x^{r(a_{p-r+1} - 1/r) - N} G_{pq}^{ln} \left(\beta x^r \left| \begin{matrix} a_1 \dots a_{p-r}, a_{p-r+1} - N/r, \dots, a_p - N/r \\ b_1 \dots b_q \end{matrix} \right. \right).$$

The fourth formula is

$$\begin{aligned}
 (4.3) \quad &\frac{d^N}{dx^N} x^{-r(b_{q-r+1} + 1/r - 1)} G_{pq}^{ln} \left(\beta x^r \left| \begin{matrix} a_1 \dots a_p \\ b_1 \dots b_q \end{matrix} \right. \right) \\
 &= r^N x^{-r(b_{q-r+1} + 1/r - 1)} G_{pr}^{ln} \left(\beta x^r \left| \begin{matrix} a_1 \dots a_p \\ b_1 \dots b_{q-r}, b_{q-r+1} + N/r \dots b_{q+N/r} \end{matrix} \right. \right)
 \end{aligned}$$

provided $q - r + 1 > l$ and the parameters $b_{q-r+1} \dots b_q$ are in A.P. with common difference $-1/r$.

The proof can be adduced on lines similar to (4.1).

Putting $N = 1$ and $x = 1/x$ in (4.2) we get

$$\begin{aligned}
 (4.4) \quad &x \frac{d}{dx} G_{pq}^{ln} \left(\beta x^r \left| \begin{matrix} a_1 \dots a_p \\ b_1 \dots b_q \end{matrix} \right. \right) = r \left(a_{p-r+1} - \frac{1}{r} \right) G_{pq}^{ln} \left(\beta x^r \left| \begin{matrix} a_1 \dots a_p \\ b_1 \dots b_q \end{matrix} \right. \right) \\
 &- r G_{pq}^{ln} \left(\beta x^r \left| \begin{matrix} a_1 \dots a_{p-r}, a_{p-r+1} - 1/r, \dots, a_p - 1/r \\ b_1 \dots b_q \end{matrix} \right. \right)
 \end{aligned}$$

Putting $N = 1$ in (4.3) we get

$$\begin{aligned}
 (4.5) \quad &x \frac{d}{dx} G_{pq}^{ln} \left(\beta x^r \left| \begin{matrix} a_1 \dots a_p \\ b_1 \dots b_q \end{matrix} \right. \right) \\
 &= r G_{pq}^{ln} \left(\beta x^r \left| \begin{matrix} a_1 \dots a_p \\ b_1 \dots b_{q-r}, b_{q-r+1} + 1/r, \dots, b_q + 1/r \end{matrix} \right. \right) \\
 &\quad + r \left(b_{q-r+1} + \frac{1}{r} - 1 \right) G_{pq}^{ln} \left(\beta x^r \left| \begin{matrix} a_1 \dots a_p \\ b_1 \dots b_q \end{matrix} \right. \right).
 \end{aligned}$$

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