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F. M. SIOSON

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by

F. M. Sioson

1. Introduction

In an earlier paper [4], M. H. A. Newman introduced an algebraic system which is characterized as the direct sum of a Boolean algebra and a non-associative Boolean ring with identity. Subsequently, the system has been studied by G. D. Birkhoff and G. Birkhoff [2] and Y. Wooyenaka [7] who called it a *Newman algebra.* The original definition of Newman states that this is an algebraic system $(N, +, \cdot)$ with two binary operations $+$ and $\cdot$ satisfying the following postulates:

P_1: For each $x, y, z \in N$, $x(y+z) = xy+xz$;

P_1': For each $x, y, z \in N$, $(x+y)z = xz+yz$;

P_2: There exists a pair of elements $0, 1 \in N$ such that for each $x \in N$, there is at least one other element $\bar{x} \in N$ with $\bar{x}x = 0$ and $\bar{x}+x = 1$;

P_3: For each $x \in N$, $0+x = x$;

P_4: For each $x \in N$, $xx = x$.

M. H. A. Newman [4], [5] showed that this and two others are independent axiomatizations of Newman algebras, while G. D. Birkhoff and G. Birkhoff observed that the same is true for their left-right symmetric counterparts.

Actually, the observation by the Birkhoff's is a case of a metatheorem which we shall now formulate. For any postulate P, conceivably involving $+$ and $\cdot$, denote by $P^{+}(P^{\cdot})$ the proposition obtained from P by commuting all additions (multiplications) occurring in P. Naturally, if no addition (multiplication) occurs in P, then $P^{+} = P(P^{\cdot} = P)$. In any case, note that $P^{++} = P = P^{\cdot\cdot}$ and $P^{+\cdot} = P^{\cdot +}$. If G_4 stands for the fourgroup of all transformations of postulates generated by $+$ and $\cdot$, then we have the following

METATHEOREM 1. *If $P_1, P_2, \ldots, P_n$ is an independent axiomatization of Newman algebras (or for that matter of any algebraic systems commutative with respect to $+$ and $\cdot$), then $P_1^t, P_2^t, \ldots, P_n^t$ for each $t \in G_4$ is also an independent axiomatization of Newman algebras (of the same algebraic systems).*

This metatheorem follows very readily by model-theoretic arguments. Since $P_1, P_2, \ldots, P_n$ is a system of axioms for Newman algebras, one should be able to derive from them the commutative laws under the operations $+$ and $\cdot$. This means that $P_1^t, P_2^t, \ldots, P_n^t$ for any $t \in G_4$ are consequences of the propositions $P_1, P_2, \ldots, P_n$ and hence every model of the latter system must also be a model of the former. Conversely, if M is a model of $P_1^t, P_2^t, \ldots, P_n^t$, then the model M^t obtained from M by transposing their t-operation tables is a model of the propositions $P_1, P_2, \ldots, P_n$. Thus, by virtue of a previous conclusion, $M = M^{tt}$ is also a model of $P_1, P_2, \ldots, P_n$. Whence the two systems of axioms possess precisely the same models. By virtue of the independence of the system $P_1, P_2, \ldots, P_n$ there exists for each $i = 1, 2, \ldots, n$ a model M_i of $P_1, \ldots, P_{i-1}, P_{i+1}, \ldots, P_n$ which fails to satisfy P_i. Then the t-transpose M_i^t of M_i also satisfies $P_1^t, \ldots, P_{i-1}^t, P_{i+1}^t, \ldots, P_n^t$ but not P_i^t. This means that $P_1^t, P_2^t, \ldots, P_n^t$ is an independent system of equations.

The preceding metatheorem, of course, applies to Boolean algebras, since these are also Newman algebras. Something more, however, is true. If for each proposition P in Boolean algebra the proposition obtained from P by replacing $+$ by $\cdot$ and $\cdot$ by $+$ is denoted by $\bar{P}$ (also called the dual of P) and G_8 is the eight-group of postulate-transformations generated by $+$, $\cdot$, and $^-$, then we have the following result whose proof will be omitted:

METATHEOREM 2. *If $P_1, P_2, \ldots, P_n$ form an independent system of axioms for Boolean algebras, then for each $t \in G_8$ $P_1^t, P_2^t, \ldots, P_n^t$ is also an independent system of axioms for Boolean algebras.*

It is easy to see that the homomorphic images, the direct products, and the subalgebras of Newman algebras are also Newman algebras. Hence, by a well-known characterization of G. Birkhoff [3] the family of all Newman algebras is a *variety* or *equational class* (i.e. a system definable in terms of equational laws). Y. Wooyenaka [7] gave two sets of generating equations for Newman algebras.

From the results of [4] it is easily seen that the following equations hold for all elements belonging to any Newman algebra:

$$N_1\colon\ x(y+z) = xy+xz;$$
$$N_2\colon\ x(y+\bar{y}) = x; \qquad \bar{N}_2\colon\ x+y\bar{y} = x;$$
$$N_3\colon\ xy = yx; \qquad \bar{N}_3\colon\ x+y = y+x;$$
$$N_4\colon\ x(y\bar{y}) = y\bar{y};$$
$$N_5\colon\ xx = x;$$
$$N_6\colon\ \bar{\bar{x}} = x;$$
$$\bar{N}_7\colon\ x+(y+z) = (x+y)+z.$$

Without considering their transforms under the members of the group G_4, these identities or equations appear to be the most natural and the most commonly known ones. A Newman algebra may be defined as an algebraic system $(N, +, \cdot, {}^-)$ satisfying the above nine equations. These equations, however, are not independent. The aim of the present communication is to extract all independent systems of equations out of the $2^9-1 = 511$ possible equation-subsets of the given nine that generate all equations in any Newman algebra. One may call such systems *equational bases.* The pertinent result is as follows:

METATHEOREM 3. *The only equational bases for Newman algebras out of the given pool of nine equations are*

$$\mathbf{I} = \{N_1, N_2, \bar{N}_2, N_3\} \text{ and } \mathbf{II} = \{N_1, N_2, N_3, N_4\}$$

and their transforms under the group G_4.

Another result is needed in order to apply the previous metatheorem.

METATHEOREM 4. *The following equations are equivalent in any Newman algebra*:

$$\bar{N}_1\colon\ x+yz = (x+y)(x+z);$$
$$\bar{N}_4\colon\ x+(y+\bar{y}) = y+\bar{y};$$
$$\bar{N}_5\colon\ x+x = x;$$
$$\bar{N}_8\colon\ x+xy = x;$$
$$N_8\colon\ x(x+y) = x.$$

To show this, consider any Newman algebra and recall that the nine equations, N_1, N_2, $\bar{N}_2$, N_3, $\bar{N}_3$, N_4, N_5, N_6, and $\bar{N}_7$, given above hold in any such algebra. The required result is then demonstrated as follows:

$$\bar{N}_1 \text{ implies } \bar{N}_4\text{: } y+\bar{y} = (y+\bar{y})+y\bar{y} = (y+\bar{y})+x(y\bar{y})$$
$$= ((y+\bar{y})+x)((y+\bar{y})+y\bar{y}) = (x+(y+\bar{y}))(y+\bar{y})$$
$$= x+(y+\bar{y}) \quad (\bar{N}_2, N_4, \bar{N}_1, \bar{N}_3{-}\bar{N}_2, N_2);$$

$$\bar{N}_4 \text{ implies } \bar{N}_5\text{: } x = x(y+\bar{y}) = x(x+(y+\bar{y}))$$
$$= xx+x(y+\bar{y}) = x+x \quad (N_2, \bar{N}_4, N_1, N_5{-}N_2);$$

$$\bar{N}_5 \text{ implies } \bar{N}_8\text{: } x = x(y+\bar{y}) = xy+x\bar{y} = (xy+xy)+x\bar{y}$$
$$= xy+(xy+x\bar{y}) = (xy+x\bar{y})+xy = x(y+\bar{y})+xy$$
$$= x+xy \quad (N_2, N_1, \bar{N}_5, \bar{N}_7, \bar{N}_3, N_1, N_2);$$

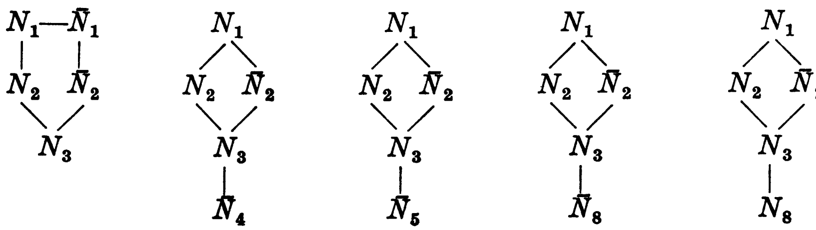
$$\bar{N}_8 \text{ implies } N_8\text{: } x = x+xy = xx+xy = x(x+y) \quad (\bar{N}_8,\ N_5,\ N_1);$$

$$N_8 \text{ implies } \bar{N}_1\text{: } (x+y)(x+z) = (x+y)x+(x+y)z$$
$$= x(x+y)+z(x+y) = x+(zx+zy) = (x+zx)+zy$$
$$= (x+xz)+yz = (xx+xz)+yz = x(x+z)+yz$$
$$= x+yz$$
$$(N_1, N_3{-}N_3,\ N_8{-}N_1,\ \bar{N}_7,\ N_3{-}N_3,\ N_5,\ N_1, N_8).$$

This completes the circle of necessary implications.

Subject to independence-proofs that will be supplied later, the following metatheorem follows from the fact (see [4] or [2]) that either **I** or **II** each gives rise to an equational basis for Boolean algebras when $\bar{N}_5$ (and hence any one of the five equations in Metatheorem 4) is added to it.

METATHEOREM 5. *The following systems of equations and all their transforms under the members of the group G_8 are equational bases for Boolean algebras*:



(A) (B) (C) (D) (E)

$$\begin{tikzcd}[column sep=small] N_1 \arrow[dr, dash] & & \bar{N}_1 \arrow[dl, dash] \\ & N_2 \arrow[d, dash] & \\ & N_3 \arrow[d, dash] & \\ & N_4 & \end{tikzcd}$$
(F)

$$\begin{tikzcd}[column sep=small] & N_1 \arrow[d, dash] & \\ & N_2 \arrow[d, dash] & \\ & N_3 \arrow[dl, dash] \arrow[dr, dash] & \\ N_4 & & N_4 \end{tikzcd}$$
(G)

$$\begin{tikzcd} N_1 \arrow[d, dash] \\ N_2 \arrow[d, dash] \\ N_3 \arrow[d, dash] \\ N_4 \arrow[d, dash] \\ \bar{N}_5 \end{tikzcd}$$
(H)

$$\begin{tikzcd} N_1 \arrow[d, dash] \\ N_2 \arrow[d, dash] \\ N_3 \arrow[d, dash] \\ N_4 \arrow[d, dash] \\ \bar{N}_8 \end{tikzcd}$$
(I)

$$\begin{tikzcd} N_1 \arrow[d, dash] \\ N_2 \arrow[d, dash] \\ N_3 \arrow[d, dash] \\ N_4 \arrow[d, dash] \\ N_8 \end{tikzcd}$$
(J).

Systems (A), (B), (F), and (G) have been shown to be equational bases for Boolean algebras in a previous article [6]. One might be tempted to conjecture at this point that every equational basis for Boolean algebras may be obtained by combining a basis for Newman algebras with some non-Newman equation of Boolean algebras. This is, however, not so. The following equational basis of Boolean algebras (derived in [6])

$$\begin{tikzcd}[column sep=small] N_1 \arrow[dr, dash] & & \bar{N}_1 \arrow[dl, dash] \\ & \bar{N}_2 \arrow[d, dash] & \\ & N_3 \arrow[d, dash] & \\ & \bar{N}_4 & \end{tikzcd}$$

contains equations $\bar{N}_1$ and $\bar{N}_4$ which are independent of Newman algebras and the subsystem consisting of N_1, $\bar{N}_2$, and N_3 does not at all form a basis for Newman algebras.

We shall now show that **I** and **II** are indeed equational bases for Newman algebras.

2. The equational basis I

To show that **I** is equationally complete for Newman algebras, the original set of axioms of Newman given in the beginning of this paper will be derived.

2.1. $(x+y)z = xz+yz$.

$$(x+y)z = z(x+y) = zx+zy = xz+yz \quad (N_3, N_1, N_3—N_3).$$

2.2. $xx = x$.

$$xx = xx+x\bar{x} = x(x+\bar{x}) = x \quad (\bar{N}_2, N_1, N_2).$$

2.3. $x+\bar{x} = y+\bar{y}$.

$x+\bar{x} = (x+\bar{x})(y+\bar{y})=(y+\bar{y})(x+\bar{x}) = y+\bar{y}$ (N_2, N_3, N_2).

2.4. $\bar{\bar{x}}x = \bar{\bar{x}}$.

$\bar{\bar{x}} = \bar{\bar{x}}(x+\bar{x}) = \bar{\bar{x}}x+\bar{\bar{x}}\bar{x} = \bar{\bar{x}}x+\bar{x}\bar{\bar{x}} = \bar{\bar{x}}x$ $(N_2, N_1, N_3, \bar{N}_2)$.

2.5. $\bar{x}+x = x+\bar{x}$, $\bar{x}x = x\bar{x}$.

First note that (a) $\bar{\bar{x}} = \bar{\bar{x}}(\bar{x}+\bar{\bar{x}}) = \bar{\bar{x}}\bar{x}+\bar{\bar{x}}\bar{\bar{x}} = \bar{\bar{x}}\bar{x}+\bar{\bar{x}} = \bar{\bar{x}}\bar{x}+\bar{\bar{x}}x = \bar{\bar{x}}(\bar{x}+x)$ (N_2, N_1, 2.2, 2.4, N_1) and (b) $\bar{x} = \bar{x}+x\bar{x} = \bar{x}\bar{x}+\bar{x}x = \bar{x}(\bar{x}+x)$ ($\bar{N}_2$, 2.2—N_3, N_1). Then $\bar{x}+x = (\bar{x}+x)(\bar{x}+\bar{\bar{x}}) = (\bar{x}+x)\bar{x}+(\bar{x}+x)\bar{\bar{x}} = \bar{x}(\bar{x}+x)+\bar{\bar{x}}(\bar{x}+x) = \bar{x}+\bar{\bar{x}} = x+\bar{x}$ (N_2, N_1, N_3—N_3, (a)—(b), 2.3). The second part of 2.5 follows from N_3.

2.6. $\bar{\bar{x}} = x$.

$\bar{\bar{x}} = \bar{\bar{x}}x = \bar{\bar{x}}x+x\bar{x} = x\bar{\bar{x}}+x\bar{x} = x(\bar{\bar{x}}+\bar{x}) = x(\bar{x}+\bar{\bar{x}}) = x$ (2.4, $\bar{N}_2, N_3, N_1$, 2.5, N_2).

2.7. $(y\bar{y})\overline{(y\bar{y})} = y\bar{y}$.

$(y\bar{y})\overline{(y\bar{y})} = (y\bar{y})\overline{(y\bar{y})}+y\bar{y} = (y\bar{y})\overline{(y\bar{y})}+(y\bar{y})^2 = (y\bar{y})(\overline{y\bar{y}}+y\bar{y}) = (y\bar{y})(y\bar{y}+\overline{y\bar{y}}) = y\bar{y}$ ($\bar{N}_2$, 2.2, N_1, 2.5, N_2).

2.8. $\overline{y\bar{y}} = y+\bar{y}$.

$\overline{y\bar{y}} = \overline{(y\bar{y})}(y\bar{y}+\overline{y\bar{y}}) = \overline{(y\bar{y})}(y\bar{y})+\overline{(y\bar{y})^2} = (y\bar{y})\overline{(y\bar{y})}+\overline{y\bar{y}} = y\bar{y}+\overline{y\bar{y}} = y+\bar{y}$ (N_2, N_1, N_3—2.2, 2.7, 2.3).

2.9. $x\bar{x} = y\bar{y}$.

$x\bar{x} = \overline{\overline{x\bar{x}}} = \overline{x+\bar{x}} = \overline{y+\bar{y}} = \overline{\overline{y\bar{y}}} = y\bar{y}$ (2.6, 2.8, 2.3, 2.8, 2.6).

2.10. $x(y\bar{y}) = y\bar{y}$.

$x(y\bar{y}) = x(x\bar{x}) = x(x\bar{x})+x\bar{x} = x(x\bar{x}+\bar{x}) = x(\bar{x}x+\bar{x}\bar{x}) = x(\bar{x}(x+\bar{x})) = x\bar{x} = y\bar{y}$ (2.9, $\bar{N}_2$, N_1, N_3—2.2, N_1, N_2, 2.9).

2.11. $y\bar{y}+x = x$.

$y\bar{y}+x = x(y\bar{y})+x(y+\bar{y}) = x(y\bar{y}+(y+\bar{y})) = x(y\bar{y}+\overline{y\bar{y}}) = x$ (2.10—N_2, N_1, 2.8, N_2).

Setting $\bar{x}x = x\bar{x} = 0$ and $\bar{x}+x = x+\bar{x} = 1$, it is easy to see that N_1, 2.1, 2.11, and 2.2 are respectively Newman's axioms P_1, P_1', P_3 and P_4. A reformulation of 2.3, 2.9, and 2.5 gives rise to postulate P_2.

The independence of system **I** is next shown as follows.

IN_1. N_1 is independent of N_2, $\bar{N}_2$, N_3, (and $\bar{N}_5$). Consider the operations defined on the set consisting of 0 and 1 by the following tables:

$+$	0	1
0	0	0
1	0	1

$\cdot$	0	1
0	0	1
1	1	0

y	$\bar{y}$
0	1
1	0

Note that $1(1+0) \neq 11+10$. The other equations are easily verified.

IN_2. N_2 is independent of N_1, $\bar{N}_2$, N_3, (and $\bar{N}_5$).

$+$	0	1
0	0	0
1	1	1

$\cdot$	0	1
0	0	0
1	0	1

y	$\bar{y}$
0	1
1	0

Here $1(0+\bar{0}) \neq 1$.

I$\bar{N}_2$. $\bar{N}_2$ is independent of N_1, N_2, N_3, (and $\bar{N}_5$).

$+$	0	1
0	0	1
1	1	1

$\cdot$	0	1
0	0	0
1	0	1

y	$\bar{y}$
0	1
1	1

Note that here $0+1\bar{1} \neq 0$.

IN_3. N_3 is independent of N_1, N_2, $\bar{N}_2$, (and $\bar{N}_5$).

$+$	0	1
0	0	0
1	1	1

$\cdot$	0	1
0	0	0
1	1	1

y	$\bar{y}$
0	0
1	1

Notice that here $01 \neq 10$.

I$\bar{N}_5$. $\bar{N}_5$ is independent of N_1, N_2, $\bar{N}_2$, and N_3.

+	0	1
0	0	1
1	1	0

•	0	1
0	0	0
1	0	1

y	$\bar{y}$
0	1
1	0

Here observe that $1+1 \neq 1$. This model shows that $\bar{N}_5$ is in fact independent of any axiom system for Newman algebras, a property that will henceforth be assumed.

3. The equational basis II

The completeness of **II** as an axiomatization of Newman algebras is shown by deriving $\bar{N}_2$ (and hence **I**) from it.

3.1. $x\bar{x} = y\bar{y}$.

$x\bar{x} = (y\bar{y})(x\bar{x}) = (x\bar{x})(y\bar{y}) = y\bar{y}$ (N_4, N_3, N_4).

3.2. $\overline{y+\bar{y}} = y\bar{y}$.

$\overline{y+\bar{y}} = \overline{(y+\bar{y})}(y+\bar{y}) = (y+\bar{y})\overline{(y+\bar{y})} = y\bar{y}$ (N_2, N_3, 3.1).

3.3. $x+y\bar{y} = x$.

$x = x((y+\bar{y})+\overline{(y+\bar{y})}) = x((y+\bar{y})+y\bar{y}) = x+y\bar{y}$

(N_2, 3.2, N_1, N_2–N_4).

IIN_1. The independence of N_1 from N_2, N_3, N_4, (and $\bar{N}_5$) is shown by the following three-element model:

+	0	1	a
0	0	1	1
1	1	1	1
a	1	1	a

•	0	1	a
0	0	0	0
1	0	1	a
a	0	a	0

y	$\bar{y}$
0	1
1	0
a	0

Here $a(a+1) \neq aa+a1$.

IIN_2. The independence of N_2 from N_1, N_3, N_4, (and $\bar{N}_5$) may be effected by the same model IN_2.

IIN_3. The independence of N_3 from N_1, N_2, N_4, (and $\bar{N}_5$) follows from the following four-element model:

+	0	1	a	b
0	0	1	a	b
1	1	1	1	1
a	a	1	a	1
b	b	1	1	b

•	0	1	a	b
0	0	0	0	0
1	0	1	0	1
a	0	a	a	0
b	0	b	0	b

y	$\bar{y}$
0	1
1	0
a	b
b	a

Observe that for this model $1a \neq a1$ and $1b \neq b1$. Of the remaining equations, only N_1 needs to be verified. Since addition is commutative, the verification is effected by the following relations: $x(y+0) = xy = xy+0 = xy+x0$, $x(y+1) = x1 = \binom{0}{x}+x = xy+x1$, $x(y+y) = xy = xy+xy$, $1(a+b) = 11 = 1 = 0+1 = 1a+1b$, $a(a+b) = a1 = a = a+0 = aa+ab$, $b(a+b) = b1 = 0+b = ba+bb$.

IIN_4. The independence of N_4 from N_1, N_2, N_3, (and $\bar{N}_5$) is evident from the following model:

+	0	1
0	0	1
1	0	1

•	0	1
0	0	1
1	1	1

y	$\bar{y}$
0	0
1	0

Note that $1(0\bar{0}) \neq 0\bar{0}$.

4. Proof of metatheorem 3

First, we observe that the independence-model IIN_3 satisfies all equations of the given pool of nine with the exception of N_3. Without too much difficulty, it is also easily seen that the following model satisfies equations N_2, $\bar{N}_2$, N_3, $\bar{N}_3$, N_4, N_5, N_6, $\bar{N}_7$, but $a(b+b) \neq ab+ab$.

+	0	1	a	b
0	0	1	a	b
1	1	1	1	1
a	a	1	1	1
b	b	1	1	1

·	0	1	a	b
0	0	0	0	0
1	0	1	a	b
a	0	a	a	0
b	0	b	0	b

y	$\bar{y}$
0	1
1	0
a	b
b	a

In a similar manner a model may be found which satisfies N_1, $\bar{N}_2$, N_3, $\bar{N}_3$, N_4, N_5, N_6, and $\bar{N}_7$ but not N_2. Consider, for instance, the collection of all finite unions of open real intervals of the forms (x, y), (x, ∞), $(-\infty, y)$, $(-\infty, \infty)$ under the operations of set-union (denoted by $+$), set-intersection (denoted by $\cdot$), and the complement of the closure of a set x (denoted by $\bar{x}$). Then note that $(1,3)\big((2,4)+\overline{(2,4)}\big) \neq (1,3)$.

These observations together show that N_1, N_2, and N_3 are each independent of the rest of the pool of nine Newman equations given in the beginning. Thus, every conceivable equational basis for Newman algebras (out of the nine) must necessarily include all three of them. Moreover, any such basis cannot properly contain either I or II, for, then the resulting systems of equations would be logically dependent. These considerations imply that no equational basis (out of the nine) can contain more than seven equations. The only system with seven equations that can possibly be a basis is

$$\begin{tikzcd}[column sep=small]
 & N_1 \arrow[d, dash] & \\
 & N_2 \arrow[dl, dash] \arrow[dr, dash] & \\
N_3 \arrow[dr, dash] & & \bar{N}_3 \arrow[dl, dash] \\
 & N_5 \arrow[d, dash] & \\
 & N_6 \arrow[d, dash] & \\
 & \bar{N}_7. &
\end{tikzcd}$$

This system of equations, on the other hand, is incomplete for Newman algebras. Equation $\bar{N}_2$, which is a known property of

Newman algebras, is independent of the said system. To see this, consider the dual of the independence-model of N_2 above, that is, the collection of all finite unions of finite, semi-infinite, and infinite open real intervals under the operations of intersection (now denoted by $+$), union (now denoted by .) and the complement of the closure of a set (denoted by $^-$). Then $(1,3+(2,4)\overline{(2,4)}$ $= (1,3) \cap ((2,4) \cup \overline{(2,4)}) \neq (1,3)$, but the rest of the equations are easily verified.

The remaining $\binom{4}{3} = 4$ equation-systems with six equations and the $\binom{4}{2} = 6$ equation-systems with five equations that are likely equational bases are all proper subsets of the incomplete system considered in the previous paragraph, and hence also incomplete. Whence **I** and **II** are the only two equational bases for Newman algebras out of the pool of natural equations given above.

5. Proof of metatheorem 5

By Metatheorem 4, it will suffice to show the independence of systems (A)—(J). The independence of (A), (B), (F), and (G) have been shown in [6] and those of (C) and (H) in sections 2 and 3. We need therefore only show the independence of (D), (E), (I), and (J).

For the independence of (D) and (E) only one new model will be needed. The models used in showing the independence of N_2, $\bar{N}_2$, and N_3 in **I** may also be used in showing that these are independent in (D) and (E). No model is necessary to prove the independence of N_8 and $\bar{N}_8$. If N_8 $(\bar{N}_8)$ were dependent in E (D), then **I** would be a basis for Boolean algebras. The independence-model of N_1 from the rest of E and D is the following:

+	0	1	a	b
0	0	1	1	0
1	1	1	1	1
a	a	1	a	1
b	b	1	1	b

•	0	1	a	b
0	0	0	0	0
1	0	1	a	b
a	0	a	a	0
b	0	b	0	b

y	$\bar{y}$
0	1
1	0
a	b
b	a

We observe here that $a(b+a) \neq ab+aa$.

This last model also shows that N_1 is independent from the rest of the systems (I) and (J). Again, the independence of N_2 from the rest of (I) and (J) is effected by the model $\mathrm{I}N_2$. For N_3, one can utilize the same model used in proving the independence of N_3 from the rest of given nine Newman equations. For the independence of N_4 from the rest of (I) and (J), we take once more the collection of all finite unions of open finite, semi-infinite, and infinite real intervals under intersection (denoted by $+$), union (denoted by $\cdot$) and the complementation of the closure of set (denoted by $^-$). In this particular instance, N_2 will be satisfied but not N_4 since $(1,5)((2,4)\overline{(2,4)}) = (1,5) \cup ((2,4) \cap \overline{(2.4)}) \neq (2,4) \cup \overline{(2,4)} = (2,4)\overline{(2,4)}$. The independence proof of N_8 $(\bar{N}_8)$ from the rest of (J) $((I))$ is exactly the same as in (D) and (E).

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University of Florida