J. G. DIJKMAN

Probability theory and intuitionism. Discrete state-space

Compositio Mathematica, tome 17 (1965-1966), p. 72-95
<http://www.numdam.org/item?id=CM_1965-1966__17__72_0>

Probability theory and intuitionism

Discrete state-space

by

J. G. Dijkman

0. Introduction

0.1 The purpose of this paper is to give an intuitionistic treatment of probability theory in the case the state-space is a numerable species. To this aim the idea of an event is defined and the basic properties of the theory are derived. Two kinds of stochastic variables are introduced. The basic results from classical theory — as there are Tchebichev — and Kolmogorov-inequality, Bernoulli's law of large numbers and Borel's law of large numbers — are proved.

In section 5 we define the notion of "strong event" and compare its properties with those of an event.

0.2. For the intuitionistic nomenclature the reader is referred to Heyting [1]), whereas for the classical results of probability theory the reader may consult Loève [2]), from which book many theorems are rewritten in an intuitionistic formulation.

1.1. Let Ω be a numerable species of mathematical entities. This species Ω will be called the state-space. We number the elements of Ω and denote them by E_n, so the space Ω can be represented by $\{E_1, E_2, \ldots\}$. The number of elements may be finite, infinite or even unknown.

We introduce the following assumption:
to every element $E_i \in \Omega$ a real number p_i is assigned such that

(α_1): $(\forall E_i \in \Omega)(0 \ngtr p_i \ngtr 1)$

(α_2): $(\forall k)(\exists N)(\forall n)(E_{N+n} \in \Omega \Rightarrow 0 \ngtr 1-\sum_{i=1}^{N+n} p_i < 2^{-k})$.

1.2.1. A subspecies of Ω consisting of only one element will be called an elementary event.

[1]) A. Heyting, Intuitionism, An Introduction, North-Holland Publishing Company, Amsterdam, 1956.

[2]) M. Loève, Probability Theory (Second Edition), D. van Nostrand Company. New York, London, 1960.

1.2.2. Definition.

A species $\Gamma \subset \Omega$ is called an event if the following condition is satisfied:

$$p_j \# 0 \Rightarrow (E_j \in \Gamma) \vee (E_j \notin \Gamma).$$

Note that, if Γ is an event, this definition does not require that

$$(E_j \in \Gamma) \vee (E_j \notin \Gamma)$$

is true for every $E_j \in \Gamma$. The disjunction is only required for those elements of Ω to which a real number p_j is assigned such that $p_j \# 0$. Trivially an elementary event is an event.

1.3.1. Definition.

If Γ is an event then the number $P(\Gamma)$ is defined by

$$P(\Gamma) = \sum_{E_i \in \Gamma} p_i.$$

This number $P(\Gamma)$ is called the probability of the event Γ.

1.3.2. To justify the definition of $P(\Gamma)$ we have to show that for every event Γ the series

$$\sum_{E_i \in \Gamma} p_i \tag{1}$$

is defined i.e. it can be calculated as accurately as desired. To prove this we consider the event Γ and choose a natural number k_1. On account of α_2 (cf. 1.1.) we can calculate a natural number N_1 such that

$$1 - \sum_{i=1}^{N_1} p_i < 2^{-k_1-2},$$

hence we have

$$\sum_{\substack{E_i \in \Gamma \\ i > N_1}} p_i < 2^{-k_1-2}. \tag{2}$$

Now we prove that

$$\sum_{\substack{E_i \in \Gamma \\ i \leq N_1}} p_i$$

can be calculated as accurately as desired. We, therefore, consider the real numbers $p_1, p_2, \ldots, p_{N_1}$ and apply the theorem [3]): If a and b are real numbers such that $a \# b$, then

$$(a \# c) \vee (b \# c) \text{ for every number } c.$$

Taking

[3]) cf. e.g. Heyting l.c. p. 20.

$$a = N_1^{-1} \cdot 2^{-k_1-3} \quad \text{and} \quad b = N_1^{-1} \cdot 2^{-k_1-2}$$

we then have:

$$(p_j \# a) \vee (p_j \# b) \text{ for every } j = 1, 2, \ldots, N_1,$$

and from $a < b$ it follows that for every $j \leqq N_1$ we can prove at least one of the inequalities: $p_j > a$ or $p_j < b$.

The species of the elements $E_1, E_2, \ldots, E_{N_1}$ is partitioned into two disjoint species K_1 and K_2 such that

$$E_j \in K_1 \Rightarrow p_j < b \qquad (j \leqq N_1)$$

$$E_j \in K_2 \Rightarrow p_j > a \qquad (j \leqq N_1).$$

More than one partition may be possible but then one of the possible ones is chosen.

Evidently:

$$(3) \qquad \sum_{E_j \in K_1} p_j < N_1 b = 2^{-k_1-2}.$$

For every $E_j \in K_2$ it follows: $p_j > a$, so $p_j \# 0$, hence

$$(E_j \in \Gamma) \vee (E_j \notin \Gamma),$$

which implies:

$$\sum_{E_j \in \Gamma \cap K_2} p_j$$

is well defined. Combining the latter result with (2) and (3) it is found:

$$\sum_{E_j \in \Gamma \cap K_2} p_j - 2^{-k_1-2} < \sum_{E_i \in \Gamma} p_i < \sum_{E_j \in \Gamma \cap K_2} p_i + 2^{-k_1-1},$$

and these inequalities prove that (1) can be calculated as accurately as desired.

1.3.3. If Γ is a species such that

$$(1) \qquad \sum_{E_i \in \Gamma} p_i$$

is positively convergent, then Γ is an event.

Proof. Let $p_\nu \# 0$, then we have to prove:

$$(E_\nu \in \Gamma) \vee (E_\nu \notin \Gamma).$$

Putting $s = \sum_{E_i \in \Gamma} p_i$ we can calculate s as accurately as desired i.e. for every natural number k we can indicate an interval with rational endpoints, with length less than 2^{-k-1}, and containing s. Let k be chosen such that $p_\nu > 2^{-k}$. From the positive convergence it follows that a natural number N can be calculated such that

(2) $$(\forall n >)(|s - N \sum_{\substack{E_i \in \Gamma \\ i=1}}^{n} p_i| < 2^{-k-1}),$$

and hence

(3) $$(\forall n > N)(\forall m)(\sum_{\substack{E_i \in \Gamma \\ i=n+1}}^{n+m} p_i < 2^{-k-1}).$$

Relation (3) implies $E_\nu \notin \Gamma$ if $\nu > N$; if $\nu \leqq N$ the disjunction

$$(E_\nu \in \Gamma) \vee (E_\nu \notin \Gamma)$$

is a consequence of (2). The latter results prove the statement.

1.3.4. Definition.

The event Γ_1 implies the event Γ_2 (notation: $\Gamma_1 \Rightarrow \Gamma_2$) if

(1) $$(p_j \# 0) \wedge (E_j \in \Gamma_1) \Rightarrow (E_j \in \Gamma_2).$$

Two events Γ_1 and Γ_2 are called equivalent if they imply each other; this will be denoted by: $\Gamma_1 \approx \Gamma_2$.

Remark. In classical probability theory the event Γ_1 implies the event Γ_2 if $\Gamma_1 \subset \Gamma_2$. In stead of this classical notation the notation $\Gamma_1 \Rightarrow \Gamma_2$ is preferred here because condition (1) only speaks of states E_j with $p_j \# 0$.

Note that in classical theory two events, which imply each other, consist of the same elementary events. In the present theory this property cannot be proven.

1.3.5. If Γ_1 and Γ_2 are two events then

$$\Gamma_1 \approx \Gamma_2 \Rightarrow P(\Gamma_1) = P(\Gamma_2).$$

Proof. The proof of

$$(\forall k)(|P(\Gamma_1) - P(\Gamma_2)| < 2^{-k})$$

runs along the same lines as in section 1.3.2.

2. Properties of events

2.1.1. If Γ_1 and Γ_2 are events, then $\Gamma_1 \cup \Gamma_2$ is an event.

Proof. Let E_j be an element of Ω with $p_j \# 0$, then

$$(E_j \in \Gamma_i) \vee (E_j \notin \Gamma_i) \qquad (i = 1, 2),$$

hence

$$(E_j \in \Gamma_1 \cup \Gamma_2) \vee (E_j \notin \Gamma_1 \cup \Gamma_2).$$

2.1.2. In the same way we have:

If Γ_1 and Γ_2 are events, then $\Gamma_1 \cap \Gamma_2$ is an event.

2.1.3. Ω is an event and $P(\Omega) = 1$.

Proof. The first part of the statement is obvious and the second part is evident on account of α_2 (1.1.).

2.2.1. Definition.

If Γ is an event, then $\bar{\Gamma}$ is defined as the species of elements E_j with $E_j \notin \Gamma$.

2.2.2. If Γ is an event, then $\bar{\Gamma}$ is an event.

Proof. From $p_j \# 0$ it follows

$$(E_j \in \Gamma) \vee (E_j \notin \Gamma),$$

hence

$$(E_j \notin \bar{\Gamma}) \vee (E_j \in \bar{\Gamma}).$$

2.2.3. If Γ is an event, then $\Gamma \approx \bar{\bar{\Gamma}}$.

Proof. $\Gamma \Rightarrow \bar{\bar{\Gamma}}$, for $E_j \in \Gamma \Rightarrow E_j \notin \bar{\Gamma} \Rightarrow E_j \in \bar{\bar{\Gamma}}$.

$$\bar{\bar{\Gamma}} \Rightarrow \Gamma, \text{ for } (p_j \# 0) \wedge (E_j \in \bar{\bar{\Gamma}}) \Rightarrow \{(E_j \in \Gamma) \vee (E_j \notin \Gamma)\} \wedge (E_j \in \bar{\bar{\Gamma}}) \Rightarrow (E_j \in \Gamma).$$

2.3.1. Let $\{\Gamma_n\}$ be a sequence of events such that Γ_i and Γ_j have no common element for every i and j with $i \neq j$. Let $\sum_{n=1}^{\infty} P(\Gamma_n)$ be positively convergent, then we have

(i): $\bigcup_{n=1}^{\infty} \Gamma_n$ is an event

(ii): $\sum_{n=1}^{\infty} P(\Gamma_n) = P(\bigcup_{n=1}^{\infty} \Gamma_n)$.

Proof. We choose a state E_j with $p_j \# 0$. On account of the positive convergence of $\sum_{n=1}^{\infty} P(\Gamma_n)$ we have:

$$(\exists N)(\forall n)(P(\Gamma_{N+n}) < p_j). \tag{1}$$

Let N_1 be a natural number satisfying (1), then

$$(\forall n)(E_j \notin \Gamma_{N_1+n}). \tag{2}$$

From $p_j \# 0$ it follows that

$$(E_j \in \Gamma_n) \vee (E_j \notin \Gamma_n)$$

can be solved for $n = 1, 2, \ldots, N_1$, hence we have

$$(E_j \in \bigcup_{n=1}^{N_1} \Gamma_n) \vee (E_j \notin \bigcup_{n=1}^{N_1} \Gamma_n). \tag{3}$$

The results (2) and (3) prove that $\bigcup_{n=1}^{\infty} \Gamma_n$ is an event.

The proof of the second part is now easy.

2.3.2. The sequence $\{\sum_{\nu=1}^{n} P(\Gamma_\nu)\}$ is monotonely non-decreasing and bounded, which implies the convergence of $\sum P(\Gamma_n)$ in classical mathematics. However, from the intuitionistic point of view the properties "monotone" and "bounded" are not sufficient to guarantee the positive convergence of $\sum P(\Gamma_n)$ and the necessity to formulate explicitly the positive convergence of $\sum P(\Gamma_n)$, as we did in section (2.3.1.), can be illustrated by the following example.

We consider the decimal expansion of π.

Let τ be the sequence 0 1 2 . . . 9. We suppose that $p_1 \neq 0$. (This is no restriction). We define the event Γ_k by

$$\Gamma_k = \{E_{k+1}\} \qquad (k = 1, 2, \ldots; E_{k+1} \in \Omega)$$

if among the first k decimals of π the sequence τ does not occur, but if τ occurs and if λ is the index of the digit 9 in the first sequence τ that occurs, then we define

$$\Gamma_\lambda = \{E_1\} \quad \text{and} \quad \Gamma_{\lambda+n} = \{E_{\lambda+n}\} \qquad (n = 1, 2, \ldots).$$

Every element Γ_n of the sequence $\{\Gamma_n\}$ is an event, but we cannot prove (nowadays) that $\bigcup_{n=1}^{\infty} \Gamma_n$ is an event for we have no proof for either the occurrence or the nonoccurrence of τ. It is easily seen that the series $\sum_{n=1}^{\infty} P(\Gamma_n)$ is twofold negatively convergent [4]) with limit values 1 and $1-p_1$.

2.3.3. Let the sequence $\{\Delta_n\}$ of events Δ_n be defined by

$$\Delta_n = \Omega - \bigcup_{i=1}^{n} \Gamma_i \qquad (\text{definition } \Gamma_n\text{: 2.3.2.}),$$

then

$$\Delta_n \supset \Delta_{n+1},$$

hence $\{\Delta_n\}$ is a monotonely decreasing sequence of events. It is, however, not allowed to say that $\bigcap_{n=1}^{\infty} \Delta_n$ is an event, for we have no proof of the disjunction

$$(E_1 \in \bigcap_{n=1}^{\infty} \Delta_n) \vee (E_1 \notin \bigcap_{n=1}^{\infty} \Delta_n).$$

From this example it becomes clear that it is not allowed to state that the intersection of a monotonely decreasing sequence of

[4]) For the definition of this notion of convergence cf. J. G. Dijkman, Recherche de la convergence négative dans les mathématiques intuitionistes. Proc. Akad. Amsterdam **51**, p. 681–692 = Indagationes math. **10**, p. 232–243.

events is an event. From the classical point of view the statement is true:

2.3.4. In the same way as in section 2.3.1. we can prove:

Let $\{\Gamma_n\}$ be a sequence of disjoint events with the property that there exists a natural number k such that $\bigcup_{n=k}^{\infty}\Gamma_n$ is an event, then

(i): $\bigcup_{n=1}^{\infty}\Gamma_n$ is an event

(ii): $P(\bigcup_{n=1}^{\infty}\Gamma_n) = \sum_{n=1}^{\infty} P(\Gamma_n)$.

2.3.5. Let Γ be an event which can be decomposed into the sequence $\{\Gamma_n\}$ of disjoint events Γ_n $(n = 1, 2, \ldots)$, then

$$P(\Gamma) = \sum_{n=1}^{\infty} P(\Gamma_n).$$

The proof is simple and will be omitted, but note that

$$(E_j \in \Gamma) \Rightarrow (\exists n(E_j \in \Gamma_n)$$

follows from $\Gamma = \bigcup_{n=1}^{\infty}\Gamma_n$ and not from the fact that $\Gamma_n (n = 1, 2, \ldots)$ is an event, for in the definition of an event we have only required

$$p_i \# 0 \Rightarrow (\forall n)[(E_i \in \Gamma_n) \vee (E_i \notin \Gamma_n)].$$

and it may happen that $E_i \in \Gamma$ with $p_i \neq 0$ without having a proof of $p_i \# 0$.

2.4.1. Let Γ be an event, then $P(\Gamma \cup \bar{\Gamma}) = 1$.

Remark. Note that it may happen that for some $E_j \in \Omega$ we have no proof of $E_j \in \Gamma \cup \bar{\Gamma}$. For this reason 2.3.5. cannot be applied here.

Theorem 2.4.1. is a special case of:

2.4.2. Let Γ be an event such that $\Omega \backslash \Gamma = \phi$, then we have:

$$P(\Gamma) = 1.$$

Proof. Let k be a natural number, then a natural number N can be calculated such that

$$1 - \sum_{i=1}^{N} p_i < 2^{-k-1}. \tag{1}$$

For every $j = 1, 2, \ldots, N$ with $p_j \# 0$, we have $E_j \in \Gamma$ and for the remaining elements E_j, for which we have no proof of $p_j \# 0$ it is allowed to state:

$$(2) \qquad p_j < N^{-1} \cdot 2^{-k-2}.$$

From (1) and (2) it follows:

$$1 - \sum_{E_i \in \Gamma} p_i < 2^{-k}.$$

The natural number k was chosen arbitrarily, hence $P(\Gamma) = 1$.

2.5. Let Γ_1 and Γ_2 be events with $\Gamma_1 \Rightarrow \Gamma_2$, then

$$P(\Gamma_1) \not> P(\Gamma_2).$$

The proof of this statement is simple and can be given as well directly as indirectly. However, we cannot use the usual classical proof, which runs as follows:

$$\Gamma_2 = \Gamma_1 \cup (\Gamma_2 - \Gamma_1) \Rightarrow P(\Gamma_2) = P(\Gamma_1) + P(\Gamma_2 - \Gamma_1),$$

for it is not allowed to apply the classical rule: $\Gamma_2 = \Gamma_1 \cup (\Gamma_2 - \Gamma_1)$.

2.6. If $\{\Gamma_n\}$ is a descending sequence of events and $\Gamma = \bigcap_{n=1}^{\infty} \Gamma_n$ is an event, then we have

$$P(\Gamma) = \lim_{n \to \infty} P(\Gamma_n).$$

The proof, which runs along the same lines as in section 2.3.1. is omitted, but note that we have to require the condition:

$$\text{“}\Gamma = \bigcap_{n=1}^{\infty} \Gamma_n \text{ is an event”}.$$

From the classical point of view this condition is always satisfied.

2.7. Remarks.

1. In the modern set up of classical probability theory (cf. Loève l.c. or Doob[5]) the triple $(\Omega, \mathfrak{A}, P(\cdot))$ is introduced, where Ω is the state-space, $\mathfrak{A}$ a σ-field and $P(\cdot)$ a normed, nonnegative and σ-additive measure function on $\mathfrak{A}$. In our treatment we have defined the function $P(\cdot)$ for every state $E_i \in \Omega$ with $P(E_i) = p_i$ and the definition of an event Γ was chosen in such a way that $P(\Gamma)$ can be calculated by additivity.

In section 2.3.2. we saw that it is not allowed to say that the species of all events is a σ-field. It is true, that the species of all events is a field. The same remark is true for strong events (cf. 5.1.1.).

2. The properties 2.1.1., 2.1.2., 2.1.3. (excepted $P(\Omega) = 1$), 2.2.2. and 2.2.3. are independent of (α_2) and depend only on (α_1) and the definition of an event.

[5]) Doob, J. L., Stochastic Processes. John Wiley & Sons. New York (1953).

3. If condition (α_2) is replaced by the weaker condition

$$(\alpha_2'): (\forall k) \neg\neg (\exists N)(\forall n)(E_{N+n} \in \Omega \Rightarrow 0 \not> 1 - \sum_{i=1}^{N+n} p_i < 2^{-k})$$

we meet difficulties in defining the probability of an event Γ. Assuming condition (α_2') the series

$$\sum_{E_i \in \Gamma} p_i$$

will not be positively convergent for every event Γ and hence it does not define a real number in general.

In the latter case a sequence $\{P_n(\Gamma)\}$ can be introduced by

$$P_n(\Gamma) = \sum_{\substack{E_i \in \Gamma \\ i=1}}^{n} p_i$$

for every event Γ.

The sequence $\{P_n(\Gamma)\}$ is bounded and monotone, hence it is non-oscillating (cf. Heyting l.c. p. 110, theorem 4).

In the case the theory is based on (α_2') it is preferable to define an event by the property given in 1.3.3. This set up of the theory will not be developed in this paper.

3. Stochastic variables

3.1.1. Definitions.

A real-valued function $x(\cdot)$ defined for every element $E_j \in \Omega$, is called a stochastic variable if for every i and j with $E_i \in \Omega$ and $E_j \in \Omega$ we have:

$$p_i + p_j \# 0 \Rightarrow [x(E_i) = x(E_j)] \vee [x(E_i) \# x(E_j)].$$

The range of $x(\cdot)$ will be denoted by $\{x_1, x_2, \ldots\}$ with $x(E_i) = x_i$ $(i = 1, 2, \ldots)$.

By $\{E_n : x(E_n) < \lambda\}$ or shortly by $\{x < \lambda\}$ is understood the species of elements $E_n \in \Omega$ which satisfy the inequality $x(E_n) < \lambda$.

3.1.2. Remarks.

1. It is required that the subspecies of the range of a stochastic variable $x(\cdot)$ corresponding to those elements $E_n \in \Omega$, for which we have a proof of $p_i \# 0$, is an ordered species [6]). Note that the definition of a stochastic variable does not imply that the range of it is an ordered species.

[6]) cf. Heyting: l.c. p. 106.

2. If $x(\cdot)$ is a stochastic variable then from the classical point of view $x^n(\cdot)$ is a stochastic variable for every natural number n. From the intuitionistic point of view we cannot prove this property. This becomes clear from the following counter-example.

Let Ω consist of only two elements E_1 and E_2 with $p_1 = p_2 = \frac{1}{2}$ and let $x(\cdot)$ by defined by

$$x(E_1) = 1 \text{ and } x(E_2) = -1-\rho,$$

where ρ is a real number for which we have not (yet) a proof of

$$(\rho > 0) \vee (\rho < 0).$$

Evidently $x(\cdot)$ is a stochastic variable but in this case it is not allowed to state:

$$[x^2(E_1) = x^2(E_2)] \vee [x^2(E_1) \# x^2(E_2)].$$

In this case it is not allowed to say that $\{E_n : x^2 < 1\}$ is an event for we have no proof of

$$[E_2 \in \{E_n : x^2 < 1\}] \vee [E_2 \notin \{E_n : x^2 < 1\}].$$

At the same time the counter-example shows that if x is a stochastic variable, then $|x|$ need not to be a stochastic variable, but it is trivially true that if $|x|$ is a stochastic variable then x is a stochastic variable. It is easily seen that if $|x|$ is a stochastic variable then x^n is a stochastic variable for every natural number n.

3.2.1. If $x(\cdot)$ is a stochastic variable then the species

$$\{E_n : x(E_n) < x_i\}$$

is an event for every fixed i with $E_i \in \Omega$.

Proof. We have to prove:

$$p_j \# 0 \Rightarrow [E_j \in \{E_n : x < x_i\}] \vee [E_j \notin \{E_n : x < x_i\}].$$

Let us choose a natural number j such that $p_j \# 0$, then $(i = j) \vee (i \neq j)$ is true for every fixed i.

If $j = i$, then from $x_i = x_j$ it follows that $E_j \notin \{E_n : x < x_i\}$.

If $j \neq i$, then the definition of a stochastic variable implies

(1) $$[x(E_i) = x(E_j)] \vee [x(E_i) \# x(E_i)].$$

Combining (1) with:

$$[x(E_i) \# x(E_j)] \Rightarrow (x_i < x_j) \vee (x_i > x_j)$$

then we obtain

$$[E_j \in \{E_n : x < x_i\}] \vee [E_j \notin \{E_n : x_n < x_i\}].$$

3.2.2. In the same way it is proven:

If $x(\cdot)$ is a stochastic variable then the species

$$\{E_n : x \nsucc x_i\}$$

is an event for every fixed i.

A simple proof can be given by applying 2.2.2.

3.3.1. Definition.

The function $\mathbf{x}(\cdot)$ is called a weak-stochastic variable if

(i) $\mathbf{x}(E_n)$ is defined for every element $E_n \in \Omega$ for which we have $p_n \# 0$

(ii): $p_i + p_j \# 0 \Rightarrow [\mathbf{x}(E_i) = \mathbf{x}(E_j)] \vee [\mathbf{x}(E_i) \# \mathbf{x}(E_j)]$ under the condition that $\mathbf{x}(E_i)$ and $\mathbf{x}(E_j)$ are defined.

3.3.2. Remarks.

1. Let $\Lambda \subset \Omega$ be the species of elements $E_j \in \Omega$ with $p_j \# 0$. Then the weak-stochastic variable $\mathbf{x}(\cdot)$ is defined for evey element $E_j \in \Lambda$, but it is possible that $\mathbf{x}(\cdot)$ is defined on a species Λ' with $\Lambda' \supset \Lambda$. This depends on the defining law of $\mathbf{x}(\cdot)$.

2. It should be noted that the definition of $\mathbf{x}(\cdot)$ does not require that $p_i + p_j \# 0$ implies that $\mathbf{x}(E_i)$ and $\mathbf{x}(E_j)$ are defined. In general it is only allowed to say that in this case at least one of these symbols is defined.

3. The range of $\mathbf{x}(\cdot)$ will be denoted by $\mathscr{R}(\mathbf{x})$. From α_2 (cf. 1.1.) it follows that $\mathscr{R}(\mathbf{x})$ contains at least one element.

4. The following statement is evident.

A stochastic variable $x(\cdot)$ is a weak-stochastic variable.

3.3.3. Let Λ be the domain of the weak-stochasic variable $\mathbf{x}(\cdot)$, then

(i): Λ is an event,

(ii): $P(\Lambda) = 1$.

Proof.

(i): $p_j \# 0 \Rightarrow \mathbf{x}(E_j)$ is defined $\Rightarrow E_j \in \Lambda$.

(ii): Let k_1 be an arbitrarily chosen natural number, then according to α_2 (cf. 1.1.) a natural number N_1 can be calculated such that

$$\sum_{i=1}^{N_1} p_i > 1 - 2^{-k_1-1}.$$

From the spread consisting of the numbers $p_1, p_2, \ldots, p_{N_1}$ we select the elements $p_{\nu_1}, p_{\nu_2}, \ldots, p_{\nu_r}$ such that

$$p_{\nu_i} \# 0 \ (i = 1, 2, \ldots, r) \text{ and } \sum_{i=1}^{r} p_{\nu_i} > 1 - 2^{-k_1}.$$

The latter result implies

$$1 \nless P(\Lambda) > 1-2^{-k_1}.$$

The natural number k_1 was chosen arbitrarily, hence $P(\Lambda) = 1$.

3.3.4. Let $\mathbf{x}(\cdot)$ be a weak-stochastic variable.

If $x \in \mathscr{R}(\mathbf{x})$, then the species $\{E_n : \mathbf{x}(E_n) < x\}$ is an event.

Proof. $x \in \mathscr{R}(\mathbf{x}) \Rightarrow (\exists i)(\mathbf{x}(E_i) = x)$. Starting from this remark the proof can be rewritten from 3.2.1.

3.3.5. The definition of a weak-stochastic variable as given in section 3.3.1. enables us to interprete the characteristic function $x_\Gamma(\cdot)$ of an event Γ as a weak-stochastic variable. This is the content of next theorem.

Theorem. For every event Γ the function $\mathbf{x}_\Gamma(\cdot)$, defined by

$$E_n \in \Gamma \Rightarrow x_\Gamma(E_n) = 1$$
$$E_n \notin \Gamma \Rightarrow x_\Gamma(E_n) = 0,$$

is a weak-stochastic variable.

Proof. Let $p_j \# 0$, then we have

$$(E_j \in \Gamma) \vee (E_j \notin \Gamma),$$

for Γ is an event, hence $\mathbf{x}(E_j)$ is well defined.

If $\mathbf{x}(E_i)$ and $\mathbf{x}(E_j)$ are defined, then evidently

$$[\mathbf{x}(E_i) = \mathbf{x}(E_j)] \vee [\mathbf{x}(E_i) \# \mathbf{x}(E_j)].$$

3.3.6. From 3.2.1. and 3.3.4. it follows that

$$P(x < x_i) \text{ and } P(\mathbf{z} < x)$$

are defined for every i and every $x \in \mathscr{R}(\mathbf{z})$, where x and $\mathbf{z}$ are stochastic resp. weak-stochastic variables. It is clear that this can be extended to the following statement:

$$P(x < \lambda) \text{ and } P(\mathbf{z} < \mu)$$

are defined for every real number λ resp. μ with the property

(1) $$p_i \# 0 \Rightarrow (\lambda = x_i) \vee (\lambda \# x_i)$$

resp.

(1′) $$p_i \# 0 \Rightarrow [\mu = \mathbf{z}(E_i)] \vee [\mu \# \mathbf{z}(E_i)].$$

Let us define the functions $F_x(\lambda)$ and $G_\mathbf{z}(\mu)$ by

$$F_{\mathbf{x}}(\lambda) = P(\mathbf{x} < \lambda)$$

resp.

$$G_{\mathbf{z}}(\mu) = P(\mathbf{z} < \mu)$$

for every number λ resp. μ which satisfies (1) resp. (1').

The following counter-example illustrates that this definition of $F_{\mathbf{x}}(\lambda)$ resp. $G_{\mathbf{z}}(\mu)$ does not allow us to pretend that $F_{\mathbf{x}}(\lambda)$ and $G_{\mathbf{z}}(\mu)$ can be defined for every real number λ resp. μ.

Counter-example.

Let $p_j \neq 0$ (such a value of j can be calculated) and let λ be a real number for which we have no proof of

$$(x_j < \lambda) \vee (x_j \not< \lambda).$$

In this case $\{E_n : \mathbf{x} < \lambda\}$ is not an event and $P(\mathbf{x} < \lambda)$ is not defined.

3.3.7. Let $D_{F_{\mathbf{x}}}$ be the domain of $F_{\mathbf{x}}(\lambda)$, where $F_{\mathbf{x}}(\lambda)$ is defined as in 3.3.6,. then $D_{F_{\mathbf{x}}}$ contains at least one real number.

Proof. On account of (α_2) we have

$$(1) \qquad (\exists i)(p_i \neq 0).$$

Let i be an integer satisfying (1), then

$$\{E_n : \mathbf{x}(E_n) < x_i\} \text{ is an event,}$$

hence $F_{\mathbf{x}}(x_i) = P(\mathbf{x} < x_i)$ is well defined, which implies

$$\lambda = x_i \in D_{F_{\mathbf{x}}}.$$

3.3.8. Let $F_{\mathbf{x}}(\cdot)$ and $D_{F_{\mathbf{x}}}$ be defined as in 3.3.6. and 3.3.7. If

$$(\forall n)(\exists \lambda \in D_{F_{\mathbf{x}}})(\lambda > n),$$

then

$$\lim_{\substack{\lambda \to \infty \\ \lambda \in D_{F_{\mathbf{x}}}}} F(\lambda) = 1.$$

Proof. We choose an arbitrary natural number k_1 and calculate a natural number N_1 according to α_2 (cf. 1.1.) such that

$$\sum_{i=1}^{N_1} p_i > 1 - 2^{-k_1-2}.$$

From $p_1, p_2, \ldots, p_{N_1}$ the elements $p_{\nu_1}, p_{\nu_2}, \ldots, p_{\nu_r}$ are selected such that

$$p_{\nu_i} \neq 0 \ (i = 1, 2, \ldots, r) \text{ and } \sum_{i=1}^{r} p_{\nu_i} > 1 - 2^{-k_1-1}.$$

Let λ_0 be defined by: $\lambda_0 = \max[x_{\nu_1}, x_{\nu_2}, \ldots, x_{\nu_r}]$, then

$$F(\lambda) > 1-2^{-k_1},$$

for every $\lambda \in D_{F_x}$ with $\lambda > \lambda_0$, from which the assertion follows.

3.3.9. Let $F^*(\lambda)$ be defined by $F^*(\lambda) = P(x \not> \lambda)$, where $x(\cdot)$ is a stochastic variable and λ satisfies (1) (3.3.6.).

Then we have:

$$(\forall k)(\exists \lambda)(1-2^k < F^*(\lambda) \not> 1).$$

The proof runs in the same way as in section 3.3.8.

3.3.10. In the same way we obtain:

If $F(\lambda)$ resp. $F^*(\lambda)$ is defined for $\lambda = \lambda_1$ and $\lambda = \lambda_2$ then from $\lambda_1 \not> \lambda_2$ it follows:

$$F(\lambda_1) \not> F(\lambda_2) \text{ resp. } F^*(\lambda_1) \not> F^*(\lambda_2).$$

3.3.11. From the classical point of view the sum of two measurable functions is a measurable function or in probability language translated: the sum of two stochastic variables is a stochastic variable. This theorem, however, has not been proved in intuitionistic mathematics (cf. A. Heyting, l.c. page 85).

In our set up we have evidently:

Let x and y be stochastic variables with range $\{x_i\}$ resp. $\{y_i\}$. Then $x+y$ is a stochastic variable if

$$p_i \# 0 \Rightarrow (\forall j)[(x_i+y_i = x_j+y_j) \vee (x_i+y_i \# x_j+y_j)]$$

and $x-y$ is a stochastic variable if

$$p_i \# 0 \Rightarrow (\forall j)[(x_i-y_i = x_j-y_j) \vee (x_i-y_i \# x_i-y_i)].$$

3.3.12. The statements in section 3.3.11. remain true if "stochastic" is replaced by "weak-stochastic".

4. Expectation

4.1. Definitions.

4.1.1. Let $x(\cdot)$ be a function defined on some subspecies $\Lambda \subset \Omega$. As before x_i means $x(E_i)$ if $E_i \in \Lambda$.

If the series

$$\sum_{E_i \in \Lambda} |x_i| p_i$$

is positively convergent then the real number defined by

$$\sum_{E_i \in \Lambda} x_i p_i$$

is called the expectation of $x(\cdot)$ with respect to Λ.

The expectation of $x(\cdot)$ with respect to Λ (if it exists) will be denoted by $E_\Lambda\{x(\cdot)\}$ or shortly by $E_\Lambda\{x\}$ and we write $E\{x\}$ if $\Lambda = \Omega$.

REMARK. On account of the absolute convergence of the series which defines the expectation, the definition is independent of the order which has been chosen in numbering the state-space Ω.

4.1.2. By $\Lambda \approx \Omega$ we express that Λ is a subspecies of Ω such that

$$p_j \# 0 \Rightarrow E_j \in \Lambda.$$

THEOREM. Let $x(\cdot)$ be defined on Ω and let Λ be a species such that
(i) $\Lambda \approx \Omega$
(ii) $E_\Lambda\{x\}$ exists,
then: $E_\Lambda\{x\} = E\{x\}$.

PROOF. We define the sequence $\{s_n\}$ by

$$s_n = \sum_{\substack{E_i \notin \Lambda \\ i=1}}^{n} x_i p_i.$$

On account of:

$$\begin{aligned}(\exists n)(s_n \# 0) &\Rightarrow (\exists j)[(x_j p_j \# 0) \wedge (E_j \notin \Lambda)] \\ &\Rightarrow (\exists j)[(p_j \# 0) \wedge (E_j \notin \Lambda)],\end{aligned}$$

contradicting $\Lambda \approx \Omega$, it follows:

$$(\forall n)(s_n = 0). \tag{1}$$

Now let us suppose:

$$E_\Lambda\{x\} \# E\{x\},$$

then we have:

$$\begin{aligned}E_\Lambda\{x\} \# E\{x\} &\Rightarrow \sum_{\substack{E_i \in \Lambda \\ i=1}}^{\infty} x_i p_i - \sum_{\substack{E_i \in \Omega \\ i=1}}^{\infty} x_i p_i \# 0 \\ &\Rightarrow (\exists n)\Big(\sum_{\substack{E_i \in \Lambda \\ i=1}}^{n} x_i p_i - \sum_{\substack{E_i \in \Omega \\ i=1}}^{n} x_i p_i \# 0\Big) \Rightarrow (\exists n)(s_n \# 0)\end{aligned} \tag{2}$$

(1) and (2) constitute a contradiction, hence (ii) follows.

4.1.3. On account of 4.1.2. it is allowed to write $E(x\}$ in stead of $E_\Lambda\{x\}$ if $\Lambda \approx \Omega$ and if $x(\cdot)$ is defined on Ω.

4.2.3. The function $x(\cdot)$ defined on $\Lambda \subset \Omega$ is called bounded if

$$(\exists M_\Lambda)(\forall E_i \in \Lambda)(|x_i| < M_\Lambda).$$

4.2.1. A bounded stochastic variable has a finite expectation.

Proof. Evident, but remember that by definition (3.1.1.) a stochastic variable is defined for every element of Ω.

4.2.2. A bounded weak-stochastic variable has a finite expectation with respect to its domain.

Proof. Let $x(.)$ be a bounded weak-stochastic variable with domain Γ, then we know:

$$(1) \qquad (\exists M)(E_i \in \Gamma \Rightarrow |\mathbf{x}(E_i)| < M).$$

Let k be an arbitrarily chosen natural number, then from (1) and α_2 (cf. 1.1.) it follows that a natural number N_1 can be calculated such that

$$(\forall n > N_1)(\forall m)(\sum_{\substack{i=n\\E_i \in \Gamma}}^{n+m} |x_i|p_i < M \cdot 2^{-k+1}),$$

and this relation proves the positive convergence of the series $\sum_{E_i \in \Gamma}|x_i|p_i$ on account of Cauchy's general convergence principle, hence $E_\Gamma\{x\}$ exists.

4.2.3. Let $\mathbf{x}_\Gamma(\cdot)$ be the characteristic function of the event Γ (cf. 3.3.5.), then

$$E\{\mathbf{x}_\Gamma\} = P(\Gamma).$$

Proof. Let Λ be the domain of $\mathbf{x}_\Gamma(\cdot)$, then $\Lambda \approx \Omega$ (cf. 3.3.3). On account of 4.2.2. we know that $E\{\mathbf{x}_\Gamma\}$ exists and from the definitions (1.3.1.) and (4.1.1.) it follows:

$$E_\Lambda\{\mathbf{x}_\Gamma\} = \sum_{E_i \in \Lambda} x_i p_i = \sum_{E_i \in \Gamma} p_i = P(\Gamma).$$

4.2.4. Though the sum of two stochastic variables need not be a stochastic variable we can speak of the expectation of the sum of two stochastic variables. The absolute convergence of the series occurring in the definition of expectation implies

$$E\{x+y\} = E\{x\}+E\{y\},$$

under the restriction that both expectations in the righthand member exist.

It is trivially true that the same remarks apply to the sum of two weak-stochastic variables.

4.3. Many theorems can be translated from the classical theory into the intuitionistic theory with only slight modifications. We give some of them.

4.3.1. Tchebichev-inequality.

Let $|x|$ be a stochastic variable and let $\lambda \neq 0$ be a real number such that

$$\{E_n : |x(E_n)| > \lambda\}$$

is an event, then

$$P\{|x| > \lambda\} \not> \lambda^{-2} \cdot E\{x^2\},$$

under condition that $E\{x^2\}$ exists.

Proof. Let us put $\Lambda_1 = \{E_n : |x(E_n)| > \lambda\}$, then Λ_1 is an event and the same is true for $\Lambda_2 = \bar{\Lambda}_1$ (cf. 2.2.2.). This implies:

$$E\{x^2\} = E_{\Lambda_1}\{x^2\} + E_{\Lambda_2}\{x^2\} \not< \lambda^2 P\{|x| > \lambda\},$$

hence

$$P\{|x| > \lambda\} \not> \frac{E\{x^2\}}{\lambda^2}.$$

Remarks.

1. The modification is:
   We have to require that $|x|$ is a stochastic variable and it is not sufficient to know that x is a stochastic variable.
2. $E\{x^2\}$ occurs in the righthand member of the inequality. It is allowed to speak of the stochastic variable x^2 (cf. 3.1.2.).
3. Obviously the inequality can be formulated for weak-stochastic variables.

4.3.2. Definitions.

The events $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$ $(n \geqq 2)$ are independent if the following equality

$$P(\bigcap_{i=1}^{k} \Gamma_{\nu_i}) = \prod_{i=1}^{k} P(\Gamma_{\nu_i})$$

holds for every choice of the natural numbers $1 \leqq \nu_1 < \nu_2 < \ldots < \nu_k \leqq n$ with $k = 2, 3, \ldots, n$.

The sequence $\{\Gamma_n\}$ is a sequence of independent events if $\Gamma_1, \ldots, \Gamma_k$ are independent for every k.

The stochastic variables x and y are independent if

$$P\{(x < x_i) \cap (y < y_j)\} = P(x < x_i)P(y < y_j)$$

for every x_i and y_j belonging to the range of x resp. y.

4.3.3. Let x and y be independent stochastic variables then

$$E\{x \cdot y\} = E\{x\} \cdot E\{y\}.$$

The proof is simple and will be omitted, but note that $x \cdot y$ need not to be a stochastic variable in general.

If x and y are stochastic variables then $x \cdot y$ is a stochastic variable if

$$p_i \# 0 \Rightarrow (\forall j)[(x_i y_i = x_j y_j) \vee (x_i y_i \# x_j y_j)].$$

4.3.4. Let $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$ be independent events and let us replace some or all Γ_ν by $\bar{\Gamma}_\nu$, then the new sequence of events consists of independent events, as can easily be proven by calculation.

4.4. Bernoulli's law of large numbers.

Let us suppose that $\{\Gamma_k\}$ is a sequence of independent events such that $P(\Gamma_k) = p$ $(k = 1, 2, \ldots)$ and let Λ be the species consisting of the elements $E_j \in \Omega$ with $p_j \# 0$.

According to the definition of an event we have

$$(E_j \in \Lambda) \Rightarrow (\forall k)[(E_j \in \Gamma_k) \vee (E_j \notin \Gamma_k)],$$

which enables us to define the sequence $\{\mathbf{x}_k(\cdot)\}$ of functions by

$$(E_j \in \Lambda) \wedge (E_j \in \Gamma_k) \Rightarrow \mathbf{x}_k(E_j) = 1$$
$$(E_j \in \Lambda) \wedge (E_j \notin \Gamma_k) \Rightarrow \mathbf{x}_k(E_j) = 0.$$

From (3.3.5.) it is clear that $\mathbf{x}_k$ is a weak-stochastic variable for every k and (4.2.2.) implies that $E_\Lambda\{\mathbf{x}_k(\cdot)\}$ exists, satisfying

$$E_\Lambda\{\mathbf{x}_k(\cdot)\} = E\{\mathbf{x}_k(\cdot)\} = P(\Gamma_k) \qquad \text{(cf. 4.2.3.).}$$

Now we define the function $\mathbf{S}_n(\cdot)$ by

$$p_j \# 0 \Rightarrow S_n(E_j) = \sum_{i=1}^{n} x_i(E_j)$$

for every natural number n. Evidently, the functions $\mathbf{S}_n(\cdot)$ are weak-stochastic variables with Λ as domain.

By $\{\varepsilon_n\}$ we indicate a sequence of real numbers with the properties:

$$^{+}\lim_{n \to \infty} \varepsilon_n = 0 \text{ and } p \pm \varepsilon_n \# \frac{m}{n} \qquad (m = 0, 1, \ldots, n)$$

for every natural number n and we introduce the species $\lambda_{n,k}$ $(n, k = 1, 2, \ldots)$ consisting of those elements $E_j \in \Omega$ for which $\mathbf{S}_n(E_j)$ is defined and which satisfy:

$$\left| \frac{\mathbf{S}_n(E_j)}{n} - p \right| > \varepsilon_k.$$

It is easily seen that $\lambda_{n,k}$ is an event for every pair of natural numbers n and k, which implies that

$$P\left(E_j : \left|\frac{\mathbf{S}_n(E_j)}{n} - p\right| > \varepsilon_k\right)$$

is well defined. Furthermore it is true that the species, consisting of those elements E_i for which we have

$$(E_i \in \Lambda) \wedge (\mathbf{S}_n(E_i) = j),$$

is an event. The independence of the sequence $\{\Gamma_k\}$ then implies:

$$P(\mathbf{S}_n(\cdot) = j) = \binom{n}{j} p^j (1-p)^{n-j},$$

which leads to:

$$E_\Lambda\{\mathbf{S}_n(\cdot)\} = \sum_{j=1}^{n} jP(\mathbf{S}_n(\cdot) = j) = np.$$

Now it follows:

$$\begin{aligned} E_\Lambda\{[\mathbf{S}_n(\cdot) - E_\Lambda\{\mathbf{S}_n(\cdot)\}]^2\} &\not< E_{\lambda_{n,k}}\{[\mathbf{S}_n(\cdot) - E_\Lambda\{\mathbf{S}_n(\cdot)\}]^2\} \\ &= E_{\lambda_{n,k}}\{[\mathbf{S}_n(\cdot) - np]^2\}, \end{aligned}$$

hence

$$P\left\{\left|\frac{\mathbf{S}_n(\cdot)}{n} - p\right| > \varepsilon_k\right\} \not> \frac{E_\Lambda\{[\mathbf{S}_n(\cdot) - np]^2\}}{n^2 \varepsilon_k^2}. \tag{1}$$

Furthermore we have:

$$\begin{aligned} E_\Lambda\{[\mathbf{S}_n - np]^2\} &= \sum_{E_i \in \Lambda} \sum_{j=0}^{n} (j-np)^2 P(\mathbf{S}_n(E_i) - np = j - np) \\ &= \sum_{j=0}^{n} (j-np)^2 \sum_{E_i \in \Lambda} P(\mathbf{S}_n(E_i) = j) \\ &= \sum_{j=0}^{n} (j-np)^2 \binom{n}{j} p^j (1-p)^{n-j} = np(1-p). \end{aligned} \tag{2}$$

Combining (1) and (2) we get

$$P\left(\left|\frac{\mathbf{S}_n(\cdot)}{n} - p\right| > \varepsilon_k\right) \not> \frac{p(1-p)}{n\varepsilon_k^2},$$

which proves

$$(\forall k)\left(\lim_{n \to \infty} P\left(\left|\frac{\mathbf{S}_n(\cdot)}{n} - p\right| > \varepsilon_k\right) = 0\right).$$

4.5.1. Let $\{|\mathbf{x}_n|\}$ be a sequence of weak-stochastic variables which satisfies the following conditions:

(i) there exists a sequence $\{\varepsilon_n\}$ of positive numbers with $\lim_{n\to\infty} \varepsilon_n = 0$ and such that

$$\Delta_{n,m} \stackrel{\text{df}}{=} \{E_k : |\mathbf{x}_n(E_k)| \nless \varepsilon_m\}$$

is an event for every pair of natural numbers n and m;

(ii) ${}^+\lim_{n\to\infty} P(\Delta_{n,m}) = 0$,

then we have

(j) $\Delta \stackrel{\text{df}}{=} \{E_k : \lim_{n\to\infty} \mathbf{x}_n(E_k) = 0\}$ is an event,

(jj) $P(\Delta) = 1$.

PROOF (j). Δ is an event if

$$p_i \# 0 \Rightarrow (E_i \in \Delta) \vee (E_i \notin \Delta).$$

Let us take a $p_i \# 0$ then $\mathbf{x}(E_i)$ is defined (cf. 3.3.1.) and (ii) implies:

$$(\forall m)(\exists N_m)(n > N_m \Rightarrow E_i \notin \Delta_{m,n}) \Rightarrow$$
$$(\forall m)(\exists N_m)(n > N_m \Rightarrow \mathbf{x}_n(E_i) < 2\varepsilon_m) \Rightarrow$$
$${}^+\lim \mathbf{x}_n(E_i) = 0 \Rightarrow E_i \in \Delta \Rightarrow \Delta \text{ is an event.}$$

(jj). From the implication

$$p_i \# 0 \Rightarrow E_i \in \Delta$$

and α_2 (cf. 1.1.) it follows

$$(\forall k)(P(\Delta) > 1-2^{-k}),$$

hence $P(\Delta) = 1$.

4.5.2. Borel's strong law of large numbers.

Let $\{\Gamma_n\}$ be a sequence of independent events with $P(\Gamma_n) = p$ $(n = 1, 2, \ldots)$. If there exists a sequence $\{\varepsilon_n\}$ of positive real numbers with $\lim_{n\to\infty}\varepsilon_n = 0$ such that $p\pm\varepsilon_n \# m/n$ $(m = 0, 1, \ldots, n)$ for every natural number n, then

$$P\left(E_k : \lim_{n\to\infty} \frac{\mathbf{S}_n(E_k)}{n} = p\right) = 1,$$

where $\mathbf{S}_n(\cdot)$ is defined as in section 4.4.

PROOF. The proof can be rewritten (using 4.5.1.) from Loève (l.c. page 19) with only slight modifications.

4.5.3. Kolmogorov-inequality.

Let $|\mathbf{x}|$ be a weak-stochastic variable with $|\mathbf{x}| \ngtr 1$, and let ε be a positive real number such that

$$\Delta \stackrel{\text{df}}{=} \{E_k : \mathbf{x}(E_k) \nless \varepsilon\}$$

is an event, then

$$P\{|\mathbf{x}| \nless \varepsilon\} \nless E\{\mathbf{x}^2\}-\varepsilon^2.$$

Proof. Remark 2 of section 3.1.2. applies to weak-stochastic variables, hence $\mathbf{x}^2$ is a weak-stochastic variable with the same domain Λ' as $|\mathbf{x}|$ and according to 4.2.2. the expectation $E\{\mathbf{x}^2\}$ is well defined. Let Λ'' represent the event

$$\{E_k : |\mathbf{x}(E_k)| < \varepsilon\}$$

and $\{x_i\}$ the range of $|\mathbf{x}|$, then it is easily seen that

$$E\{\mathbf{x}^2\} = \sum_{E_i \in \Lambda'} x_i^2 p_i = \sum_{E_i \in \Delta} x_i^2 p_i + \sum_{E_i \in \Lambda''} x_i^2 p_i$$
$$\not> \sum_{E_i \in \Delta} x_i^2 p_i + \varepsilon^2 \not> \sum_{E_i \in \Delta} p_i + \varepsilon^2 = P(\Delta) + \varepsilon^2.$$

5.0. The purpose of this section is to investigate an other notion of event and to compare it with the results obtained in the foregoing sections.

5.1.1. Definition.

The species $\Lambda \subset \Omega$ is called a strong event if the following condition is satisfied

$$p_j \neq 0 \Rightarrow (E_j \in \Lambda) \vee (E_j \notin \Lambda).$$

5.1.2. The method developed in 1.3.2. can be applied to prove: The series $\sum_{E_i \in \Lambda} p_i$ is positively convergent and defines a real number which we shall indicate by $P(\Lambda)$. The number $P(\Lambda)$ will be called the probability of the strong event Λ.

5.1.3. A strong event is an event.

Proof. This is an immediate consequence of: $p_j \# 0 \Rightarrow p_j \neq 0$.

5.1.4. The converse of 5.1.3. cannot be proven from the intuitionistic point of view because we can construct real numbers p with $p \neq 0$, but for which we are not able to prove $p \# 0$ [7]).

5.1.5. Let Λ be a strong event and let Γ be defined as the subspecies of Λ, consisting of the states $E_i \in \Lambda$ with $p_i \# 0$, then

(i) Γ is an event

(ii) $P(\Gamma) = P(\Lambda)$.

Proof. (i) The implications

$$p_j \# 0 \Rightarrow p_j \neq 0 \Rightarrow (E_j \in \Lambda) \vee (E_j \notin \Lambda),$$

and the definition of Γ prove the statement.

[7]) cf. L. E. J. Brouwer: Essentieel negative eigenschappen. Proc. Akad. A'dam **51** p. 963–965 = Indagationes Math. **10** p. 322–324.

D. van Dantzig: Comments on Brouwer's theorem on essentially negative predicates. Prov. Akad., A'dam **52**, p. 949–957 = Indagationes Math. **11**, p. 347–355.

Heyting (l.c. chapter VIII) discusses both papers.

(ii) From (i) it follows that $P(\Gamma)$ is a well defined real number and from the proof given in (1.3.2.) it is easily seen that $P(\Gamma) = P(\Lambda)$.

Note that this theorem states, that every strong event contains an event which possesses the same probability.

5.1.6. The following properties are easily verified.

1. If Λ_1 and Λ_2 are strong events, then $\Lambda_1 \cup \Lambda_2$ and $\Lambda_1 \cap \Lambda_2$ are strong events (cf. 2.1.1. and 2.1.2.)

2. If Λ is a strong event then $\bar{\Lambda}$ is a strong, where the definition of $\bar{\Lambda}$ is given in the same way as in 2.2.1.

3. If Λ is a strong event, then $(E_j \in \Lambda) \Rightarrow (E_j \in \bar{\bar{\Lambda}})$.

Remark. Note that $(E_j \in \bar{\bar{\Lambda}}) \Rightarrow (E_j \in \Lambda)$ need not to be true, for let E_j be a state such that we have no prove of: $(p_j = 0) \vee (p_j \neq 0)$ an let Λ be the species consisting of those elements $E_n \in \Omega$ for which

$$(p_n = 0) \vee (p_n \neq 0).$$

Applying the theorem:

For every real number a we have: $\neg\neg [(a = 0) \vee (a \neq 0)]$, it follows: $E_j \in \bar{\bar{\Lambda}}$, but we have no proof of $E_j \in \Lambda$.

Evidently the following statement is true:

$$p_j \neq 0 \Rightarrow [(E_j \in \Lambda) \Rightarrow (E_j \in \bar{\bar{\Lambda}})].$$

5.1.7. Comparing the foregoing sections 5.1.... with those concerning events it becomes clear that events and strong events possess analogous properties and at a first glance there is no reason to prefer the concept of event to that of strong event. However, in 2.3.1. an important property of events was proved but a counter-example illustrates that strong events do not possess that property.

Counter-example.

Let E_1 be a state such that $p_1 \neq 0$, but we suppose that we have no proof of $p_1 \# 0$. Furthermore we consider the decimal expansion of π and to τ the same meaning as in 2.3.2. is given.

The strong-event Λ_n is defined by:

$$\Lambda_n = \{E_{n+1}\} \qquad (n = 1, 2, \ldots)$$

if n is not the index of the digit 9 in the first sequence τ (if it exists) in the decimal expansion of π,

$$\Lambda_n = \{E_1\}$$

if n is the index of the digit 9 in the first sequence τ occuring in the decimal expansion of π.

We cannot prove (nowadays) that

$$\Lambda \overset{\text{df}}{=} \bigcup_{n=1}^{\infty} \Lambda_n$$

is a strong event for we have no proof of

$$(E_1 \in \Lambda) \vee (E_1 \notin \Lambda)$$

though $p_1 \neq 0$.

On the other hand $\sum_{n=1}^{\infty} P(\Lambda_n)$ in positively convergent. This counterexample shows that theorem 2.3.1. does not remain true if "event" is replaced by "strong event".

5.1.8. From the intuitionistic point of view the theorems 2.3.1. and 2.3.4. are equivalent and from the classical point of view the requirement are even superfluous, but, surprisingly the equivalence of these theorems cannot be proven if we replace "event" by "strong event" for theorem 2.3.1. is not true for strong events. On the other hand theorem 2.3.4. remains true under this replacement as is expressed by the following theorem:

Let $\{\Lambda_n\}$ be a sequence of disjoint strong events. If there exists a natural number k such that $\bigcup_{n=k}^{\infty} \Lambda_n$ is a strong event, then

(i) $\bigcup_{n=1}^{\infty} \Lambda_n$ is a strong event

(ii) $P(\bigcup_{n=1}^{\infty} \Lambda_n) = \sum_{n=1}^{\infty} P(\Lambda_n)$.

Proof. (i). Let $p_i \neq 0$ and let us define the species Λ and Λ' by

$$\Lambda = \bigcup_{n=1}^{\infty} \Lambda_n \text{ and } \Lambda' = \bigcup_{n=k}^{\infty} \Lambda_n.$$

Evidently

$$(1) \qquad \Lambda = \Lambda_1 \cup \Lambda_2 \cup \ldots \cup \Lambda_{k-1} \cup \Lambda'.$$

From

$$(E_i \in \Lambda_j) \vee (E_i \notin \Lambda_j) \quad j = 1, 2, \ldots, k-1$$

and

$$(E_i \in \Lambda') \vee (E_i \notin \Lambda')$$

it follows in connection with (1):

$$(E_i \in \Lambda) \vee (E_i \notin \Lambda),$$

which proves that Λ is a strong event.

Applying (2.3.1.) and (5.1.5.) the second part (ii) is easily proven.

5.1.9. Up to section 5. a theory has been built up with definition (1.2.2.) of an event as starting point. As becomes clear from sections (1.2.2.), (1.3.1.), (1.3.2.) and (1.3.3.) a completely equivalent theory is obtained if theorem 1.3.3. is chosen as definition of an event.

However, this equivalence is broken up if "event" is replaced by "strong event", for let Δ be a species such that

$$\sum_{E_i \in \Delta} p_i$$

is positively convergent, then this positive convergence does not imply that the species Δ is a strong event.

Acknowledgement

I wish to thank Prof. dr. A. Heyting for his help during the preparation of this paper.

The interesting comments of Prof. dr. ir. J. W. Cohen were of great value for me.

(Oblatum 27-2-63).

Technological University Delft
Department of Mathematics.