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## Bounds for the solutions of $\Delta\omega \cong P(r)f(\omega)$

by

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Let  $\Delta = \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2 + \dots + \partial^2/\partial x_n^2$  be the  $n$ -dimensional Laplace operator and let  $D_r$  and  $S_r$  denote the open sphere  $x_1^2 + x_2^2 + \dots + x_n^2 < r^2$  ( $r > 0$ ) and its boundary  $x_1^2 + x_2^2 + \dots + x_n^2 = r^2$ , respectively. The aim of this paper is to find explicit bounds for the max.  $\omega(Q)$  where  $\omega(Q)$  satisfies the differential equation

$$(1) \quad \Delta\omega = P(r)f(\omega)$$

or, more generally, the differential inequality

$$(2) \quad \Delta\omega \cong P(r)f(\omega)$$

where  $Q \in D_R$  with  $R > r$  and  $P(r)$  is positive, monotonically increasing and twice continuously differentiable. In theorem 1 an upper bound is obtained where  $P(r)$  is either  $C_1 e^{C_2 r^2}$  or  $A r^n$  where  $C_1, C_2$  and  $A$  are arbitrary constants,  $C_1$  and  $A$  being positive and  $C_2$  non-negative. However, in 2-dimensional case, an upper bound is given if  $P(r)$  is  $\alpha r^\beta$  where  $\alpha$  and  $\beta$  are arbitrary positive constants. In theorem 2 we find a lower bound in case  $P(r)$  is  $\alpha r^\beta$  and  $\Delta$  is a 2-dimensional Laplace operator. Also, the behaviour of these solutions at an isolated singularity is investigated.

**THEOREM 1.** *Let  $f(\omega)$  be positive, non-decreasing, differentiable function in  $(-\infty, \infty)$ , for which*

$$\int_{\omega}^{\infty} \frac{dt}{f(t)} \quad (\omega > -\infty)$$

*exists and*

$$(3) \quad f'(\omega) \int_{\omega}^{\infty} \frac{dt}{f(t)} \leq 1.$$

*If*

$$(G) \quad u(r) = \sup_{Q \in S_r} \omega(Q)$$

where  $\omega(Q)$  ranges over all functions of class  $C^2$  in  $D_r$  which satisfy (2), then

$$(4) \quad \frac{C_1 e^{c_2 r^2} (R^2 - r^2)^2}{4nR^2} \leq \int_{u(r)}^\infty \frac{dt}{f(t)}$$

provided  $P(r) = C_1 e^{c_2 r^2}$  ( $C_1 > 0$  and  $C_2 \geq 0$ ) and

$$(5) \quad \frac{Ar^n (R^2 - r^2)^2}{4nR^2} \leq \int_{u(r)}^\infty \frac{dt}{f(t)}$$

in case  $P(r) = Ar^n$  ( $A > 0$ ). Also if  $\Delta$  is a 2-dimensional Laplace operator and  $P(r) = \alpha r^\beta$ , then

$$(6) \quad \frac{\alpha r^\beta (R^2 - r^2)^2}{8R^2} \leq \int_{u(r)}^\infty \frac{dt}{f(t)}$$

$\alpha$  and  $\beta$  being arbitrary positive constants. The inequalities (4), (5) and (6) are sharp.

PROOF: Consider the function  $g = g(r)$  defined by

$$(7) \quad \frac{C(R^2 - r^2)^2}{R^2} P(r) = \int_g^\infty \frac{dt}{f(t)}$$

where  $C$  is a positive constant to be chosen later. Denoting by  $x$  one of the variables  $x_\alpha$  and differentiating with respect to  $x$ , we have

$$(8) \quad -\frac{4CxP(r)(R^2 - r^2)}{R^2} + \frac{2Cx\dot{P}(r)(R^2 - r^2)^2}{R^2} = -\frac{g_x}{f(g)}$$

where the dot denotes differentiation with respect to  $r^2$ . Differentiating again with respect to  $x$

$$\begin{aligned} & -\frac{4CP(r)(R^2 - r^2)}{R^2} + \frac{8Cx^2P(r)}{R^2} - \frac{16Cx^2\dot{P}(r)(R^2 - r^2)}{R^2} \\ & + \frac{2C\dot{P}(r)(R^2 - r^2)^2}{R^2} + \frac{4Cx^2\ddot{P}(r)(R^2 - r^2)^2}{R^2} = -\frac{g_{xx}}{f(g)} + \frac{g_x^2}{f^2(g)} f'(g). \end{aligned}$$

With the help of (7) and (8), we obtain,

$$\begin{aligned} \frac{g_{xx}}{P(r)f(g)} &= -\frac{8Cx^2}{R^2} + \frac{4C(R^2 - r^2)}{R^2} + \frac{16Cx^2\dot{P}(r)(R^2 - r^2)}{R^2P(r)} + \frac{4Cx^2}{R^2} \\ & \cdot \frac{CP(r)(R^2 - r^2)^2 f'(g)}{R^2} \cdot \left[ 2 - \frac{\dot{P}(r)(R^2 - r^2)}{P(r)} \right]^2 \\ & - \frac{2}{P^2(r)} (2x^2\ddot{P}(r) + \dot{P}(r)) \int_g^\infty \frac{dt}{f(t)}. \end{aligned}$$

Summing over all  $x_k$ , we have

$$\begin{aligned} \frac{\Delta g}{P(r)f(g)} = & -\frac{8Cr^2}{R^2} + \frac{4nC(R^2-r^2)}{R^2} + \frac{16Cr^2\dot{P}(r)(R^2-r^2)}{R^2P(r)} + \frac{4Cr^2}{R^2} \\ & \cdot \frac{CP(r)(R^2-r^2)^2f'(g)}{R^2} \cdot \left[2 - \frac{\dot{P}(r)(R^2-r^2)}{P(r)}\right]^2 \\ & - \frac{2}{P^2(r)} (2r^2\ddot{P}(r) + \dot{P}(r)n) \int_a^\infty \frac{dt}{f(t)}. \end{aligned}$$

Using (3) it reduces to

$$(9) \quad \frac{\Delta g}{P(r)f(g)} \leq 4C \left[ n - \frac{r^2}{R^2} (n-2) \right] - \frac{2C(R^2-r^2)^2}{R^2} \left\{ \frac{2r^2\ddot{P}(r) + n\dot{P}(r)}{P(r)} - \frac{2r^2\dot{P}^2(r)}{P^2(r)} \right\}.$$

Now we consider three cases:

*Case I:* Choose  $P(r)$  such that  $\ddot{P}(r) - \dot{P}^2(r)/P(r) = 0$  or  $P = C_1 e^{c_2 r^2}$  where  $C_1 > 0$  and  $C_2 \geq 0$  are arbitrary constants. Then (9) reduces to

$$\frac{\Delta g}{C_1 e^{c_2 r^2} f(g)} \leq 4C \left\{ n - \frac{r^2}{R^2} (n-2) \right\}.$$

If,  $n \geq 2$  and  $C = 1/4n$ , it follows that

$$\Delta g \leq C_1 e^{c_2 r^2} f(g).$$

Since  $g(0) = 0$  and  $g(r)$  increases to  $\infty$  as  $r \rightarrow R$  the proof of (4) will follow from the following lemma:

**LEMMA:** Let  $f(t)$  be monotonically increasing continuous function defined for all  $t$ . Suppose the functions  $g$  and  $\omega$  are subject to the inequalities

$$\Delta g \leq P(r)f(g)$$

and

$$\Delta \omega \geq P(r)f(\omega)$$

respectively, for  $0 \leq r_0 < r < R$ . If  $g \rightarrow \infty$  for  $r \rightarrow R$ , then

$$\omega \leq g$$

for  $r_0 < r < R$ .

A proof of this lemma (for  $r_0 = 0$ ) can be found in [3]. The changes required to provide it for  $r_0 > 0$  are obvious.

Case II. Assume  $P(r)$  to satisfy

$$\frac{\dot{P}(r)}{P(r)} \left( \frac{2r^2 \dot{P}(r)}{P(r)} - n \right) = 0.$$

(i) If  $\dot{P}(r)/P(r) = 0$  then  $P = k$  where  $k$  is an arbitrary positive constant. This case is implied by Case I if we choose  $C_1 = k$  and  $C_2 = 0$ .

(ii) If  $2r^2 \dot{P}(r)/P(r) - n = 0$  or,  $P = Ar^n$  ( $A$  being an arbitrary positive constant) the inequality (9) becomes

$$\frac{\Delta g}{Ar^n f(g)} \leq 4C \left\{ n - \frac{r^2}{R^2} (n-2) \right\}.$$

Again, if  $n \geq 2$  and  $C = 1/4n$ , we have

$$\Delta g \leq Ar^n f(g).$$

Now the proof of (5) will follow from the above lemma.

Case III: Choose  $P$  such that  $2r^2 P(r) \ddot{P}(r) + nP(r) \dot{P}(r) - 2r^2 \dot{P}^2(r) = 0$  or,  $P = \alpha r^\beta$  where  $\alpha$  and  $\beta$  are arbitrary positive constants and  $n = 2$ . Then (9) gives

$$\frac{\Delta g}{\alpha r^\beta f(g)} \leq 8C.$$

If  $C = \frac{1}{8}$ , it follows that

$$\Delta g \leq \alpha r^\beta f(g).$$

Again, with the help of the Lemma, we get (6).

This completes the proof of theorem 1. Now, we derive the following corollaries.

COROLLARY 1. In case of a function  $\omega$  satisfying

$$\Delta\omega = C_1 e^{\omega + c_2 r^2}$$

which is regular in  $D_R$

$$\omega \leq 2 \log \frac{2\sqrt{n}R}{\sqrt{C_1}(R^2 - r^2)} - C_2 r^2 \quad (n \geq 2)$$

where  $C_1 > 0$  and  $C_2 \geq 0$  are arbitrary constants.

Indeed, setting  $f(t) = e^t$  in (4), we get, where  $\omega = u$

$$\frac{C_1 e^{c_2 r^2} (R^2 - r^2)^2}{4nR^2} \leq e^{-\omega}.$$

Taking logarithm on both sides

$$\omega \leq 2 \log \frac{2\sqrt{n}R}{\sqrt{C_1}(R^2-r^2)} - C_2 r^2.$$

REMARK: Nehari [2] proved that in case of a solution  $u$  of

$$\Delta u = e^u$$

which is regular in  $D_R$

$$\phi(r) \leq 2 \log \frac{2\sqrt{n}R}{(R^2-r^2)}$$

where  $\phi(r) = \sup_{Q \in S_r} u(Q)$ .

If we take  $C_1 = 1$  and  $C_2 = 0$  the above result becomes a particular case of this corollary.

COROLLARY 2: If  $\omega$  satisfies the equation

$$\Delta \omega = Ar^n e^\omega$$

where  $A$  is an arbitrary positive constant, it is subject to the inequality

$$(10) \quad \omega \leq 2 \log \frac{2\sqrt{n}R}{\sqrt{A}r^{n/2}(R^2-r^2)}.$$

At an isolated singularity of  $\omega$ , the behaviour of  $\omega$  is such that

$$\overline{\lim}_{r \rightarrow 0} \left( \frac{\omega}{\log \frac{1}{r}} \right) \leq n$$

set  $f(t) = e^t$  in (5). With  $u = \omega$ , we obtain

$$e^{-\omega} \geq \frac{Ar^n(R^2-r^2)^2}{4nR^2}.$$

Taking logarithm

$$\omega \leq 2 \log \frac{2\sqrt{n}R}{\sqrt{A}(R^2-r^2)r^{n/2}}.$$

Now (10) could be written

$$\omega \leq \log \frac{4nR^2}{A(R^2-r^2)^2} + n \log \frac{1}{r}.$$

Dividing by  $\log 1/r$  and letting  $r \rightarrow 0$

$$\overline{\lim}_{r \rightarrow 0} \left( \frac{\omega}{\log \frac{1}{r}} \right) \leq n.$$

**REMARK:** This is a generalisation of and an improvement upon a result of the author [1], namely, if  $\omega = \omega(x_1, x_2, \dots, x_n)$  is a solution of

$$\Delta\omega \geq r^2e^\omega$$

which is regular for  $0 < r < R$ , then

$$\omega \leq \log \frac{4(n+4)R^4}{r^4(R^2-r^2)^2}$$

and at an isolated singularity of  $\omega$ ,

$$\overline{\lim}_{r \rightarrow 0} \left( \frac{\omega}{\log \frac{1}{r}} \right) \leq 4.$$

**COROLLARY 3.** Every solution  $\omega$  of

$$\Delta\omega = \alpha r^\beta e^\omega$$

where  $\alpha$  and  $\beta$  are arbitrary positive constants and  $\Delta$  is 2-dimensional Laplace operator, satisfies

$$\omega \leq 2 \log \frac{2\sqrt{2}R}{\sqrt{\alpha r^\beta} (R^2-r^2)}.$$

At an isolated singularity of  $\omega$

$$\overline{\lim}_{r \rightarrow 0} \left( \frac{\omega}{\log \frac{1}{r}} \right) \leq \beta.$$

Setting  $f(t) = e^t$  in (6), it could be proved exactly as Corollary 2.

In the next theorem we find a lower bound for the maximum of the solutions of the differential inequality

$$(11) \quad \Delta\omega \geq \alpha r^\beta f(\omega)$$

where  $\Delta$  is a 2-dimensional Laplace operator and  $\alpha, \beta$  are arbitrary positive constants.

**THEOREM 2.** Let  $f(\omega)$  satisfy the conditions of theorem 1 with (3) replaced by

$$(3') \quad f'(\omega) \int_\omega^\infty \frac{dt}{f(t)} = 1.$$

If

$$(G') \quad v(r) = \sup_{Q \in S_r} \omega(Q)$$

where  $\omega(Q)$  ranges over all functions of class  $C^2$  in  $D_r$  ( $r^2 = x_1^2 + x_2^2$ ) and which satisfy the inequality (11), then

$$\int_v^\infty \frac{dt}{f(t)} \leq \frac{\alpha r^\beta (R^2 - r^2)}{4}.$$

PROOF: Consider the function  $h = h_\rho(r)$  defined by

$$(12) \quad \frac{\rho^2 - r^2}{4} = \frac{1}{P(r)} \int_h^\infty \frac{dt}{f(t)} \quad (\rho > R > r)$$

where  $P(r)$  is positive, monotonically increasing and twice continuously differentiable. Clearly,  $h_\rho(r)$  belongs to the class  $C^2$  in  $D_R$  and satisfies the differential inequality (11) if  $P(r) = \alpha r^\beta$ . Differentiating (12) twice with respect to  $x = x_k$  ( $k = 1, 2$ ), we obtain

$$(13) \quad -\frac{x}{2} = -\frac{hx}{f(h)P(r)} - \frac{2x\dot{P}(r)}{P^2(r)} \int_h^\infty \frac{dt}{f(t)}$$

$$-\frac{1}{2} = -\frac{h_{xx}}{f(h)P(r)} + \frac{2xh_x\dot{P}(r)}{f(h)P^2(r)} + \frac{h_x^2}{f^2(h)P(r)} f'h - \frac{2\dot{P}(r)}{P^2(r)} \int_h^\infty \frac{dt}{f(t)}$$

$$- \frac{4x^2}{P^2(r)} \ddot{P}(r) \int_h^\infty \frac{dt}{f(t)} + \frac{8x^2\dot{P}(r)}{P^3(r)} \int_h^\infty \frac{dt}{f(t)} + \frac{2x\dot{P}(r)}{P^2(r)} \cdot \frac{h_x}{f(h)}.$$

Using (13) and rearranging, we get,

$$-\frac{1}{2} = -\frac{h_{xx}}{P(r)f(h)} + \frac{2x^2\dot{P}(r)}{P(r)}$$

$$+ x^2P(r)f'(h) \left[ \frac{\rho^2 - r^2}{2} \cdot \frac{\dot{P}(r)}{P(r)} - \frac{1}{2} \right]^2 - \frac{2x^2\ddot{P}(r) + \dot{P}(r)}{P(r)} \cdot \frac{\rho^2 - r^2}{2}.$$

Summing over both  $x_k$ , we have

$$\frac{\Delta h}{P(r)f(h)} = 1 + \frac{2r^2\dot{P}(r)}{P(r)}$$

$$+ r^2P(r)f'(h) \left[ \frac{\rho^2 - r^2}{2} \cdot \frac{\dot{P}(r)}{P(r)} - \frac{1}{2} \right]^2 - \frac{2r^2\ddot{P}(r) + 2\dot{P}(r)}{P(r)} \cdot \frac{\rho^2 - r^2}{2}.$$

Since  $f' > 0$ , we obtain with the help of (3')

$$\frac{\Delta h}{P(r)f(h)} \geq 1 - \frac{2r^2\ddot{P}P + 2P\dot{P} - 2r^2\dot{P}^2}{P^2(r)} \cdot \frac{\rho^2 - r^2}{2}.$$

Now choose  $P(r)$  such that  $2r^2P(r)\ddot{P}(r) + 2P(r)\dot{P}(r) - 2r^2\dot{P}^2(r) = 0$  or,  $P = \alpha r^\beta$  where  $\alpha$  and  $\beta$  are arbitrary positive constants. Hence,

$$(14) \quad \Delta h \geq \alpha r^\beta f(h).$$



Consequently  $(G')$  and (14) imply

$$h(r) \leq v(r).$$

Since we can take  $\rho$  arbitrary, close to  $R$ , we have

$$(15) \quad \int_0^\infty \frac{dt}{f(t)} \leq \frac{\alpha r^\beta (R^2 - r^2)}{4}$$

which proves theorem 2.

**COROLLARY 4:** *If  $\omega$  satisfies the equation*

$$(16) \quad \Delta\omega = \alpha r^\beta e^\omega \quad \left( \Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right)$$

*and is regular in  $D_r$ , then*

$$(17) \quad \log \frac{4}{\alpha r^\beta (R^2 - r^2)} \leq \omega.$$

*Moreover, at an isolated singularity of  $\omega$ , the behaviour of  $\omega$  is such that*

$$\overline{\lim}_{r \rightarrow 0} \left( \frac{\omega}{\log \frac{1}{r}} \right) \geq \beta.$$

Setting  $f(t) = e^t$  in (15), this could be proved exactly as Corollary 2.

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