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Certain properties of Meijer-Laplace transform

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'Certain properties of Meijer-Laplace transform'

by

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1

A generalization of the classical Laplace transform

$$(1.1) \qquad \phi(p) = p\int_0^\infty e^{-pt}f(t)dt$$

has been introduced by the author [1, p. 57] in the form

$$(1.2) \qquad \phi(p) = p\int_0^\infty G^{m+1,0}_{m,m+1}\left(pt\Big/\begin{matrix} n_1+a_1, \ldots, n_m+a_m \\ n_1, \ldots\ldots, n_m, \sigma \end{matrix}\right) f(t)dt$$

and denoted symbolically by $\phi(p) = G[f(t); a_m, n_m, \sigma]$. Certain rules and recurrence relations have also been established [2] for the transform (1.2), which we shall call as Meijer-Laplace transform. Setting $a_j = 0, j = 1, 2, \ldots, m-1$.

(i) and $a_m = 0, \sigma = 0$, (1.2) reduces to (1.1)

(ii) and $a_m = -m'-k$, $n_m = m'-k$, $\sigma = -m'-k$ we get (1.2) reduced to Meijer's transform [3, p. 730],

$$(1.3) \qquad \phi(p) = p\int_0^\infty (pt)^{-k-\frac{1}{2}}e^{-\frac{1}{2}pt}W_{k+\frac{1}{2},m'}(pt)f(t)dt$$

(iii) and $a_m = \frac{1}{2}-m'-k$, $n_m = 2m'$, $\sigma = 0$ we get (1.2) reduced to Verma's transform [4, p. 209]

$$(1.4) \qquad \phi(p) = p\int_0^\infty (pt)^{m'-\frac{1}{2}}e^{-\frac{1}{2}pt}W_{k,m'}(pt)f(t)dt$$

We denote (1.1), (1.3) and (1.4) symbolically by

$$\phi(p) \doteqdot f(t),\ f(t) \xrightarrow[m']{k+\frac{1}{2}} \phi(p) \text{ and } \phi(p) = W[f(t); k, m']$$

respectively.

In this paper we have obtained a theorem for the Meijer-Laplace transform (1.2), which is a generalization of the well-known Tricomi's theorem [5, p. 564]. Various particular cases have been given and the theorem is illustrated by an example.

2

Theorem: If

$$\phi(p) = G[f(y); a_m, n_m, \sigma]$$

then,

$$(2.1) \qquad p^{n/s-\lambda}\phi(p^{-n/s}) = G\left[t^\lambda \int_0^\infty F(t, y) f(y) dy; \alpha_\mu, \eta_\mu, \rho\right]$$

where [1]

$$F(t, y) = \frac{(2\pi)^{(n-s)/2} \cdot s^{\sigma-\Sigma a_i+\frac{1}{2}}}{n^{\rho-\Sigma \alpha_j+\frac{1}{2}+\lambda}} G^{sm+s,\, n\mu}_{n\mu+sm,\, sm+s+n\mu+n}$$

$$\times \left(\frac{y^s t^n}{s^s n^n} \middle/ \begin{matrix} \dfrac{-\eta_j-\alpha_j-\lambda+l}{n}, \dfrac{n_i+a_i+\nu}{s} \\ \dfrac{n_i+\nu}{s}, \dfrac{\sigma+\nu}{s}, \dfrac{-\eta_j-\lambda+l}{n}, \dfrac{-\rho-\lambda+l}{n} \end{matrix}\right)$$

provided n, s are positive integers $s > n$; $|\arg y^s/p^n| < (s-n)\pi/2$; min (Re η_j, Re ρ) + n min (Re n_i/s, Re σ/s) $> -1 -$ Re $\lambda >$ $-$ Re$(\eta_j+\alpha_j+\lambda)-2$ for $i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, \mu$; $l = 0, 1, \ldots, n-1$; $\nu = 0, 1, \ldots, s-1$. the integral in (2.1) is absolutely and uniformly convergent and the Meijer-Laplace transform of $|t^\lambda \int_0^\infty F(t, y) dy|$ exists.

Proof: From (1.2), replacing p by $p^{-n/s}$ and multiplying by $p^{n/s-\lambda}$, we obtain

$$(2.2) \qquad p^{n/s-\lambda}\phi(p^{-n/s}) = \int_0^\infty p^{-\lambda} G^{m+1,0}_{m,m+1}\left(yp^{-n/s} \middle/ \begin{matrix} n_i+a_i \\ n_i, \sigma \end{matrix}\right) f(y) dy;$$

$$i = 1, 2, \ldots, m.$$

But we have from [6]

[1] For the sake of brevity the symbol

$$G^{2\beta,\beta n}_{\beta n+m,\, 2\beta+sm}\left(x \middle/ \begin{matrix} \dfrac{a_j+\nu}{\beta}, c_i+d_i \\ \dfrac{\pm b+\nu}{\beta}, \dfrac{\alpha_i+l}{s} \end{matrix}\right); \; \begin{matrix} i = 1, 2, \cdots, m;\ j = 1, 2, \cdots, n; \\ \nu = 0, 1, \cdots, \beta-1;\ l = 0, 1, \cdots, s-1 \end{matrix}$$

has been employed to denote

$$G^{2\beta,\beta n}_{\beta n+m,\, 2\beta+sm}\left(x \middle/ \begin{matrix} \dfrac{a_1}{\beta}, \dfrac{a_1+1}{\beta}, \cdots, \dfrac{a_1+\beta-1}{\beta}, \cdots, \dfrac{a_n}{\beta}, \cdots, \dfrac{a_n+\beta-1}{\beta}, c_1+d_1, \cdots, c_m+d_m \\ \dfrac{b}{\beta}, \dfrac{b+1}{\beta}, \cdots, \dfrac{b+\beta-1}{\beta}, \dfrac{-b}{\beta}, \cdots, \dfrac{-b+\beta-1}{\beta}, \dfrac{\alpha_1}{s}, \cdots\cdots, \dfrac{\alpha_m+s-1}{s} \end{matrix}\right).$$

(2.3)
$$p^{-\lambda} G^{m+1,0}_{m,m+1}\left(yp^{-n/s}\Big/ {n_i+a_i \atop n_i,\ \sigma}\right)$$
$$= p\int_0^\infty G^{\mu+1,0}_{\mu,\mu+1}\left(pt\Big/ {\eta_j+\alpha_j \atop \eta_j,\ \rho}\right) t^\lambda F(t, y)dt$$
$$i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, \mu.$$

Hence substituting from (2.3) in the integrand on the R.H.S. of (2.2), and changing the order of integration we get the result (2.1).

To justify the change of order of integration we see that the t-integral in (2.3) is absolutely and uniformly convergent as n and s are positive integers,

$$s > n,\quad \left|\arg \frac{y^s}{p^n}\right| < \left(\frac{s-n}{2}\right)\pi;$$

$$\min (\text{Re}\ \eta_j, \text{Re}\ \rho)+n \min \left(\frac{\text{Re}\ n_i}{s}, \frac{\text{Re}\ \sigma}{s}\right)$$
$$> -1- \text{Re}\ \lambda > -\text{Re}\ (\eta_j+\alpha_j+\lambda)-2$$
$$\text{for } i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, \mu.$$

the y-integral in (2.1) is absolutely and uniformly convergent and the repeated integral is absolutely convergent as the Meijer-Laplace transform of $|t^\lambda \int_0^\infty F(t, y)dy|$ exists. Hence by De La Vallee Poussin's Theorem [7, p. 504] the inversion of the order of integration is justified.

3. Particular cases

In all the following cases we assume n, s as positive integers with $s > n$, $|\arg y^s/p^n| < (s-n)\pi/2$ and the integrals involved are absolutely and uniformly convergent.

(i) With $a_j = 0$, $j = 1, 2, \ldots, m-1$; $a_m = -m'-k$; $\sigma = -m'-k$, $n_m = m'-k$ we have;

If

$$f(y) \xrightarrow[m']{k+\frac{1}{2}} \phi(p)$$

then

$$p^{n/s-\lambda}\phi(p^{-n/s}) = \frac{(2\pi)^{(n-s)/2} s^{k+m'}}{n^{\rho-\Sigma\alpha_j+\frac{1}{2}+\lambda}} G\left[t^\lambda \int_0^\infty F(t, y)f(y)dy;\ \alpha_\mu, \eta_\mu, \rho\right]$$

where

$$F(t,y)=G^{2s,\,n\mu}_{n\mu+s,\,2s+n\mu+n}\left(\frac{y^s t^n}{s^s n^n}\middle|\begin{matrix}\frac{-\eta_j-\alpha_j-\lambda+l}{n},\ \frac{-2k+\nu}{s}\\ \frac{+m'-k+\nu}{s},\ \frac{-\eta_j-\lambda+l}{n},\ \frac{-\rho-\lambda+l}{n}\end{matrix}\right)$$

$$\min(\operatorname{Re}\eta_j,\operatorname{Re}\rho)+\frac{n}{s}\operatorname{Re}(-k\pm m')>-1-\operatorname{Re}\lambda$$

$$>-\operatorname{Re}(\eta_j+\alpha_j+\lambda)-2;$$

for $j=1,2,\ldots,\mu$; $l=0,1,\ldots,n-1$; $\nu=0,1,\ldots,s-1$.

(ia) Further with $k=\pm m'$ we get
If

$$\phi(p)\doteqdot f(y)$$

then

$$(3.1)\quad p^{n/s-\lambda}\phi(p^{-n/s})=\frac{(2\pi)^{(n-s)/2}\cdot s^{\frac{1}{2}}}{n^{\rho-\Sigma\alpha_j+\frac{1}{2}+\lambda}}\,G\left[t^\lambda\int_0^\infty F(t,y)f(y)dy;\alpha_\mu,\eta_\mu,\rho\right]$$

where

$$(3.2)\quad F(t,y)=G^{s,\,n\mu}_{n\mu,\,s+n\mu+n}\left(\frac{y^s t^n}{s^s n^n}\middle|\begin{matrix}\frac{-\eta_j-\alpha_j-\lambda+l}{n}\\ \frac{\nu}{s},\ \frac{-\eta_j-\lambda+l}{n},\ \frac{-\rho-\lambda+l}{n}\end{matrix}\right)$$

$$\min(\operatorname{Re}\eta_j,\operatorname{Re}\rho)+\operatorname{Re}\lambda+1>0,\ \operatorname{Re}(\eta_j+\alpha_j)>-1$$

for $j=1,2,\ldots,\mu$; $l=0,1,\ldots,n-1$; $\nu=0,1,\ldots,s-1$.

(ib) Further with $k=\pm m'$ and $\alpha_j=0$, $j=1,2,\ldots,\mu$; $\rho=0$ we obtain
If

$$\phi(p)\doteqdot f(y)$$

then

$$p^{n/s-\lambda}\phi(p^{-n/s})\doteqdot\frac{(2\pi)^{(n-s)/2}\cdot s^{\frac{1}{2}}}{n^{\frac{1}{2}+\lambda}}\,t^\lambda\int_0^\infty G^{s,\,0}_{0,\,s+n}\left(\frac{y^s t^n}{s^s n^n}\middle|\frac{\nu}{s},\ \frac{-\lambda+l}{n}\right)f(y)dy$$

$$\operatorname{Re}\lambda+1>0;\ l=0,1,\ldots,n-1;\ \nu=0,1,\ldots,s-1.$$

With $n=s=1$ we get the well-known Tricomi's theorem [5, p. 564].

(ii) Setting $a_j=0$, $j=1,2,\ldots,m-1$; $a_m=\frac{1}{2}-k-m'$; $n_m=2m'$, $\sigma=0$ we have
If

$$\phi(p)=W[f(y);k,m']$$

then

$$(3.3)\quad p^{n/s-\lambda}\phi(p^{-n/s}) = \frac{(2\pi)^{(n-s)/2}\cdot s^{\frac{1}{2}}}{n^{\rho-\Sigma\alpha_j+\frac{1}{2}+\lambda}} G\left[t^{\lambda}\int_0^{\infty} F(t, y)f(y)dy; \alpha_\mu, \eta_\mu, \rho\right]$$

where

$$(3.4)\quad F(t, y) = G^{2s,\, n\mu}_{n\mu+s,\, 2s+n\mu+n}\left(\frac{y^s t^n}{s^s n^n}\middle/ \begin{matrix} \dfrac{-\eta_j-\alpha_j-\lambda+l}{n}, \dfrac{\frac{1}{2}-k+m'+\nu}{s} \\ \dfrac{m'\pm m'+\nu}{s}, \dfrac{-\eta_j-\lambda+l}{n}, \dfrac{-\rho-\lambda+l}{n}\end{matrix}\right)$$

$$\min(\operatorname{Re}\eta_j, \operatorname{Re}\rho) + \frac{n}{s}\operatorname{Re}(m'\pm m')$$

$$> -1 - \operatorname{Re}\lambda > -\operatorname{Re}(\eta_j+\alpha_j+\lambda)-2$$

for $j = 1, 2, \ldots, \mu$; $l = 0, 1, \ldots, n-1$; $\nu = 0, 1, \ldots, s-1$.

With $\alpha_j = 0, j = 1, 2, \ldots, \mu-1$; $\alpha_\mu = \frac{1}{2}-\mu'-\lambda'$; $\eta_\mu = 2\mu'$; $\rho = 0$ and $n = s = 1$ we get after some simplification the result by *R.* Narain [8, p. 316].

EXAMPLE: The function $K(z)$ of Boersma J. [9] is defined by

$$K\begin{bmatrix} a_r, \mu_r \\ b_t, \nu_t \end{bmatrix}; z\Bigg] = \frac{1}{2\pi i}\int \frac{\Gamma(-s)\prod_{k=1}^{r}\Gamma(a_j+\mu_j s)}{\prod_{j=1}^{t}\Gamma(b_k+\nu_k s)}(-z)^s ds$$

$$= (\text{const.})\, G^{1,\, \Sigma\mu_j}_{\Sigma\mu_j,\, 1+\Sigma\nu_k}\left(-z\middle/ \begin{matrix} \dfrac{1-a_j+i}{\mu_j} \\ 0, \dfrac{1-b_k+l}{\nu_k}\end{matrix}\right)$$

where μ_j, ν_k are natural numbers; $i = 0, 1, \ldots, \mu_j-1$; $j = 1, 2, \ldots, \gamma$; $l = 0, 1, \ldots, \nu_k-1$; $k = 1, 2, \ldots, t$ and const.

$$= \frac{(2\pi)^{\frac{1}{2}(r-t+\Sigma\nu_k-\Sigma\mu_j)}\prod_{j=1}^{r}\mu_j^{a_j+\mu_j s-\frac{1}{2}}}{\prod_{k=1}^{t}\nu_k^{b_k+\nu_k s-\frac{1}{2}}}$$

Let

$$f(y) = y^{-h}K\begin{bmatrix} a_r, \mu_r \\ b_t, \nu_t \end{bmatrix}; -y\Bigg] \doteqdot p^h K\begin{bmatrix} 1, 1; a_r, \mu_r \\ b_t, \nu_t\end{bmatrix}; -\frac{1}{p}\Bigg] = \phi(p)$$

$$\operatorname{Re} h < 1;\ \operatorname{Re} p > 0;\ \sum\mu_j \leqq 1+\sum\nu_k,$$

in case (ia) of the theorem.

Then (3.1) gives,

$$p^{n/s(1-h)-\lambda}K\begin{bmatrix}1,1;a_r,\mu_r; & -p^{n/s}\\ b_t,\nu_t & \end{bmatrix}$$

$$=\frac{(2\pi)^{(n-s)/2}\cdot s^{\frac{1}{2}}}{\eta^{\rho-\sum\alpha_j+\frac{1}{2}+\lambda}}G\left[t^{\lambda}\int_0^{\infty}F(t,y)K(y)dy;\alpha_\mu,\eta_\mu,\rho\right]$$

where $F(t, y)$ is given by (3.2), provided n, s are positive integers,

$$\text{Re}\,h<1;\ \text{Re}\,p>0;\ \min(\text{Re}\,\eta_j,\text{Re}\,\rho)+\text{Re}\,\lambda+1>0;$$
$$\text{Re}\,(\eta_j+\alpha_j)>-1$$

for $j=1,2,\ldots,\mu$; $l=0,1,\ldots,n-1$; $\nu=0,1,\ldots,s-1$; μ_r, ν_t are natural numbers, $\sum\mu_j\leqq 1+\sum\nu_k$; $s>n$.

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