COMPOSITIO MATHEMATICA

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Compositio Mathematica, tome 18, nº 1-2 (1967), p. 55-60
<http://www.numdam.org/item?id=CM_1967__18_1-2_55_0>

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by

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1. Introduction

In some of his papers (see [1], [2]) the author attempted to investigate various properties of a function $f \in L_1$, when some, rather simple, assumptions are made on the Fourier transform $T[f] = \hat{f}(t) = \int_{-\infty}^{\infty} f(x) e^{itx} dx$ of f. More particularly, several results concerning a function whose Fourier transform has nonnegative real part have been obtained. Also, numerous applications of these results in the theory of holomorphic functions, summability of series, tauberian theorems, etc., have been given. We state one of them as a sample: Let $G(z) = \sum_{n=0}^{\infty} a_n z^n$ be a holomorphic function in the unit circle $|z| < 1$, $(z = re^{it})$. For fixed r, put $E_r = \{t \in [0, 2\pi]; R_e G(re^{it}) < 0\}$. Then:

$$|a_n| \, r^n \leqq 2R_e a_0 + \sup_{t \in E_r} |R_e G(re^{it})|, \qquad (n = 1, 2, \ldots).$$

For $E_r = \emptyset$ and $a_0 = 1$, we get the well known inequality $|a_n| \leqq 2$, due to Carathéodory.

The large number of these significant applications suggests a further investigation of the class of functions whose Fourier transforms have nonnegative real part. In this paper we consider the class $\mathscr{D}$ (often used in the above applications) defined as follows: A function f, non zero almost everywhere, is said to belong to $\mathscr{D}$ if: a) $f \in L_1$; b) $f(x) = 0$ for $x < 0$; c) there is a number c such that $\lim_{h \to 0} 1/h \int_0^h |f(x) - c \mid dx = 0$; d) $\mathrm{R}_e \hat{f}(t) \geqq 0$. We prove that for each $f \in \mathscr{D}$, the corresponding number c is unique, real and positive, and that for a certain equivalence relation $\sim$, in $\mathscr{D}$, the quotient $\mathscr{D}/\sim$ is a group, isomorphic to the multiplicative group of the positive real numbers. Also, as a consequence of these facts, we give a sufficient condition for the positiveness of a function in L_1.

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Definition: *A function f belongs to the class Δ if: a) $f \in L_1$; b) $f(x) = 0$ for $x < 0$; c) there exist $M > 0, h > 0$ such that $|f(x)| \leqq M$ a.e. for $0 < x < h$; d) $R_e\hat{f}(t) \geqq 0$.*

Theorem 1. *If $f \in \Delta$, $g \in \Delta$ then $fg \in \Delta$.*

Proof. We first notice that every function f in Δ is essentially bounded. In fact, since $f(x)+\overline{f(-x)}$ is essentially bounded in a neighbourhood of the origin and $T[f(x)+\overline{f(-x)}] = 2R_e\hat{f}(t) \geqq 0$, it follows [see [3] p. 20] that $T[f(x)+\overline{f(-x)}] \in L_1$. This implies that the inversion holds for $f(x)+\overline{f(-x)}$, i.e.,

$$f(x)+\overline{f(-x)} = \frac{1}{\pi}\int_{-\infty}^{\infty} R_e\hat{f}(t)e^{-itx}\,dt \quad \text{a.e.},$$

so that for $x \geqq 0$,

$$|f(x)| \leqq \frac{1}{\pi}\int_{-\infty}^{\infty} R_e\hat{f}(t)dt \quad \text{a.e.}$$

Next we prove that $fg \in \Delta$. Obviously fg satisfies conditions a), b), c) of the above definition. We show that condition d) is also satisfied.

Put $g_1(x) = \overline{g(x)}$. We have:

$\hat{g}_1(t) = \bar{\hat{g}}(t)$ and $R_e\hat{g}_1(t) = R_e\hat{g}(-t) \geqq 0$. Let now τ, ρ be any two real numbers with $\rho > 0$. Put:

$$\begin{aligned}
K_\rho(\tau) &= \frac{1}{\pi}\int_{-\infty}^{\infty} e^{-\rho t^2}\cdot\hat{f}(t+\tau)R_e\hat{g}_1(t)dt = \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\rho t^2}\hat{f}(t+\tau)[\hat{g}_1(t)+\bar{\hat{g}}_1(t)]dt = \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\rho t^2}\hat{f}(t+\tau)\hat{g}_1(t)dt + \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\rho t^2}\hat{f}(t+\tau)\bar{\hat{g}}_1(t)dt \\
&= A_\rho + B_\rho.
\end{aligned}$$

Put

$$h(x) = (fe^{i\tau x}) * g_1 = \int_0^x f(y)e^{i\tau y}\cdot g_1(x-y)dy.$$

We have:

$$\hat{h}(t) = T[f\cdot e^{i\tau x}]\cdot T[g_1] = \hat{f}(t+\tau)\hat{g}_1(t).$$

Also:

$$\bar{\hat{g}}_1(t) = \int_0^{\infty} g(x)e^{-itx}dx = \int_{-\infty}^{0} g(-x)e^{itx} = T[g(-x)].$$

Put:

$$h_1(x) = (fe^{i\tau x}) * g(-x) = \int_x^\infty f(y)e^{i\tau y} \cdot g(y-x)dx.$$

We have:

$$\hat{h}_1(t) = T[fe^{i\tau x}] \cdot T[g(-x)] = \hat{f}(t+\tau) \cdot \bar{g}_1(t).$$

From the fact that f and g are essentially bounded follows that h and h_1 are continuous.
Therefore:

$$\lim A_\rho = h(0) = 0$$

$$\lim_{\rho\to 0} B_\rho = h_1(0) = \int_0^\infty f(y) \cdot g(y)e^{i\tau y}dy.$$

But it is easily seen that $R_e K_\rho(\tau) \geqq 0$.
This implies that:

$$R_e[\lim_{\rho\to 0} K_\rho(t)] = R_e \int_0^\infty f(y) \cdot g(y)e^{i\tau y}dy = R_e\, T[fg] \geqq 0.$$

The theorem is proved.

Theorem 2. *Hypothesis.* (i) $f \in L_1$; (ii) $R_e\hat{f}(t) \geqq 0$. (iii) *There exists a number c such that*

$$\lim_{h\to 0} \frac{1}{h} \int_0^h |f(x)+f(-x)-c|\, dx = 0.$$

Conclusion. For every real number τ:

$$R_e\{f(x)e^{i\tau x}+f(-x)e^{-i\tau x}\} \leqq R_e c \qquad \text{a.e.}$$

Proof: We first notice that the function $F(x) = f(x)e^{i\tau x}$ satisfies the hypothesis of the theorem. In fact, that (i) and (ii) are satisfied is obvious. That (iii) is satisfied follows from the following inequality:

$$\frac{1}{h}\int_0^h |F(x)+F(-x)-c|\, dx \leqq \frac{1}{h}\int_0^h |f(x)+f(-x)-c|\, dx$$
$$+\frac{1}{h}\int_0^h |\sin x\tau||\, f(x)-f(-x)\, |dx+|\, c|\, \frac{1}{h}\int_0^h (1-\cos x\tau)dx.$$

Next, for $R > 0$, put:

$$I_R(x) = \frac{1}{2\pi}\int_{-\infty}^\infty \hat{F}(t)e^{-(t^2/R^2)}(2-e^{itx}-e^{-itx})dt.$$

Since F satisfies condition (iii), we have:

$$\lim_{R\to\infty} I_R(x) = c - F(x) - F(-x) \text{ a.e.}$$

Also:

$$I_R(x) = \frac{2}{\pi}\int_{-\infty}^{\infty} \hat{F}(t)e^{-(t^2/R^2)} \sin^2 \frac{tx}{2}\, dx \text{ and } R_e\hat{F}(t) \geqq 0.$$

It follows that:

$$R_e[\lim_{R\to\infty} I_R(x)] \geqq 0 \text{ or, } (*)\ R_e\ \{f(x)e^{i\tau x} + f(-x)e^{-i\tau x}\} \leqq R_e c \quad \text{a.e.}$$

The theorem is proved.

Corollary . We have $\mathscr{D} \subset \Delta$.

Proof. Let $f \in \mathscr{D}$. Then f satisfies the hypothesis of Th. 2. For $x > 0$ and $\tau = (-\arg f(x))/x$ we get from (*), $|f(x)| < R_e c$, a.e., which implies that $f \in \Delta$.

Note:

Theorem 2 remains true if assumption (iii) is replaced by

$$\int_0^1 |f(x) + f(-x) - c| \frac{dx}{x} < +\infty. \text{ (see [1]).}$$

The following theorem summarizes some properties of the functions of the class $\mathscr{D}$.

Theorem 3.

I. *To each f in $\mathscr{D}$ corresponds only one number c; noted c_f.*

II. *The class $\mathscr{D}$ is closed under the ordinary addition and multiplication. For $f \in \mathscr{D}$, $g \in \mathscr{D}$ we have:*

$$c_{fg} = c_f c_g,\ c_{f+g} = c_f + c_g$$

III. *For $f \in \mathscr{D}$, the number c_f is real and positive.*

Proof. To prove I, let c_1, c_2 be two numbers corresponding to a function f in $\mathscr{D}$. We have by assumption

$$\lim_{h\to 0}\frac{1}{h}\int_0^h |f(x) - c_1|\, dx = \lim_{h\to 0}\frac{1}{h}\int_0^h |f(x) - c_2|\, dx = 0$$

Therefore the inversion holds, so that

$$c_1 = c_2 = \lim_{R\to\infty}\frac{1}{\pi}\int_{-\infty}^{\infty} f(t)e^{-(t^2/R^2)} dt.$$

We now prove II. Let $f \in \mathscr{D}$, $g \in \mathscr{D}$. For $x < 0$, we have $f(x) \cdot g(x) = 0$, and by the corollary of Th. 2 $|g(x)| \leqq Rc_g$ a.e.
Therefore:

$$|f(x)g(x)-c_f c_g| \leqq |g(x)||f(x)-c_f| + |c_f||g(x)-c_g| \leqq Rc_g|f-c_f|+|c_f||g-c_g|$$

which implies

$$\lim_{h\to 0} \frac{1}{h}\int_0^h |f(x)g(x)-c_f c_g|\, dx = 0.$$

Also, by Th. 1, we have $R_e\widehat{fg} \geqq 0$. It follows that $fg \in \mathscr{D}$ and $c_{fg} = c_f c_g$.
The proof of $c_{f+g} = c_f + c_g$ is trivial.
We finally prove III. First notice that for each $f \in \mathscr{D}$ we have $R_e c_f > 0$. For if $R_e c_f = 0$, then the inequality $|f(x) \leqq Rc_f$ a.e., would imply $f(x) = 0$ a.e., which would contradict the definition of the class $\mathscr{D}$.
Next, we have part II that $f^2 \in \mathscr{D}$ and $c_{f2} = c_f^2$.
Put $c_f = \mu + i\nu$. Then $R_e c_f^2 = R_e c_f^2 = \mu^2 - \nu^2 > 0$ or $|\nu/\mu| < 1$. It follows that $-\pi/4 > \arg c_f < \pi/4$.
Let now n be an arbitrary positive integer. Since $f^n \in \mathscr{D}$ and $c_{f^n} = c_f^n$ we also have:

$$-\frac{\pi}{4} < \arg c_f^n < \frac{\pi}{4}.$$

Since n is arbitrary, the last inequality holds if and only if c_f is a positive real number. The theorem is proved.

Let now Φ be a mapping from $\mathscr{D}$ to the set of positive real numbers, defined by $\Phi(f) = c_f$. The mapping Φ is onto. For if c is any positive real number and f_1 the function defined to be equal to ce^{-x} for $x \geqq 0$ and to zero for $x < 0$, then $f_1 \in \mathscr{D}$ and $c_f = c$. On the other hand Φ is not one-to-one. For let f_2 be the function equal to ce^{-x^2} for $x \geqq 0$ and to zero for $x < 0$. We have $f_2 \in \mathscr{D}$, $f_1 \neq f_2$ and $c_{f_1} = c_{f_2} = c$.

Define now the binary relation $\sim$ in $\mathscr{D}$ as follows. For $f \in \mathscr{D}$, $g \in \mathscr{D}$ put $f \sim g$ if $c_f = c_g$. Then it is easy to see that $\sim$ is an equivalence relation in $\mathscr{D}$, and that there is a one-to-one correspondence between the quotient $\mathscr{D}/\sim$ and the set of positive real numbers. Next define the operation $\otimes$ in $\mathscr{D}/\sim$ as follows: for any two elements A_a, A_b in $\mathscr{D}/\sim$, corresponding to the positive real numbers a and b respectively, set $A_a \otimes A_b = A_{ab}$. The definition is justified by part I of Th. 3. We then easily prove that:

THEOREM 4. *$\mathscr{D}/\sim$, provided with the operation $\otimes$, is an abelian group isomorphic to the multiplicative group of the positive real numbers.*

The following theorem provides a sufficient condition for a function in L_1, to be positive. The problem of determining the positiveness of f from the properties of $\hat{f}$ does not have a simple solution. Bochner proved that f is positive if and only if $\hat{f}$ is positive definite. However, in practice, it is not generally easy to check whether or not $\hat{f}$ is positive definite. The condition given below, far from being a necessary one, seems to be useful for application purposes.

THEOREM 5. *Let $f \in L_1$ non zero a.e., and α a Lebesgue point of f, (i.e. $\lim_{h\to 0} \int_\alpha^{\alpha+h} |f(x)-f(\alpha)|\,dx$). Suppose $R_e \int_0^\infty f(x+\alpha)e^{itx}dx \geqq 0$. Then $f(\alpha)$ is a positive real number.*

PROOF. It can be easily seen that the function F defined to be equal to $f(x+\alpha)$ for $x \geqq 0$ and to zero for $x < 0$ belongs to the class $\mathscr{D}$, with $c_f = f(\alpha)$. It follows from part III of Th. 3 that $f(\alpha)$ is real and positive.

NOTE. Since for $f \in L_1$ almost all of its points are Lebesgue points, it follows from Th. 5 that if for every Lebesgue point α, $R_e \int_0^\infty f(x+\alpha)e^{itx}dx \geqq 0$ then $f(x) > 0$ a.e.

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(Oblatum 28-3-1966)

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