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On the Fourier coefficients of a simple discontinuous function

by

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1

Let $f(x)$ be a function integrable in the sense of Lebesgue over the interval $(-\pi, \pi)$ and defined outside this by periodicity. Let its Fourier series be

$$\tfrac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n \cos nx+b_n \sin nx) \equiv \sum A_n(x).$$

Then

$$\sum_{n=1}^{\infty}(b_n \cos nx-a_n \sin nx) \equiv \sum B_n(x)$$

is the conjugate series of the Fourier series of $f(x)$.

For a fixed x, we write

$$\psi(t) = \psi_x(t) = f(x+t)-f(x-t)-l.$$

2

Let $\sum_{n=1}^{\infty} a_n$ be an infinite series with the partial sums $\{s_n\}$. Let σ_n be the first arithmetical means of s_n, that is, $\sigma_n = (\sum_{\nu=1}^{n} s_\nu)/n$. Let $A : (a_{n,m})$ $(m = 1,2, \ldots; n \leqq m)$ be a regular triangular matrix. If

$$\lim_{m\to\infty} \sum_{n=1}^{m} a_{n,m}\sigma_n = \tau,$$

then we say that the series $\sum a_n$ or the sequence $\{s_n\}$ is summable A. $(C, 1)$ to the sum τ.

3

In this note, we prove a theorem for the A method of summation of the Fourier coefficients of $f(t)$ connected with its jump at the point $t = x$.

THEOREM. *Let the jump of $f(t)$ at $t = x$ be l, i.e.,*

$$f(x+0)-f(x-0) = l.$$

If A satisfies

(i) $$\sum_{n=1}^{m} n \mid a_{n,m}-a_{n+1,m} \mid = o(1) \quad [2],$$

and

(ii) $$\sum_{n=1}^{m} |a_{n,m}| = o(1)$$

as $m \to \infty$, then the sequence

$$\{nB_n(x)\} \equiv \{n(b_n \cos nx - a_n \sin nx)\}$$

is summable A. (C, 1) to l/π.

4

If we denote by σ_n the $(C, 1)$-transformation of $\{nB_n(x)\}$, we have, after Mohanty-Nanda [1],

$$\sigma_n - \frac{l}{\pi} = \frac{1}{\pi}\int_0^1 \psi(t)\left\{\frac{\sin nt}{nt^2} - \frac{\cos nt}{t}\right\} dt + o(1),$$

by Riemann-Lebesgue's theorem.

On account of the regularity of the A method, we need establish that

$$I = \frac{1}{\pi}\sum_{n=1}^{m} a_{n,m}\int_0^1 \psi(t) g_n(t) dt = o(1)$$

as $m \to \infty$, where

$$g_n(t) = \frac{\sin nt}{nt^2} - \frac{\cos nt}{t},$$

which is $O(n^2t)$ by expanding $\sin nt$ and $\cos nt$ into the power series of n and t. It is known that [3, p. 5]

$$\sum_{\nu=1}^{n} \frac{\sin \nu t}{\nu} = O(1).$$

Thus, if we write

$$I = \frac{1}{\pi}\sum_{n=1}^{m} a_{n,m}\left\{\int_0^{n^{-1}} + \int_{n^{-1}}^1\right\} \psi(t) g_n(t) dt$$

$$= \frac{1}{\pi}\sum_{n=1}^{m} a_{n,m}\{P+Q\},$$

say, then,

$$
\begin{aligned}
|P| &\leqq \Big(\sum_{n=1}^{m} |a_{n,m}|\Big)\int_0^{n^{-1}} |\psi(t)|\,|g_n(t)|dt \\
&= o(1).O\left(n^2\int_0^{n^{-1}} t\,dt\right) \\
&= o\left(n^2\int_0^{n^{-1}} t\,dt\right) \\
&= o(1).
\end{aligned}
$$

We denote

$$
\begin{aligned}
G_\nu(t) &= g_1(t)+g_2(t)+\dots+g_\nu(t) \\
&= \frac{1}{t^2}\sum_{\mu=1}^{\nu}\frac{\sin \mu t}{\mu} - \frac{1}{t}\sum_{\mu=1}^{\nu}\cos \mu t \\
&= O\left(\frac{1}{t^2}\right) - \frac{1}{t}D_\nu(t) \\
&= O\left(\frac{1}{t^2}\right),
\end{aligned}
$$

where $D_\nu(t)$ is the Dirichlet kernel of the Fourier series of $f(t)$. It is known that [3, p. 49] $D_\nu(t) = O(t^{-1})$.

By considering that

$$
\sum_{n=1}^{m} |a_{n,m}| = \sum_{n=1}^{m-1} n(|a_{n,m}|-|a_{n+1,m}|)+m|a_{m,m}|
$$

and

$$
\left|\sum_{n=1}^{m-1} n(|a_{n,m}|-|a_{n+1,m}|)\right| \leqq \sum_{n=1}^{m} |a_{n,m}-a_{n+1,m}|,
$$

we see that the conditions (i) and (ii) imply that $ma_{m,m} = o(1)$ as $m \to \infty$. So that, we write

$$
\begin{aligned}
\left|\frac{1}{\pi}\sum_{n=1}^{m} a_{n,m}Q\right| &= \left|\frac{1}{\pi}\sum_{n=1}^{m} a_{n,m}\int_{n^{-1}}^{1}\psi(t)\{G_n(t)-G_{n-1}(t)\}dt\right| \\
&\leqq \left|\frac{1}{\pi}\sum_{n=1}^{m-1}(a_{n,m}-a_{n+1,m})\int_{n^{-1}}^{1}\psi(t)G_n(t)\,dt\right| \\
&\quad + \left|\frac{1}{\pi}\sum_{n=2}^{m} a_{n,m}\int_{n^{-1}}^{(n-1)^{-1}}\psi(t)G_{n-1}(t)\,dt\right| + o(1) \\
&= J_1+J_2+o(1),
\end{aligned}
$$

say, by Abel's transformation. Now,

$$J_1 = O\left\{\sum_{n=1}^{m-1} |a_{n,m} - a_{n+1,m}| \int_{n^{-1}}^{1} \frac{dt}{t^2}\right\}$$

$$= O\left\{\sum_{n=1}^{m} n|a_{n,m} - a_{n+1,m}|\right\}$$

$$= o(1)$$

by the condition (i). And,

$$J_2 = O\left\{\sum_{n=2}^{m} |a_{n,m}| \int_{n^{-1}}^{(n-1)^{-1}} \frac{dt}{t^2}\right\}$$

$$= O\left\{\sum_{n=1}^{m} |a_{n,m}|\right\}$$

$$= o(1)$$

as $m \to \infty$ by the condition (ii) of the theorem.

The theorem is thus completely proved.

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