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by

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Introduction

It is known that a filter is a proper dual ideal in the Boolean ring of all subsets of a set. See Kelley [3,83]. In this paper filters, ideals in semigroups, and ideals in rings are discussed and related to each other. In this connection the chief unifying concept will be the f-ideal (filter motivated ideal).

DEFINITION 1. *An ideal A is an f-ideal in a commutative semigroup S if for $a, b \in A$, there exists $c \in A$ such that $cx = a$ and $cy = b$ have solutions in S. If S itself is an f-ideal, then S will be called an f-semigroup.*

All semigroups and rings are commutative. It will be proved that the filters are identical with the f-ideals of the semigroup of all non-null subsets of a set with the operation of set union; an f-ideal in a semi-group which is an ideal of another semigroup is also an f-ideal of the latter semigroup; a cancellation semigroup has an identity iff all its principal ideals are f-ideals; an f-ideal in the semigroup under multiplication of a ring is always an ideal in the ring; and if a semigroup or ring ideal is an f-ideal with a finite number of generators, then the semigroup or ring ideal is a principal ideal. Finally it will be proved that the semigroup ideals in a ring are identical with the ring ideals iff the semigroup ideals form a linearly ordered set under set inclusion. This latter section is related to work by Aubert [1].

Notation

The notation for filters, semigroups, rings is from Kelley [3], Clifford and Preston [2], and McCoy [4] respectively unless otherwise noted. All semigroup and rings are commutative. When the semigroup in a ring is referred to, it will mean the semigroup of the

[1] Part of the research done at Kent State University.

multiplication operation of the ring. ab will always mean a is multiplied by b in either the semigroup or the ring. AB will mean all products ab such that $a \in A$ and $b \in B$. In these terms A is an ideal in a semigroup S if $AS \subset A$. If S were a ring, in addition $a-b \in A$ in order for A to be an ideal of the ring. The semigroup ideal generated by a set G is $G \cup GS$ and if the S has an identity it is simply GS. If R is a ring, the ring ideal generated by $g_1, g_2, \ldots, g_m$ consists of all elements of the form $\sum_{i=1}^{m}(n_i g_i + r_i g_i)$ where $(n_i \in I, r_i \in R)$. If the elements of an ideal can be expressed in the above form, the ideal will be said to have a finite number of generators. In case R has an identity the elements of the ideal can be expressed in the form $\sum_{i=1}^{m} r_i g_i$. If there exists a single element that generates a semigroup or ring ideal, the ideal is called a principal semigroup or ring ideal respectively. The term identity will refer to the identity of the semigroup or the multiplicative identity or unity of the ring; the additive identity of the ring will be called the zero of the ring.

f-semigroups and filters

THEOREM 1. *Let $\mathscr{S}$ be the semigroup of all non-null subsets of a given set X, under the operation of set union. Then a subfamily of $\mathscr{S}$ is a filter iff it is an f-ideal.*

PROOF: Since for a filter $\mathscr{F}$, $F \cup S \in \mathscr{F}$ for $F \in \mathscr{F}$ and $S \in \mathscr{S}$, $\mathscr{F}$ is an ideal in $\mathscr{S}$. Let $F_1, F_2 \in \mathscr{F}$. Since $(F_1 \cap F_2) \cup F_1 = F_1$ and $(F_1 \cap F_2) \cup F_2 = F_2$, $\mathscr{F}$ is an *f*-ideal in $\mathscr{S}$. Conversely, let $\mathscr{A}$ be an *f*-ideal in $\mathscr{S}$. For $A \in \mathscr{A}$ and $S \in \mathscr{S}$, $A \cup S \in \mathscr{A}$. For $A, B \in \mathscr{A}$, there exists $C \in \mathscr{A}$ such that $C \cup X = A$ and $C \cup Y = B$ have solutions in S. $C \subset A$ and $C \subset B$, so $C \subset A \cap B$ and $C \cup (A \cap B) = A \cap B$. So $A \cap B \in \mathscr{A}$. Since $C \in \mathscr{A}$, $A \cap B \neq \emptyset$, so $\mathscr{A}$ is a filter.

Not every ideal in $\mathscr{S}$ is a filter.

EXAMPLE 1. *Any ideal in $\mathscr{S}$ generated by two disjoint sets A and B is clearly not a filter.*

Properties of *f*-ideals

Not every ideal in the semigroup of integers under multiplication is an *f*-ideal.

EXAMPLE 2. *The ideal consisting of all multiples of either* 2 *or* 3 *is not an f-ideal.*

It is possible for an ideal A to be an ideal in a semigroup T, which is in turn an ideal of a semigroup S and yet A, may not be an ideal in S.

Example 3. *Let S consist of the integers, T the even integers and A the integers congruent to $0(mod\ 8)$ and 4.*

A is an ideal in T, in fact a principal ideal in T, but A is not an ideal in S.

However for A an f-ideal, the above situation can not happen.

Theorem 2. *Let T be an ideal of a commutative semigroup S and let A be an f-ideal in T, then A is an f-ideal in S.*

Proof. Let $a \in A$ and $s \in S$. There exists a' and t, $a' \in A$, $t \in T$ such that $a't = a$. Set $t' = ts$. Since $t' \in T$, then $as = a'ts = a't' = a''$. Clearly $a'' \in A$. So A is an ideal in S and the equations given in definition 1 have solutions in T; so they have solutions in S. Hence A is an f-ideal in S.

All, some or none of the ideals of a commutative semigroup may be f-ideals.

Example 4. *Let S consist of the non-negative reals under addition. All the ideals are f-ideals.*

Example 5. *Let S consist of the positive reals under addition. Ideals of the form $x > a$ for fixed $a \in S$ are f-ideals, but ideals of the form $x \geqq a$ are not f-ideals.*

Example 6. *Let S consist of the even integers, excluding zero, under multiplication. No ideal is an f-ideal.*

Example 7. *Let S consist of the positive reals with $ab = max(a, b)$. Every ideal is an f-ideal but unlike Example 4, there is no identity for S.*

The next three theorems throw some light on these examples.

Theorem 3. *Every ideal in a commutative semigroup S is an f-ideal iff at least one of the equations $ax = b$ or $by = a$ have solutions for every $a, b \in S$.*

Proof: Let A be an ideal in S satisfying the condition of the theorem. Let $a, b \in A$. Clearly either a or b is the c of the definition 1. So A is an f-ideal. Conversely let every ideal be an f-ideal. If the condition is not satisfied there exist two elements a and b (possibly identical) such that neither $ax = b$ or $by = a$ have solutions. Consider the ideal generated by a and b. There must exist $c \in A$ such that $cs = a$ and $ct = b$ have solutions in S.

Since $c \in A$, either $az = c$ or $bw = c$ have solutions. Then either $ax = b$ has a solution zs or $by = a$ has a solution wt.

Examples 4 and 7 illustrate this theorem. In example 5, $bx = a$ does not have a solution for $b \geqq a$, for any $a \in S$, so the principal ideals are not f-ideals.

THEOREM 4. *A principal ideal A in a commutative semigroup S is an f-ideal iff there exists $t \in S$ such that for $b \in A$, $bt = b$ where t is independent of b.*

PROOF: Let A be generated by a. If A is an f-ideal, there exists $c \in A$, $x \in S$ such that $cx = a$. Also there exists $y \in S$ such that $ay = c$. Set $t = xy$; $at = a$. Let $b \in A$; $b = as$ for some $s \in S$. $bt = ast = ats = as = b$, so the condition of the theorem is satisfied. Clearly if the condition is satisfied, a generator of the principal ideal will serve as the c of definition 1, and A will be an f-ideal.

COROLLARY 4. *A commutative cancellation semigroup has an identity iff every principal ideal is an f-ideal.*

THEOREM 5. *For $a, b \in S$, S a commutative f-semigroup, let $ax = b$ have at most a finite number of solutions. Then all the principal ideals of S are f-ideals iff S has an identity.*

PROOF. By Theorem 4, if S has an identity the principal ideals are f-ideals. If every principal ideal is an f-ideal, for $a \in S$, $ax = a$ has a solution by Theorem 4. Let $n(a)$ denote the number of solutions of $ax = a$. Let k be the minimum value taken on by $n(a)$. $k \geqq 1$. Assume $k > 1$. Then for some a there exists k solutions. $x_1, x_2, \ldots x_k$ of $ax = a$. Any solution of $x_m y = x_m$, $m = 1, 2, \ldots, k$ is a solution of $ax = a$ for $ay = ax_m y = ax_m = a$. If $x_1 y = x_1$ has k solutions, $x_2 y = x_2$ can have at most $k-1$ solutions since $x_1 x_2 = x_1$, contrary to k being the minimum value of $n(a)$. So $k = 1$. Hence there exists $b \in S$ such that $bx = b$ has a unique solution which will be designated as e. We proceed to show that e is an identity for S. Let $a \in S$. There exists $c \in S$, such that $cx = a$ and $cy = b$ have solutions $x, y \in S$. $cz = c$ has a solution f. $bf = cyf = cfy = cy = b$ so $f = e$; $ae = cxe = cex = cx = a$; so e is an identity.

Clearly in any commutative semigroup with identity the principal ideals are f-ideals. In Example 7, there is no identity but the equations have an infinite number of solutions for $b = a$. On the other hand in Example 5, there is no identity and there are at most a finite number of solutions of $ax = b$ for each a and b, but none of the principal ideals are f-ideals. Example 6 is a cancellation semigroup without identity and without f-ideals.

Example 2 shows an ideal in a cancellation semigroup with identity that is not an f-ideal. Clearly this ideal is not a principal ideal. The next theorem clarifies the relation between principal and f-ideals.

THEOREM 6. *Let A be an f-ideal with a finite number of generators. Then A is principal ideal.*

PROOF: Let $a_1, a_2, \ldots, a_m$ be the n generators of A and assume $n > 1$. There exists $c \in A$ such that $cx_1 = a_1$, and $cx_2 = a_2$ have solutions in S. Then $c, a_3, \ldots a_n$ generates S. Hence one may show by mathematical induction that A is a principal ideal.

However an f-ideal may have an infinite number of generators.

EXAMPLE 8. *Let $\mathscr{F}$ be the filter on an infinite set X such that $F \in \mathscr{F}$ if the complement of F is finite. By theorem 1, $\mathscr{F}$ is an f-ideal in $\mathscr{S}$, the family of all non-null subsets of X.*

f-ideals and rings

It can be easily shown that an f-ideal in the semigroup of a ring under the ring multiplication is an ideal in the ring but a stronger result can be proved.

THEOREM 7. *Let R be imbedded in a commutative ring R'. Let A be an f-ideal in the semigroup of R' such that $A \subset R$. Then A is a ring ideal in R.*

PROOF: Let $a, b \in A$. There exists $c \in A$ such that $cx = a$ and $cy = b$ have solutions $x, y \in R'$. So $c(x-y) = a-b$, and hence $a-b \in A$ and since A is also a semigroup ideal of R, A is a ring ideal of R.

A ring ideal may not be an f-ideal.

EXAMPLE 9. *Let R be the ring of even integers and R' the ring of integers. No ring ideal of R except the ideal consisting of 0 is an f-ideal, but in R' every ring ideal is an f-ideal.*

Analogous to Theorem 6, we have the following theorem.

THEOREM 8. *Let an ideal A in a commutative R ring have a finite number of generators, If A is an f-ideal, A is a principal ideal in R.*

Because of the similarity to the proof of Theorem 6, we omit the proof.

It might be noted that in commutative rings without identity, the principal semigroup ideals may not be the principal ring ideals.

The semigroup ideal A in example 3 is a principal semigroup ideal in T but is not an f-ideal. On the other hand the integers congruent to 0 modulo 4 are a principal ring ideal of T the even integers, but are not a principal semigroup ideal. Considered as a semigroup ideal, it requires an infinite number of generators. However if $ax = a$ has a solution, both semigroup and ring ideals generated by a are given by aR.

Ideals in semigroups and rings

A ring ideal is always a semigroup ideal. Theorem 7 gives a sufficient condition for a semigroup ideal to be a ring ideal in a commutative ring. Here necessary and sufficient conditions are given for all the semigroup ideals to be ring ideals.

THEOREM 9. *In a commutative ring R of order other than 2, the following are equivalent.*

(a) *The semigroup ideals are linearly ordered by inclusion.*

(b) *Every semigroup ideal is a ring ideal.*

(c) *Every semigroup ideal is an f-ideal.*

(d) *For $a, b \in R$ either $ax = b$ or $by = a$ have solutions in R.*

PROOF: From Theorem 3, (c) and (d) are equivalent, and from theorem 7, (b) follows from (c). We proceed to show that (b) $\rightarrow$ (a). Let A and B be two ideals such that neither $A \subset B$ or $B \subset A$. Then there exist elements $a, b \in R$ such that $a \in A$, $a \notin B$, $b \in B$, $b \notin A$. $A \cup B$ is a semigroup ideal but not a ring ideal. For if $a-b \in A \cup B$, $a-b \in A$ or $a-b \in B$. Then $b \in A$ or $a \in B$. Either one is a contradiction so the semigroup ideals are linearly ordered.

(a) $\rightarrow$ (d) for $a \neq b$ either $A(a) \subset A(b)$ or $A(b) \subset A(a)$ so that either a is in the principal ideal of b or b is in the principal ideal of a. Hence $by = a$ or $ax = b$ have solutions in R for $a \neq b$.

We wish to show that $ax = a$ has solutions for rings not of order 2. Clearly $ax = a$ has a solution in the ring containing one element. Let R contain at least three elements. Let a be an element. If $a = 0$, $ax = a$ has a solution. Let $a \neq 0$ and let $b \neq a$ and $b \neq 0$. If $A(b) \subset A(a)$ and $A(a-b) \subset A(a)$, $aw = b$ and $az = a-b$ have solutions. So $aw+az = a$ and then $ax = a$ has a solution. Since $bx = awx = axw = aw = b$, $bx = b$ has a solution.

By interchanging letters $ax = a$ has a solution when $A(a) \subset A(b)$

and $A(a-b) \subset A(b)$. If $A(a) \subset A(a-b)$ and $A(b) \subset A(a-b)$, $(a-b)y = a$ and $(a-b)z = b$ have solutions so $(a-b)x = a-b$ has a solution and it follows that $ax = a$, and the theorem is proved.

(a) and (b) are equivalent to each other in any ring, but (c) and (d) do not necessarily follow from them, the sole exception being the ring of order two such that all products are zero. The equivalence of (a), (b) and (d) for commutative rings with identity was shown by Aubert [1,54].

In condition (a) of the theorem one can not replace semigroup ideal by ring ideal.

EXAMPLE 10. *Let R consist of the subring of the integers modulo 8 consisting of* 2, 4, 6 *and* 0.

The only proper ring ideal consists of the elements 4 and 0, so these ideals are linearly ordered. But there are two additional proper semigroup ideals consisting of 2, 4, and 0 and 4, 6, and 0 respectively.

However the following theorem is true.

THEOREM 10. *Let the ring ideals of a commutative ring R be linearly ordered by inclusion and let $ax = a$ have a solution for each a. Then the semigroup ideals and the ring ideals of R are identical.*

PROOF: Let A be a semigroup ideal. Let $a, b \in A$. Either $a \in A[b]$ the ring ideal generated by b or $b \in A[a]$. Either $a-b \in A[b] \subset A$ or $a-b \in A[a] \subset A$, since $A(a) = A[a]$ and $A(b) = A[b]$.

COROLLARY 10. *Let the ring ideals of a commutative ring R with identity be linearly ordered by inclusion, then the semigroup ideals are identical with the ring ideals.*

The ideals in a field are trivially linearly ordered. The ideals in the rings of integers modulo p^k are also linearly ordered. The example below shows a ring where the semigroup ideals are linearly ordered and there is an infinite number of distinct ideals.

EXAMPLE 11. *From the ring of rationals, delete fractions of the form a/b, $(a, b) = 1$ such that b is even.*

The ideals are all principal ideals generated by elements of the form 2^k, $k = 0, 1, 2, \ldots$.

It can be shown that, if all the semigroup ideals are principal ideals, then the ideals are linearly ordered. For a discussion of the theorem and a restricted converse, see Aubert [1,44]. However the ring of integers is an example where all the ring ideals are principal ideals, but are not linearly ordered.

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